J. KSIAM Vol.13, No.3, 217–224, 2009

APPROXIMATION OF QUADRIC SURFACES USING SPLINES

YOUNG JOON AHN

DEPT OF MATHEMATICS EDUCATION, CHOSUN UNIVERSITY, GWANGJU, 501–759, SOUTH KOREA
E-mail address: ahn@chosun.ac.kr

ABSTRACT. In this paper we present an approximation method of quadric surface using quartic spline. Our method is based on the approximation of quadratic rational Bézier patch using quartic Bézier patch. We show that our approximation method yields G^1 (tangent plane) continuous quartic spline surface. We illustrate our results by the approximation of helicoid-like surface.

1. INTRODUCTION

Approximations of conic section and quadric surface by Bézier curve and surface are important tasks in CAGD (Computer Aided Geometric Design) or CAD/CAM. In the recent twenty years, many works on the approximation of conic section including circular arc or quadric surface by Bézier curve or surface with high order approximation have been developed [1, 6, 7, 8, 17, 19, 20, 22].

In particular, Fang presented the quintic Bézier curve approximation of circular arc[10] and of conic section[11], and presented rational quartic representation for conic sections[12]. Floater found the approximation of conic section by quadratic Bézier curve[14] and Bézier curve of odd degree n with error bound analysis[15], and presented the approximation of rational curve by Bézier curve having the optimal approximation order $2n$[16]. Recently, Ahn has presented the quartic Bézier curve approximation of conic section with error bound analysis[3].

In this paper we present an approximation method of quadric surface using quartic spline using the approximation method in [3]. We find the necessary and sufficient condition that the quartic approximate spline is G^1 (tangent plane) continuous. We also obtain the error bound of our approximation method using the error analysis in [15].

In §2, we present the method of the quartic spline approximation of the quadric surface, and find the some properties of our approximation method. In §3, we illustrate our assertions by some examples.

Received by the editors July 26 2009; Accepted September 10 2009.

2000 *Mathematics Subject Classification.* 41A05, 41A10, 41A15 .

Key words and phrases. quadric surface, quartic Bézier surface, quadratic rational Bézier surface, tangent plane continuity, conic section, Hausdorff distance.

This work was supported by the Korea Research Foundation Grant funded by the Korean Government(MOEHRD) (KRF-2006-311-C00231) .

2. APPROXIMATION METHOD OF QUADRIC SURFACE USING QUARTIC BÉZIER SURFACE

In this section we present an approximation method of quadric spline by the G^1 quartic spline. Quadric spline consists of quadratic rational Bézier patches, and the quadratic rational Bézier patch is the extension of quadratic rational Bézier curves to two variables, which is also called by conic section. Conic section is represented in the standard rational quadratic Bézier form

$$\mathbf{r}(t) = \frac{\mathbf{p}_0 B_0^2(t) + \mathbf{p}_1 w B_1^2(t) + \mathbf{p}_2 B_2^2(t)}{B_0^2(t) + w B_1^2(t) + B_2^2(t)}, \qquad t \in [0,1]$$

where $\mathbf{p}_0,\ \mathbf{p}_1,\ \mathbf{p}_2 \in \mathbb{R}^3$ are the control points, $w > 0$ is the weight associated with $\mathbf{p}_1$, and $B_i^n(t)$ is the Bernstein polynomial of degree n given by

$$B_i^n(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}, \qquad t \in [0,1].$$

(Refer to [5, 13, 15].) Also the quadratic rational Bézier patch $\mathbf{R}(t,s)$ has the representation

$$\mathbf{R}(t,s) = \frac{\sum_{i=0}^2 \sum_{j=0}^2 \mathbf{p}_{ij} w_{ij} B_i^2(t) B_j^2(s)}{\sum_{i=0}^2 \sum_{j=0}^2 w_{ij} B_i^2(t) B_j^2(s)}, \qquad t,s \in [0,1] \tag{2.1}$$

where $\mathbf{p}_{ij}, (0 \le i,j \le 2)$ are control points, and w_{ij} are weights. In this paper we assume that

$$w = \left[\ w_{ij}\ \right]_{i=0,1,2}^{j=0,1,2} = \begin{bmatrix} 1 & w_2 & 1 \\ w_1 & w_1 w_2 & w_1 \\ 1 & w_2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ w_1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & w_2 & 1 \end{bmatrix}. \tag{2.2}$$

Quartic spline consists of quartic Bézier patches. The quartic Bézier patch $\mathbf{S}(t,s)$ can be expressed by

$$\mathbf{S}(t,s) = \sum_{i=0}^4 \sum_{j=0}^4 \mathbf{b}_{ij} B_i^4(t) B_j^4(s), \qquad t,s \in [0,1]$$

where $\mathbf{b}_{ij},\ 0 \le i,j \le 4$, are the control points, and $\mathbf{b}_{ij}$ also called control net of $\mathbf{S}(t,s)$. Actually, the quartic Bézier approximation method for given quadratic rational Bézier patch means looking for the control net $\mathbf{b}_{ij}$ for the given control points $\mathbf{p}_{ij}$ and weights w_{ij}.

Recently, we found an approximation method of conic section by quartic Bézier curve in [3]. In the method, for given conic section $\mathbf{r}(t)$ having the control points $\mathbf{p}_i$, $i = 0,1,2$, and weight w, the quartic Bézier approximation $\mathbf{b}(t) = \sum_{i=0}^4 B_i^4(t)\mathbf{b}_i$ has the control points

$$\begin{aligned} \mathbf{b}_0 &= \mathbf{p}_0 \\ \mathbf{b}_1 &= (1-\alpha)\mathbf{p}_0 + \alpha \mathbf{p}_1 \\ \mathbf{b}_2 &= \frac{1-\beta}{2}\mathbf{p}_0 + \beta \mathbf{p}_1 + \frac{1-\beta}{2}\mathbf{p}_2 \\ \mathbf{b}_3 &= \alpha \mathbf{p}_1 + (1-\alpha)\mathbf{p}_2 \\ \mathbf{b}_4 &= \mathbf{p}_2 \end{aligned} \tag{2.3}$$

where

$$\begin{aligned}\alpha &= \frac{2w}{w+1} - \frac{3}{4}\beta \\ \beta &= \frac{2(w^2 - w + 2 + 2(w-1)\sqrt{w+1})}{3w(w+1)}.\end{aligned}$$

Also, we had the error bound analysis[3],

$$d_H(\mathbf{b}, \mathbf{r}) \le E_4(w)|\mathbf{p}_0 - 2\mathbf{p}_1 + \mathbf{p}_2| \tag{2.4}$$

where

$$E_4(w) = \frac{1}{2^8} \max\left(\frac{1}{w^2}, 1\right) \frac{|w-1|^3(w+2-2\sqrt{w+1})^2}{w^2(w+1)}$$

for $0 < w < \frac{7+\sqrt{17}}{2} \approx 5.562$.

Now, we construct the quartic Bézier surface approximation of quadratic rational Bézier surface. Let $\mathbf{R}(t, s)$ be given quadratic rational Bézier patch as in Equation (2.1). We use the quartic Bézier curve approximation method[3] of conic section as two variables case. Thus we have the matrix form of the quartic Bézier curve approximation in Equation (2.3)

$$(\mathbf{b}_0, \cdots, \mathbf{b}_4)^T = A_w(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2)^T \tag{2.5}$$

or equivalently, $(\mathbf{b}_0, \cdots, \mathbf{b}_4) = (\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2)A_w^T$, where A_w is 5×3 matrix

$$A_w = \begin{pmatrix} 1 & 0 & 0 \\ 1-\alpha & \alpha & 0 \\ \frac{1-\beta}{2} & \beta & \frac{1-\beta}{2} \\ 0 & \alpha & 1-\alpha \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally, we present the quartic Bézier surface approximation

$$\mathbf{S}(t, s) = \sum_{i=0}^{4} \sum_{j=0}^{4} \mathbf{b}_{ij} B_i(t) B_j(s)$$

where the control net $\mathbf{b}_{ij}$ is

$$(\mathbf{b}_{ij})_{i=0,\cdots,4}^{j=0,\cdots,4} = A_{w_1} (\mathbf{p}_{ij})_{i=0,1,2}^{j=0,1,2} A_{w_2}^T.$$

As an example, we plot the quadratic rational Bézier patch and its quartic Bézier approximation in Figure 1. For more detailed description, we define the intermediate surface $\mathbf{H}(t, s)$ as the quartic Bézier curve approximation of $\mathbf{R}(t, s)$ in view point of the variable s. Then

$$\mathbf{H}(t, s) = \sum_{j=0}^{4} \frac{\sum \mathbf{h}_{ij} w_{i0} B_i^2(t)}{\sum w_{i0} B_i^2(t)} B_j(s) \tag{2.6}$$

where the control points $(\mathbf{h}_{i0}, \cdots, \mathbf{h}_{i4})^T = A_{w_2}(\mathbf{p}_{i0}, \mathbf{p}_{i1}, \mathbf{p}_{i2})^T$ for $i = 0, 1, 2$, or equivalently,

$$(\mathbf{h}_{i0}, \cdots, \mathbf{h}_{i4}) = (\mathbf{p}_{i0}, \mathbf{p}_{i1}, \mathbf{p}_{i2})A_{w_2}^T. \tag{2.7}$$

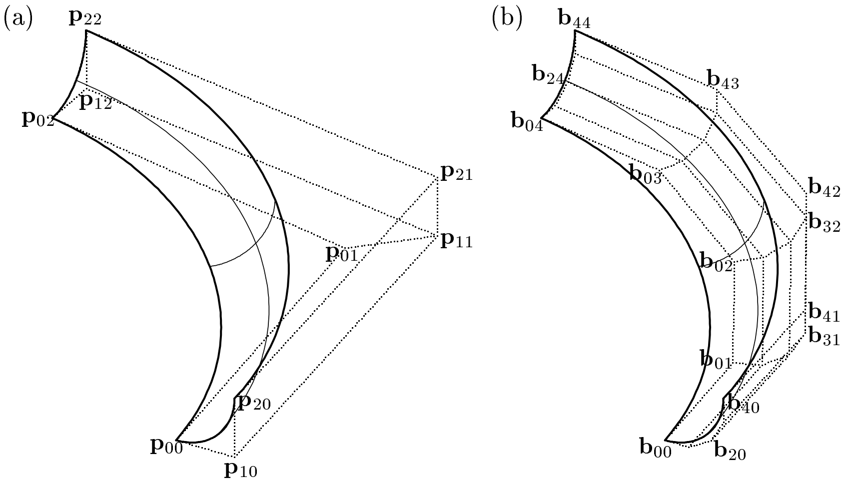

FIGURE 1. (a) The quadratic rational Bézier patch having the control net $(\mathbf{p}_{ij})_{i=0,1,2}^{j=0,1,2}$. (b) The quartic approximate Bézier patch having $(\mathbf{b}_{ij})_{i=0,\cdots,4}^{j=0,\cdots,4}$. The boundary curves of both surface are plotted by thick lines.

$\mathbf{H}(t,s)$ is the conic section in direction of t and quartic Bézier curve in direction of s. Also, $\mathbf{S}(t,s)$ is obtained as the quartic Bézier curve approximation of $\mathbf{H}(t,s)$ in view point of the variable t. Thus $(\mathbf{b}_{0j},\cdots,\mathbf{b}_{4j})^T = A_{w_1}(\mathbf{h}_{0j},\mathbf{h}_{1j},\mathbf{h}_{2j})^T$ for $j=0,\cdots,4$.

We will find a sufficient condition that our approximation method yields G^1 continuous quartic Bézier surface. Let $\mathbf{R}^l(t,s)$ and $\mathbf{R}^r(t,s)$ be two consecutive quadratic rational Bézier patches with common boundary curve $\mathbf{R}^l(1,s)=\mathbf{R}^r(0,s)$, $s\in[0,1]$. Let $\mathbf{p}_{ij}^l$ and $\mathbf{p}_{ij}^r$ be the control points of $\mathbf{R}^l(t,s)$ and $\mathbf{R}^r(t,s)$, respectively. Clearly, $\mathbf{p}_{2j}^l=\mathbf{p}_{0j}^r$, $j=0,1,2$. If the two patches satisfy

$$\begin{aligned} \mathbf{p}_{1j}^l\mathbf{p}_{2j}^l &= \lambda\mathbf{p}_{0j}^r\mathbf{p}_{1j}^r, \qquad (j=0,1,2) \\ w_1^l = w_1^r, &\quad w_2^l = w_2^r \end{aligned} \tag{2.8}$$

for some constant $\lambda>0$, then our approximation method yields G^1 continuous quartic spline.

Proposition 2.1. *If any consecutive quadratic rational Bézier patches satisfy Equation (2.8) on their common boundary, then our approximation method yields G^1 continuous quartic spline.*

Proof. Let $\mathbf{S}^l(t,s)$ and $\mathbf{S}^r(t,s)$ be the quartic Bézier surface approximations of $\mathbf{R}^l(t,s)$ and $\mathbf{R}^r(t,s)$, respectively, with common boundary curve $\mathbf{S}^l(1,s)=\mathbf{S}^r(0,s)$, $s\in[0,1]$. To show that they are G^1 continuous at the common boundary, it is sufficient[13, 18, 21] to show that $\mathbf{b}_{3j}^l\mathbf{b}_{4j}^l = \lambda\mathbf{b}_{0j}^r\mathbf{b}_{1j}^r$, $(j=0,\cdots,4)$, where $\mathbf{b}_{ij}^l$ and $\mathbf{b}_{ij}^r$ are the control nets of $\mathbf{S}^l$ and $\mathbf{S}^r$, respectively.

Let $\mathbf{h}^l_{ij}$ and $\mathbf{h}^r_{ij}$, ($i = 0, 1, 2$ and $j = 0, \cdots, 4$), be the control net of intermediate surfaces $\mathbf{H}^l$ and $\mathbf{H}^r$, respectively. Since $w^l_2 = w^r_2$, we obtain $A_{w^l_2} = A_{w^r_2}$ and $\mathbf{h}^l_{2j} = \mathbf{h}^r_{0j}$ for $j = 0, \cdots, 4$. Let $A_{w_2} = (a_{ij})^{j=0,1,2}_{i=0,\cdots,4}$.

By Equation (2.7), for $i = 0, 1, 2$ and $j = 0, \cdots, 4$, $\mathbf{h}_{ij} = \sum_{k=0}^{2} a_{jk}\mathbf{p}_{ik}$. Thus for each $j = 0, \cdots, 4$,

$$\begin{aligned} \mathbf{h}^l_{1j}\mathbf{h}^l_{2j} &= \sum_{k=0}^{2} a_{jk}\mathbf{p}^l_{2k} - \sum_{k=0}^{2} a_{jk}\mathbf{p}^l_{1k} = \sum_{k=0}^{2} a_{jk}\mathbf{p}^l_{1k}\mathbf{p}^l_{2k} \\ &= \lambda \sum_{k=0}^{2} a_{jk}\mathbf{p}^r_{0k}\mathbf{p}^r_{1k} = \lambda \mathbf{h}^r_{0j}\mathbf{h}^r_{1j}. \end{aligned} \tag{2.9}$$

For $i = 0, \cdots, 4$ and $j = 0, \cdots, 4$, $\mathbf{b}_{ij} = \sum_{k=0}^{2} a_{ik}\mathbf{h}_{kj}$. And for each $j = 0, \cdots, 4$,

$$\begin{aligned} \mathbf{b}^l_{3j}\mathbf{b}^l_{4j} &= \sum_{k=0}^{2} a_{4k}\mathbf{h}^l_{kj} - \sum_{k=0}^{2} a_{3k}\mathbf{h}^l_{kj} = \sum_{k=0}^{2} (a_{4k} - a_{3k})\mathbf{h}^l_{kj} \\ \mathbf{b}^r_{0j}\mathbf{b}^r_{1j} &= \sum_{k=0}^{2} a_{1k}\mathbf{h}^r_{kj} - \sum_{k=0}^{2} a_{0k}\mathbf{h}^r_{kj} = \sum_{k=0}^{2} (a_{1k} - a_{0k})\mathbf{h}^r_{kj} \end{aligned}$$

Since $a_{40} - a_{30} = a_{12} - a_{02} = 0$, $a_{41} - a_{31} = a_{10} - a_{00} = -\alpha$ and $a_{42} - a_{32} = a_{11} - a_{01} = \alpha$, we have

$$\begin{aligned} \mathbf{b}^l_{3j}\mathbf{b}^l_{4j} &= \alpha(\mathbf{h}^l_{2j} - \mathbf{h}^l_{1j}) = \alpha\mathbf{h}^l_{1j}\mathbf{h}^l_{2j} \\ \mathbf{b}^r_{0j}\mathbf{b}^r_{1j} &= \alpha(\mathbf{h}^r_{1j} - \mathbf{h}^r_{0j}) = \alpha\mathbf{h}^r_{0j}\mathbf{h}^r_{1j} \end{aligned}$$

By Equation (2.9) we finally have

$$\mathbf{b}^l_{3j}\mathbf{b}^l_{4j} = \lambda \mathbf{b}^r_{0j}\mathbf{b}^r_{1j}.$$

□

Corollary 2.2. *If we use this approximation method for a quadric surface, then the G^1 continuity at the boundary is automatically achieved.*

We also present the error analysis of our approximation method as the following proposition. Under the assumption Equation (2.2), the proof of the following proposition can be obtained by the similar way in that of Theorem 4.2 in Floater[15] for odd degree. Thus we omit the proof.

Proposition 2.3. *For given quadric surface $\mathbf{R}(t, s)$, the quartic Bézier surface approximation $\mathbf{S}(t, s)$ has the error bound*

$$d_H(\mathbf{R}, \mathbf{S}) \leq E_4(w_2) \max_{i=0,1,2} |\mathbf{p}_{i0} - 2\mathbf{p}_{i1} + \mathbf{p}_{i2}| + E_4(w_1) \max_{j=0,1,2} |\mathbf{p}_{0j} - 2\mathbf{p}_{1j} + \mathbf{p}_{2j}|,$$

where $d_H(\mathbf{R}, \mathbf{S})$ is the Hausdorff distance between two surfaces $\mathbf{R}$ and $\mathbf{S}$. (For more knowledge of the Hausdorff distance refer to [4, 9, 15].*)*

Proof. See Theorem 4.2 in Floater[15]. □

3. Examples and comments

In this section we apply our approximation method to two examples. One is to approximate the quadratic rational Bézier patch by quartic Bézier surface. Let $\mathbf{R}(t,s)$ be the quadratic rational Bézier patch with the control points

$$(\mathbf{p}_{ij}) = \begin{pmatrix} (0,5,\frac{\pi}{2}-1) & (0,6,\frac{\pi}{2}-1) & (0,6,\frac{\pi}{2}) \\ (-5,5,\frac{3\pi}{4}-1) & (-6,6,\frac{3\pi}{4}-1) & (-6,6,\frac{3\pi}{4}) \\ (-5,0,\pi-1) & (-6,0,\pi-1) & (-6,0,\pi) \end{pmatrix}$$

and $w_1 = w_2 = 1/\sqrt{2}$, as shown in Figure 1(a). Using our approximation method, the quartic Bézier surface $\mathbf{S}(t,s)$ is obtained as shown in Figure 1(b), and Proposition 2.2 yields the upper bound of error

$$d_H(\mathbf{R},\mathbf{S}) \le E_4(\frac{1}{\sqrt{2}}) \times (6\sqrt{2}+\sqrt{3}) \approx 2.08 \times 10^{-5}.$$

The other is to approximate the helix-like surface by quartic spline. The helix-like surface was constructed by the quadratic rational spline such as in [2]. The quadratic rational spline satisfies the condition in (2.8) with $\lambda = 1$ and $w_0 = w_1 = \frac{1}{\sqrt{2}}$. As shown in Figure 2(a), the quadratic rational spline consists of 4×2 quadratic rational Bézier patches. Using our approximation method, we have the quartic spline consisting of 4×2 quartic Bézier patches as shown in Figure 2(b). On each common boundaries between consecutive quartic Bézier patches, they are G^1 continuous. Thus the quartic Bézier spline is an G^1 approximation of the quadric spline.

Our approximation method can be also applied to approximate spheres and torii. It is easy to see that the approximate quartic splines of these surface are also G^1 continuous and have very small upper bounds of error.

Acknowledgements

The authors are very grateful to the anonymous referees for their valuable suggestions.

References

[1] Y.J. Ahn. Conic approximation of planar curves. Comp. Aided Desi. 33 (2001) 867-872.
[2] Y.J. Ahn. Helix approximations with conic and quadratic Bézier curves. Comp. Aided Geom. Desi. 22 (2005) 551-565.
[3] Y.J. Ahn. Approximation of conic sections by curvature continuous quartic splines. submitted.
[4] Y.J. Ahn, H.O. Kim. Approximation of circular arcs by Bézier curves. J. Comp. Appl. Math. 81 (1997) 145–163.
[5] Y.J. Ahn, H.O. Kim. Curvatures of the quadratic rational Bézier curves. Comp. Math. Appl. 36 (1998) 71–83.

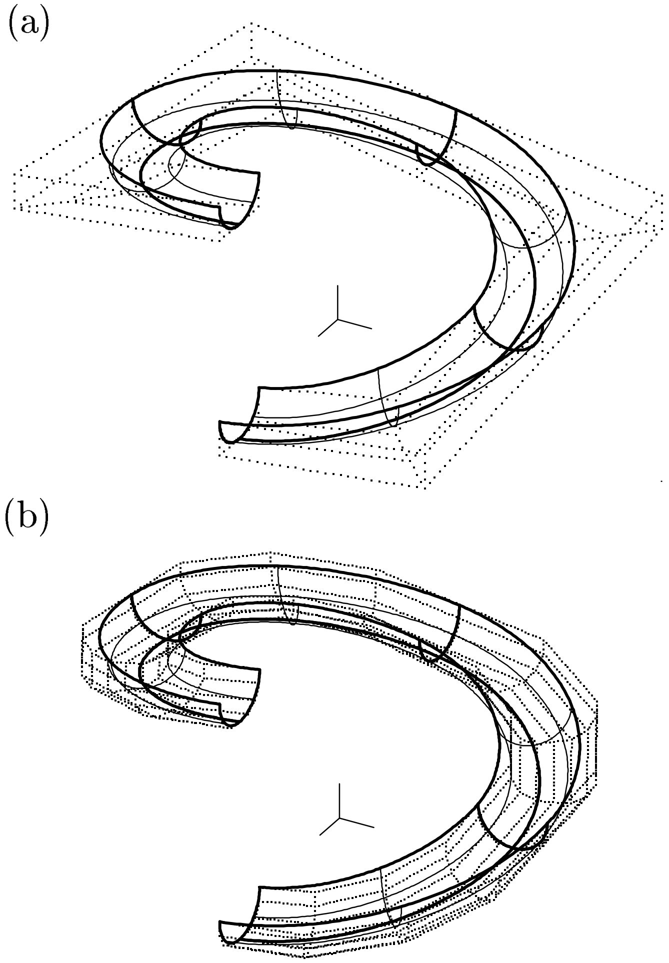

FIGURE 2. (a) The helix-like surface constructed by 4×2 quadratic rational Bézier patches, whose boundaries are plotted by thick lines. (b) The quartic approximate spline constructed by 4×2 quartic Bézier patches, whose boundaries are also plotted by thick lines. The quartic spline is G^1 continuous on every common boundaries of the consecutive quartic Bézier patches.

[6] Y.J. Ahn, Y. S. Kim, Y. S. Shin. Approximation of circular arcs and offset curves by Bezier curves of high degree. J. Comp. Appl. Math. 167 (2004) 181–191.
[7] C. de Boor, K. Höllig, M. Sabin. High accuracy geometric Hermite interpolation. Comp. Aided Geom. Desi. 4 (1987) 169–178.

[8] T. Dokken, M. Dæhlen, T. Lyche, K. Mørken. Good approximation of circles by curvature-continuous Bézier curves. Comp. Aided Geom. Desi. 7 (1990) 33–41.

[9] J. D. Emery. The definition and computation of a metric on plane curves. Comp. Aided Desi. 18 (1986) 25-28.

[10] L. Fang. Circular arc approximation by quintic polynomial curves. Comp. Aided Geom. Desi. 15 (1998) 843-861.

[11] L. Fang. G^3 approximation of conic sections by quintic polynomial curves. Comp. Aided Geom. Desi. 16 (1999) 755-766

[12] L. Fang. A rational quartic Bezier representation for conics. Comp. Aided Geom. Desi. 19 (2002) 297–312.

[13] G. Farin. *Curves and Surfaces for Computer Aided Geometric Design.* Morgan-Kaufmann, San Francisco, 2002.

[14] M. Floater. High order approximation of conic sectons by quadratic splines. Comp. Aided Geom. Desi. 12 (1995) 617–637.

[15] M. Floater. An $\mathcal{O}(h^{2n})$ Hermite approximation for conic sectons. Comp. Aided Geom. Desi. 14 (1997) 135–151.

[16] M. Floater. High order approximation of rational curves by polynomial curves. Comp. Aided Geom. Desi. 23 (2006) 621-628.

[17] M. Goldapp. Approximation of circular arcs by cubic polynomials. Comp. Aided Geom. Desi. 8 (1991) 227–238.

[18] J. A. Gregory. 1989. Geometric continuity. In T. Lyche and L.L. Schumaker, editors, *Mathematical Methods in CAGD*, pages 353–371, Nashville, Academic Press.

[19] Q. Q. Hu, G. J. Wang. Necessary and sufficient conditions for rational quartic representation of conic sections. J. Comput. Appl. Math. 203 (2007) 190-208.

[20] S. H. Kim, Y. J. Ahn. Approximation of circular arcs by quartic Bézier curves. Comp. Aided Desi. 39 (2007) 490–493.

[21] L. Lu. Approximating tensor product Bezier surfaces with tangent plane continuity. J. Comp. Appl. Math. 231 (2009) 412–422.

[22] K. Mørken. 1990. Best approximation of circle segments by quadratic Bézier curves. In P.J. Laurent, A. Le Méhauté, and L.L. Schumaker, editors, *Curves and Surfaces*, New York, Academic Press.