J. KSIAM Vol.13, No.2, 161–167, 2009

SIMULATIONS OF INK DIFFUSION ON PAPER USING VISIT COUNTS FROM RANDOM WALK SESSIONS

HEECHANG KIM[1] AND MYUNGJOO KANG[2,†]

[1] SYSTEMES INTELLIGENTS DE PERCEPTION UNIVERSITE RENE DESCARTES PARIS 5, PARIS, FRANCE

[2] DEPARTMENT OF MATHEMATICAL SCIENCE, SEOUL NATIONAL UNIVERSITY, SEOUL 151-547, KOREA

ABSTRACT. An animated ink diffusion on paper is simulated through making a multiple sessions of random walks. The simulated random walk is built and validated against the diffusion model, then animated by varying the intensity thresholds of the accumulated visit counts on each pixels on an image. Two different random walk models are built one of which is a free random walk in that the walker has exactly same probability to move in any four or eight directions in each step. The other is a biased random walk that has a higher chance to go to a pixel that has more similar intensity value. The latter can be used to simulate an ink diffusion radiating through different texture of paper.

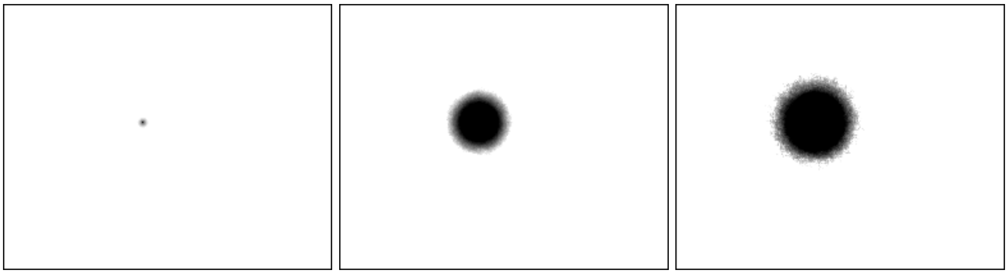
1. INTRODUCTION

Computer animations of ink painting and its rendering have been presented in the literature [1], [2], and [3]. It is done mostly, however, to simulate the diffusion process after brush strokes are applied or a blot of ink is dropped on a piece of paper, except for [3] which deals with diffusion of dyes on different textures of various types of fabric models. In this paper, we simulate a diffusion of ink from a micropipette which radiates from a single point on paper to the designated boundaries given by an image without numerically solving a partial differential equation. In section **Motivation**, we present the motivation behind our method. In section **Random Walk Simulations**, we explain how the random walk simulation is built and validate the model built. In section **Results**, we show the results of the animated diffusion. Finally in section **Conclusion**, we conclude and state our future works.

2. MOTIVATION

The motivation of this paper is simply to simulate and render a diffusion of ink from a single spot which radiates out from it as shown in Fig. 1 The first inclination is either to solve

Received by the editors May 27 2009; Accepted June 1 2009.
2000 *Mathematics Subject Classification.* 65C20, 65D10.
Key words and phrases. Diffusion process, Random walk.
† Corresponding Author. mkang@snu.ac.kr.



(a) (b) (c)

FIGURE 1. Ink diffusion in sequence. (a) S1. (b) S2. (c) S3.

numerically the heat equation

$$\frac{\partial \phi(\vec{r},t)}{\partial t} = \nabla \cdot \bigg(D(\phi,\vec{r})\, \nabla \phi(\vec{r},t) \bigg). \tag{2.1}$$

where $\phi(\vec{r},t)$ is the density of the diffusing material at location $\vec{r}$ and time t and $D(\phi,\vec{r})$ is the diffusion coefficient or to iteratively calculate the transition matrix with columns of random variable vectors in a Markov chain

$$Pr(X_{n+1} = x | X_n = x_n, ..., X_1 = x_1) = Pr(X_{n+1} = x | X_n = x_n). \tag{2.2}$$

However, what we aim to have in this paper is the fine intensity difference along the front of the diffusion as shown in Fig. 2. The first method from Eqn. (2.1) will yield only the location of the front of the diffusion at certain times and will not show what the weaker intensities of the boundary that follows the front of the diffusion. And the second method from Eqn. (2.2) gives the probability of a particle being at certain locations after a fixed time steps which means that the probabilities around the starting position tend toward the probability at the starting position. This results in a ink blot spreading wider along with time steps rather than a single point spreading out. Therefore, rather than the above two methods, we build a random walk simulation model in which case we keep track of the visit counts at each pixels which in the end will represent the intensities of the diffusion at a given time.

3. RANDOM WALK SIMULATIONS

Two different types of random walk model were created. The first model, the classical random walk model, imitates the Brownian motion in that for each step that the walker takes, the probabilities of taking a direction from four choices of north, south, east, and west are the same. Consider a simple one dimensional random walk

$$S_k = \chi_1 + \ldots + \chi_k \tag{3.1}$$

where χ_k are independent random variables taking values +1 or -1 with probabilities p and $q = 1 - p$. Without loss of generality, assume that the walk always begins at the origin x_0 and that the walk has an equal chance to go either left or right, $p = 1/2$. It is obvious then that, for

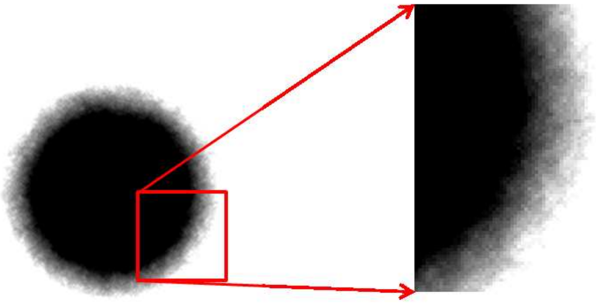


FIGURE 2. The first figure in the paper

any single session of a random walk, the probability of the walk visiting the origin is always 1 (i.e. $P(\chi_{x_0}) = 1$). Also for $x \neq 0$ the density function of the number of visits to positions with fixed $k \geq 1$ is [4]:

$$P(\xi_x) = \frac{1}{4x^2}\left(1 - \frac{1}{2\|x\|}\right)^{k-1}. \tag{3.2}$$

We can represent a probability distribution of this function in terms of integral:

$$\int_a^b P(\xi_x)\,dx. \tag{3.3}$$

Note that Eq. 3.3 is a representation of a random walk visit records for each step from which we can calculate discretely:

$$\sum_{i=a}^{b} P(\xi_i) \tag{3.4}$$

for a given number of steps k and where a and b are the boundary of the walk on the left. And on the right, Fig. 3 shows the distribution of expected visit counts at positions with lattice size and number of steps ranging from 25 to 100 and 50 to 200 respectively. In case of $p \neq 1/2$:

$$P(\xi_x) = \begin{cases} \frac{|p-q|^2}{|1-\left(\frac{q}{p}\right)^x|^2}\left[1 - \frac{|p-q|}{1-\left(\frac{q}{p}\right)^x}\right]^{k-1}, \\ for\ p > q, x > 0\ or\ p < q, x < 0 \\ \frac{|p-q|^2}{|1-\left(\frac{q}{p}\right)^x||1-\left(\frac{p}{q}\right)^x|}\left[1 - \frac{|p-q|}{1-\left(\frac{p}{q}\right)^x}\right]^{k-1}, \\ for\ p > q, x < 0\ or\ p < q, x > 0 \end{cases} \tag{3.5}$$

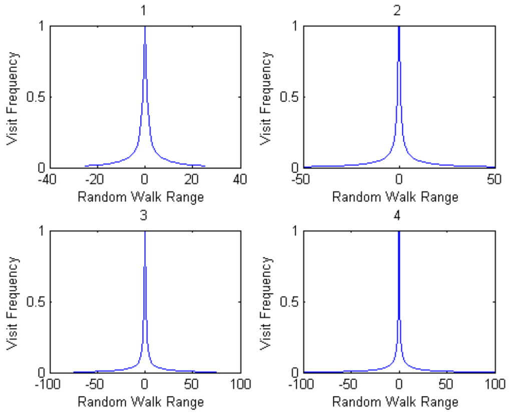


FIGURE 3. Random Walk Frequency

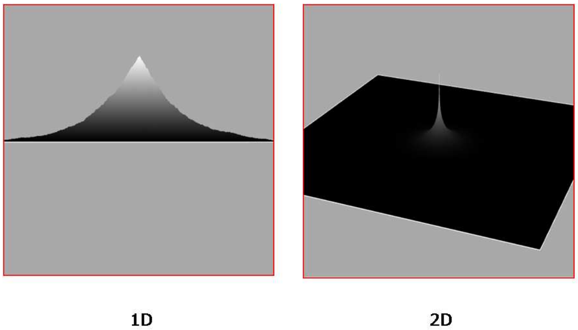


FIGURE 4. Visualized Visit Counts

where the parameters are the same as in Eqn. 3.2. Fig. 4 shows the distributions of one dimension and two dimension visit counts obtained from the module. The comparison of the computed distribution computed from Eq. 3.2 and the simulated distribution from Fig. 4 with corresponding lattice sizes and the number of steps is done and verified. Also, the plot of square distance against the time steps is plotted in Fig. 5 with ten thousand iterations and ten thousand steps taken. Notice the linear relationship between the square distance and the time steps from which the slope of the line represent the empirically determined diffusion coefficient in Eqn. (2.1).

The second model on the other hand can be considered a biased random walk which the directions the walker takes at each step depend on the intensities of the neighbor pixels. In deciding the probabilities of the directions taken, we take the typical Gaussian weighting function given by:

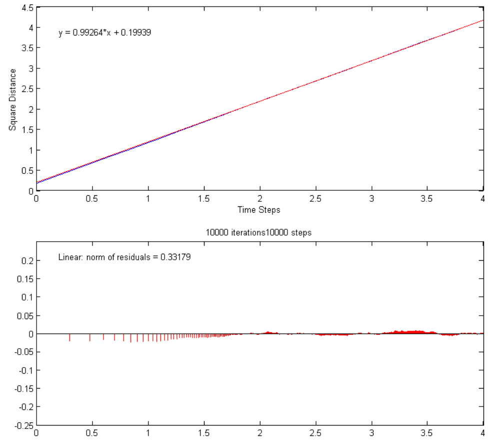


FIGURE 5. Square Distance vs. Time Steps

$$w_{ij} = exp(-\beta(g_i - g_j)^2). \quad (3.6)$$

where g_i indicates the image intensity at pixel i. The value β is used to decide how much of a factor the difference in intensity should be. Thus, this model will more likely let the walker step to directions that have similar levels of intensities with the intensities of the current position. Surely, this behavior can be altered easily by taking another weighting function. It seems sufficient that we have the first model to simulate the ink diffusion on paper but as we'll show in section **Results**, it becomes rather convenient to use the biased random walk simulations to mimic the texture patterns of the diffusion using an image with different intensity regions. As mentioned earlier, both methods keep track of the walker's visit counts at each pixel. We state here how the visit count can be used to represent the diffusion process. It was mentioned in section **Introduction** that the ink diffusion on paper fits better the diffusion from a micropipette model. The accumulated visit counts on each pixel has strong correlation with the concentration or the number of particles transferred from the micropipette onto the paper. Because the distribution of random walk frequencies remain the same regardless of the number of steps with enough iterations, we take series of upper and lower bounds of visit counts from the highest value of the visit counts to the lowest value to be the current state of diffusion. By taking subsequent values of upper and lower bounds down the visit counts, we are able to create an animated ink diffusion on paper as will be shown in section **Results**. At the time of submission, we do not have the definite correlations between the number of particles, the speed of the diffusion, and the visit counts. This will be made clear in our following papers in the future.

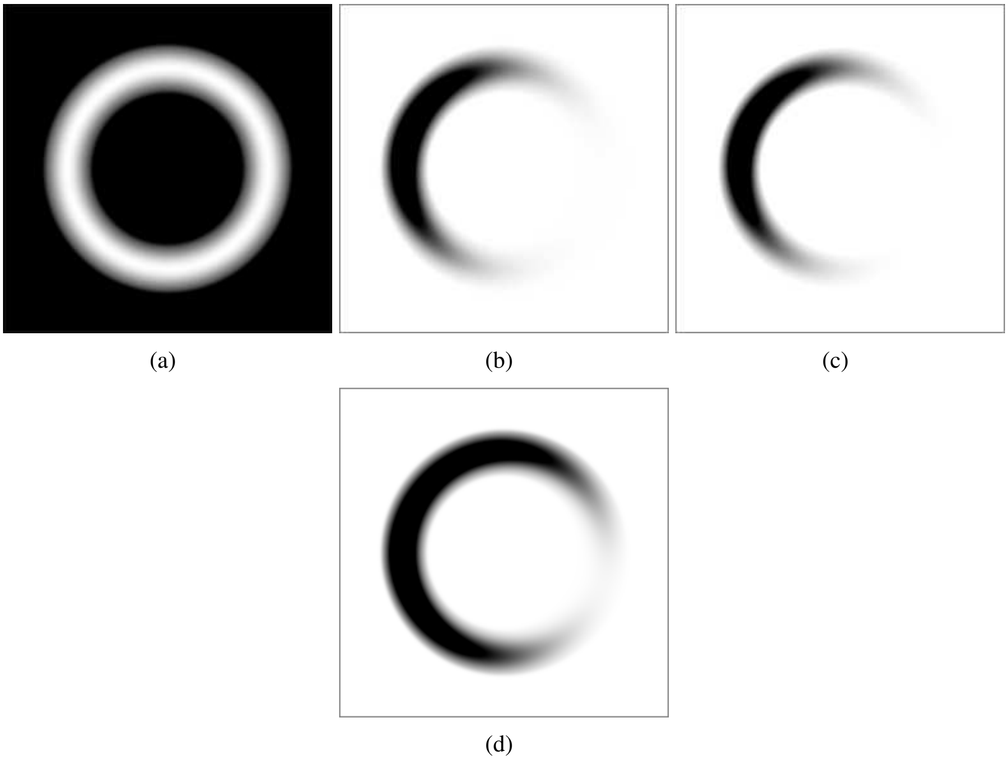


FIGURE 6. Rendering of ink diffusion. (a) Original Image. (b) Time Step 1. (c) Time Step 2. (d) Time Step 3.

4. RESULTS

Since the results from classical random walk model is shown in Fig. 1, we show here the results from a biased random walk model on an image. The original image of a donut shape with light gradient from the middle of the donut area to both outer and inner boundaries is used. The walker is released from the starting position which is in upper left corner. Because the model is built such that it will take favorable directions to similar intensity of the current position, we expect that it will visit the middle positions more often than either boundary positions. On Fig. 6(b), 6(c), and 6(d), we can see that the diffusion front takes the middle positions first then the positions near either boundaries follow the front. Also note that since there's a clear drop in intensities along the boundaries, the diffusion front never crosses over to the outer nor the inner regions.

5. Conclusion and Future Works

An animated ink diffusion on paper was studied in paper. By using an accumulated visit counts from a biased random walk model, the simulation of ink diffusion was made. With a model that satisfies the heat equation, we were able to convert the accumulated visit counts to corresponding intensities at all positions at all given time. For future works, we plan to explain the varying speeds of the moving front from the starting position to the boundary.

Acknowledgement

Myungjoo Kang was supported by the Korea Science and Engineering Foundation(KOSEF) grant funded by the Korea government(MEST)(R01-2008-000-10612-0)

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