J. KSIAM Vol.13, No.2, 109-121, 2009

SOLVING HIGHER-ORDER INTEGRO-DIFFERENTIAL EQUATIONS USING HE'S POLYNOMIALS

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ABSTRACT. In this paper, we use He's polynomials for solving higher order integro differential equations (IDES) by converting them to an equivalent system of integral equations. The He's polynomials which are easier to calculate and are compatible to Adomian's polynomials are found by using homotopy perturbation method. The analytical results of the equations have been obtained in terms of convergent series with easily computable components. Several examples are given to verify the reliability and efficiency of the proposed method.

1. INTRODUCTION

The higher-order integro differential equations arise in mathematical, applied and engineering sciences, astrophysics, solid state physics, astronomy, fluid dynamics, beam theory, fiber optics and chemical reaction-diffusion models; see [1, 2, 5, 12, 13, 28] and the references therein. Several techniques including decomposition and variational iteration have been used to investigate these problems [1, 2, 5, 12, 13, 28]. He [6-12] developed the homotopy perturbation technique based on the introduction of a homotopy, artificial or Book-keeping parameter for the solution of algebraic and ordinary differential equations. Such a technique is based on the expansion of the dependent variables and, in some cases, even constants that may appear in the governing equation, and provides series solutions. The technique has been applied with great success to obtain the solution of a large variety of nonlinear problems, see [4-12, 13-24] and the references therein. Although when it appeared, the homotopy perturbation method was believed to be a new technique, such a method has been previously used in, for example, numerical analysis and continuation algorithms whereby a parameter is introduced and increased from a value for which the problem to be solved has an easily obtainable solution, to its true valuable. In a later work Ghorbani et. al. [3, 4] split the nonlinear term into a series of polynomials calling them as the He's polynomials. The He's polynomials are calculated by using homotopy perturbation method, easier to calculate and are compatible with the Adomian's polynomials. The basic

Received by the editors December 1, 2008. Accepted May 15, 2009

2000 *Mathematics Subject Classification*. 65N10.

Key words and phrases. He's polynomials, Homotopy perturbation method, Higher-order integro differential equations.

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motivation of this paper is to use He's polynomials (which are calculated by homotopy perturbation method) for solving higher order integro differential equations by converting them into a system of integral equations. It is shown that the higher-order integro-differential equations are equivalent to the system of integral equations by using a suitable transformation. This alternate transformation plays a pivotal and fundamental role in solving the higher-order integro-differential equations. The He's polynomials [3, 4, 19-26] are introduced and used in the equivalent system of integral equations. Several examples are given to illustrate the performance of the method.

2. HOMOTOPY PERTURBATION METHOD

To explain the homotopy perturbation method, we consider a general equation of the type,

$$L(u)=0, \tag{1}$$

where L is any integral or differential operator. We define a convex homotopy H (u, p) by

$$H(u,p)=(1-p)F(u)+pL(u), \tag{2}$$

where F (u) is a functional operator with known solutions v_0, which can be obtained easily. It is clear that, for

$$H(u,p)=0, \tag{3}$$

we have

$$H(u,0)=F(u),\quad H(u,1)=L(u).$$

This shows that $H(u,p)$ continuously traces an implicitly defined curve from a starting point H (v_0, 0) to a solution function H (*f*, 1). The embedding parameter monotonically increases from zero to unit as the trivial problem F (u) = 0 is continuously deforms the original problem L (u) = 0. The embedding parameter $p \in (0, 1]$ can be considered as an expanding parameter [3, 4, 6-12, 15-26]. The homotopy perturbation method uses the homotopy parameter *p* as an expanding parameter [6-12] to obtain

$$u=\sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots, \tag{4}$$

if p $\to$ 1, then (4) corresponds to (2) and becomes the approximate solution of the form,

$$f=\lim_{p\to 1} u = \sum_{i=0}^{\infty} u_i. \tag{5}$$

It is well known that series (5) is convergent for most of the cases and also the rate of convergence is dependent on L (u); see [5-10]. We assume that (5) has a unique solution. The comparisons of like powers of *p* give solutions of various orders. In sum, according to [3, 4], He's HPM considers the solution, $u(x)$, of the homotopy equation in a series of p as follows:

$$u(x)=\sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2 u_2 + \ldots,$$

the method considers the nonlinear term $N(u)$ as

$$N(u)=\sum_{i=0}^{\infty} p^i H_i = H_0 + pH_1 + p^2 H_2 + \dots,$$

where H_n 's are the so-called He's polynomials [3, 4], which can be calculated by using the formula

$$H_n(u_0,\dots,u_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(N(\sum_{i=0}^{n} p^i u_i)\right)_{p=0}, \quad n=0,1,2,\dots .$$

3. NUMERICAL APPLICATIONS

In this section, we first show that the higher order integro differential equations can be written in the form of a system of integral equations by using a suitable transformation. The He's polynomials which are calculated by homotopy perturbation method are used for solving the reformulated system of integral equations.

EXAMPLE 3.1 [27, 28] Consider the linear boundary value problem for the fourth-order integro differential equation

$$y^{(iv)}(x)=x(1+e^x)+3e^x+y(x)-\int_0^x y(t)dt, \qquad 0<x<1$$

with boundary conditions

$$y(0)=1, \quad y'(0)=1, \quad y(1)=1+e, \quad y'(1)=2e.$$

Using the transformation $\frac{dy}{dx}=q(x),\ \frac{dq}{dx}=f(x),\ \frac{df}{dx}=z(x),$ the above boundary value problems can be transformed as:

$$\begin{cases} \frac{dy}{dx}=q(x), & \frac{dq}{dx}=f(x), \\ \frac{df}{dx}=z(x), & \frac{dz}{dx}=x(1+e^x)+3e^x+y(x)+\int_0^x y(t)dt, \end{cases}$$

with boundary conditions

$$y(0)=1, \quad q(0)=1, \quad f(0)=A, \quad z(0)=B.$$

The exact solution of the above boundary value problem is

$$y(x)=1+xe^x.$$

The above system of differential equations can be written as the following system of integral equations

$$\begin{cases} y(x) = 1 + \int_0^x q(t)dt, & q(x) = -1 + \int_0^x f(t)dt, \\ f(x) = A + \int_0^x z(t)dt, & z(x) = B + \int_0^x \left(\left(x(1+e^x) + 3e^x + y(x) \right) dx + \int_0^x y(x) \right) dx, \end{cases}$$

where

$$A = y''(0), \qquad B = y'''(0).$$

Applying the convex homotopy and using He's polynomials

$$\begin{cases} y_0 + p\, y_1 + p^2\, y_2 + \cdots = 1 + p \int_0^x \left(q_0 + p\, q_1 + p^2\, q_2 + \cdots \right) dx, \\ q_0 + p\, q_1 + p^2\, q_2 + \cdots = 1 + p \int_0^x \left(f_0 + p\, f_1 + p^2\, f_2 + \cdots \right) dx, \\ f_0 + p\, f_1 + p^2\, f_2 + \cdots = A + p \int_0^x \left(z_0 + p\, z_1 + p^2\, z_2 + \cdots \right) dx, \\ z_0 + p\, z_1 + p^2\, z_2 + \cdots = B + p \int_0^x \begin{pmatrix} x(1+e^x) + 3e^x + \left(y_0 + p\, y_1 + \cdots \right) \\ + \int_0^x \left(y_0 + p\, y_1 + p^2\, y_2 + \cdots \right) dx \end{pmatrix} dx. \end{cases}$$

Comparing the co-efficient of like powers of p

$$p^{(0)} : y_0(x) = 1, \qquad q_0(x) = 1, \qquad f_0(x) = A, \qquad z_0(x) = B,$$

$$p^{(1)} : \begin{cases} y_1(x) = x, & q_1(x) = Ax, \\ f_1(x) = B\,x, & z_1(x) = -2 + x + x^2 + 2e^x + xe^x, \end{cases}$$

$$p^{(2)} : \begin{cases} y_2(x) = \frac{1}{2} Ax^2, & q_2(x) = \frac{1}{2} Bx^2, \\ f_2(x) = -1 - 2x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + e^x + xe^x, & z_2(x) = -\frac{1}{2!}x^2 - \frac{1}{3!}x^3, \end{cases}$$

$$p^{(3)} : \begin{cases} y_3(x) = \frac{3}{3!} Bx^3, & q_3(x) = -x - x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + xe^x, \\ f_3(x) = -\frac{1}{3!}x^3 - \frac{1}{4!}x^4, & \\ z_3(x) = \frac{1}{3!}Ax^3 + \frac{1}{4!}Ax^4, & \end{cases}$$

$$p^{(4)}:\begin{cases} y_4(x)=1-\frac{1}{2}x^2-\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5-e^x+xe^x, \\ q_4(x)=-\frac{1}{4!}x^4-\frac{1}{5!}x^5, \\ f_4(x)=\frac{1}{4!}Ax^4+\frac{1}{5!}Ax^5, \\ z_4(x)=\frac{3}{4!}Bx^4+\frac{3}{5!}Bx^5, \end{cases}$$

$$p^{(5)}:\begin{cases} y_5(x)=-\frac{1}{5!}x^5-\frac{1}{6!}x^6, \\ q_5(x)=\frac{1}{5!}x^5+\frac{1}{6!}x^6, \\ f_5(x)=\frac{3}{5!}Bx^5+\frac{3}{6!}Bx^6, \\ z_5(x)=5+3x+\frac{1}{2!}x^2-\frac{1}{3!}x^3-\frac{2}{4!}x^4+\frac{2}{6!}x^6+\frac{1}{7!}x^7-5e^x+xe^x, \end{cases}$$

$\vdots$

The series solution is given as

$$\begin{aligned} y(x)&=1+x+\frac{1}{2!}Ax^2+\frac{1}{6}Bx^3+\frac{1}{6}x^4+\frac{1}{24}x^5+\left(\frac{1}{720}A+\frac{1}{180}\right)x^6 \\ &+\left(\frac{1}{840}-\frac{1}{5040}A+\frac{1}{5040}B\right)x^7+\left(\frac{11}{40320}-\frac{1}{40320}B\right)x^8+\frac{1}{40320}x^9 \\ &+\left(\frac{1}{453600}+\frac{1}{3628800}A\right)x^{10}+\left(-\frac{1}{19958400}A+\frac{1}{39916800}B+\frac{1}{3326400}\right)x^{11} \\ &+\left(\frac{1}{479001600}A-\frac{1}{239500800}B+\frac{1}{29937600}\right)x^{12}+\cdots, \end{aligned}$$

which is in full agreement with [28] where the same problem was solved by Adomian's decomposition method. Imposing the boundary conditions at x = 1, we obtained

$$A=1.999999953, \qquad B=3.000000151.$$

$$\begin{aligned} y(x)&=1+x+0.9999999765x^2+0.5000000252x^3+\frac{1}{6}x^4+\frac{1}{24}x^5 \\ &+0.008333333269x^6+0.001388888928x^7+0.0001984126946x^8 \\ &+\frac{1}{40320}x^9+0.27557310\times10^{-5}x^{10}+0.2755731983\times10^{-6}x^{11} \\ &+0.2505210766\times10^{-7}x^{12}+\cdots. \end{aligned}$$

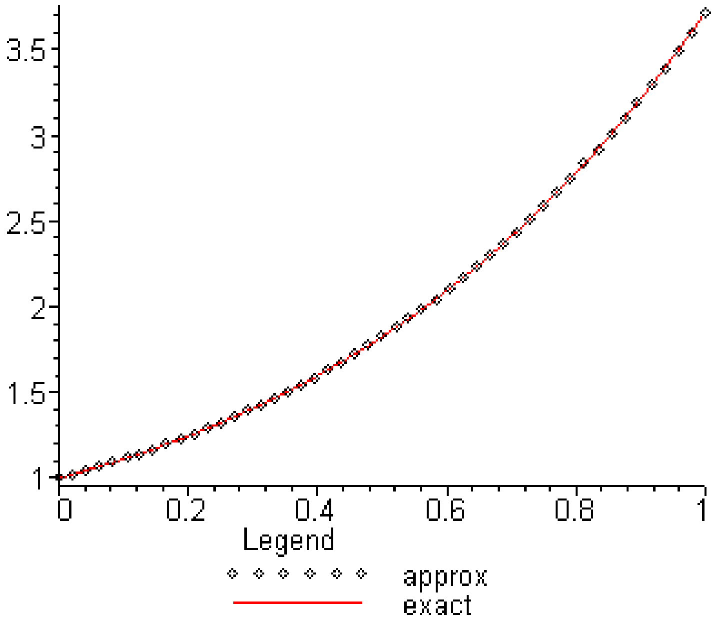
TABLE 3.1 (ERROR ESTIMATES)

x	Exact solution	Series solution	*Errors
0.0	1.000000000	1.000000000	0.00000
0.1	1.11105170920	1.1105170920	2.0 E-10
0.2	1.2442805520	1.1.2442805510	6.09 E-10
0.3	1.4049576420	1.4049576410	1.4 E-9
0.4	1.5967298790	1.5967298780	1.2 E-9
0.5	1.8243606360	1.8243606320	3.5 E-9
0.6	2.0932712800	2.0932712780	2.0 E-9
0.7	2.4096268950	2.4096268920	3.0 E-9
0.8	2.7804327420	2.7804327410	1.0 E-9
0.9	3.2136428000	3.2136427980	2.0E-9
1.0	3.7182818280	3.7182818290	1.0E-9

$* \, Error = \left| Exactsolution - Series\, solution \right|.$

Table 3.1 exhibits the errors obtained by applying the homotopy perturbation method. Higher accuracy can be obtained by using some more terms of the series solution.

FIGURE 3.1



Example 3.2 [27, 28] Consider the nonlinear boundary value problem for the integro differential equation

$$y^{(iv)}(x) = 1 + \int_0^x e^{-x} y^2 dx, \qquad 0 < x < 1$$

with boundary conditions

$$y(0)=1, \qquad y'(0)=1, \qquad y(1)=e, \qquad y'(1)=e.$$

The exact solution of the above boundary value problem is

$$y(x)=e^{x}.$$

Using transformation $\frac{dy}{dx}=q(x),\ \frac{dq}{dx}=f(x),\ \frac{df}{dx}=z(x),$ the above boundary value problems can be transformed as the following system of differential equations

$$\begin{cases} \frac{dy}{dx}=q(x), & \frac{dq}{dx}=f(x), \\ \frac{df}{dx}=z(x), & \frac{dz}{dx}=1+\int_0^x e^{-x}y^2(x)dx, \end{cases}$$

with boundary conditions

$$y(0)=1, \qquad q(0)=1, \qquad f(0)=A, \qquad z(0)=B.$$

The above system of differential equations can be written as the following system of integral equations

$$\begin{cases} y(x)=1+\int_0^x q(x)dx, & q(x)=1+\int_{0x}^x f(x)dx, \\ f(x)=A+\int_0^x z(x)dx, & z(x)=B+\int_0^x\left(1+\int_0^t e^{-x}(y(x))^2dx\right)dx. \end{cases}$$

Applying the convex homotopy and using He's polynomials

$$\begin{cases} y_0+py_1+p^2y_2+\cdots=1+p\int_0^x\left(q_0+pq_1+p^2q_2+\cdots\right)dx, \\ q_0+pq_1+p^2q_2+\cdots=1+p\int_0^x\left(f_0+pf_1+p^2f_2+\cdots\right)dx, \\ f_0+pf_1+p^2f_2+\cdots=A+p\int_0^x\left(z_0+pz_1+p^2z_2+\cdots\right)dx, \\ z_0+pz_1+p^2z_2+\cdots=B+p\int_0^x\left(1+\int_0^x e^{-x}\left(y_0+py_1+p^2y_2+\cdots\right)^2dx\right)dx. \end{cases}$$

Comparing the co-efficient of like powers of p

$$p^{(0)}: y_0(x)=1, \qquad q_0(x)=1, \qquad f_0(x)=A, \qquad z_0(x)=B,$$

$$p^{(1)}: \begin{cases} y_1(x)=x, & q_1(x)=Ax, \\ f_1(x)=Bx, & z_1(x)=4-4e^{-x}, \end{cases}$$

$$p^{(2)}:\begin{cases} y_2(x)=\frac{1}{2!}Ax^2, & q_2(x)=\frac{1}{3!}Bx^2, \\ f_2(x)=-4+4x+4e^{-x}, & z_2(x)=4-4e^{-x}-4xe^{-x}, \end{cases}$$

$$p^{(3)}\begin{cases} y_3(x)=\frac{3}{4!}Bx^3, & q_3(x)=4-4x+\frac{4}{2!}-4e^{-x}, \\ f_3(x)=4x+4e^{-x}-4xe^{-x}, & \\ z_3(x)=\frac{5}{2!}A\left(2-2e^{-x}-2xe^{-x}-x^2e^{-x}\right), & \end{cases}$$

$$\vdots$$

The series solution is given by:

$$\begin{aligned} y(x)=&1+x+\frac{1}{2!}Ax^2+\frac{1}{3!}Bx^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\left(\frac{1}{2520}A-\frac{1}{1680}\right)x^7 \\ &+\left(-\frac{1}{6720}A+\frac{1}{20160}B+\frac{1}{8064}\right)x^8+\left(\frac{1}{30240}A-\frac{1}{45360}B+\frac{1}{72576}\right)x^9 \\ &+\left(-\frac{1}{181440}A+\frac{1}{181440}B+\frac{1}{10!}\right)x^{10}+\left(\frac{1}{1330560}A-\frac{1}{997920}B+\frac{1}{7983360}\right)x^{11} \\ &+\left(-\frac{1}{11404800}A+\frac{1}{6842880}B+\frac{1}{12!}\right)x^{12}+\cdots, \end{aligned}$$

which is in full agreement with [28] where the same problem was solved by Adomian's decomposition method. Imposing the boundary conditions at $x=1$, we obtained

$$A=0.9970859583, \qquad B=1.010994057.$$

The series solution is given as

$$\begin{aligned} y(x)=&1+x+0.4985429792\,x^2+0.1684990095\,x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6 \\ &+0.0001995690641x^7+0.00002578056453x^8 \\ &+0.00002446285010\,x^9+0.3522271752\times10^{-6}x^{10} \\ &-0.1384676012\times10^{-6}x^{11}+0.624047474716\times10^{-7}x^{12}+\cdots. \end{aligned}$$

TABLE 3.2 (ERROR ESTIMATES)

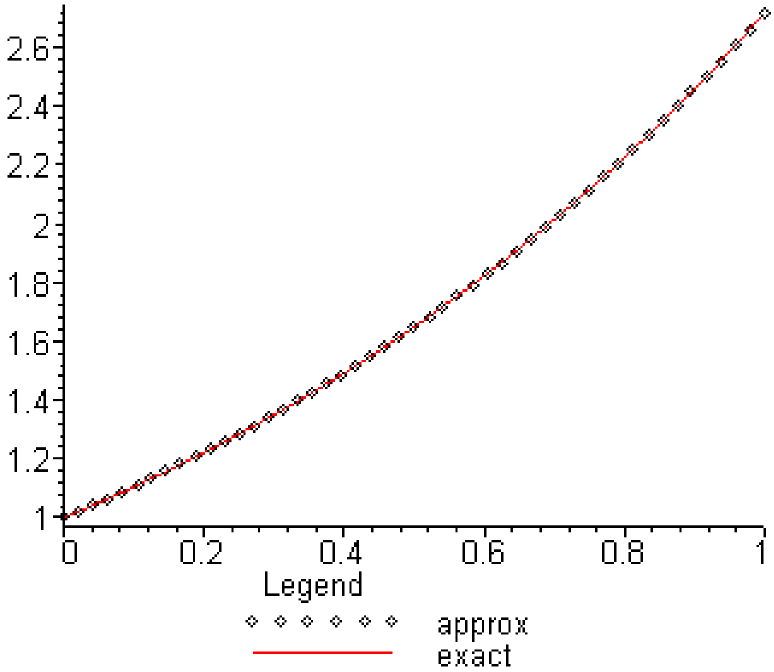
x	Exact solution	Series solution	*Errors
0.0	1.000000000	1.000000000	0.00000
0.1	1.1051581800	1.1051581800	1.27 E-5
0.2	1.2214027580	1.2213591310	4.36 E-5
0.3	1.3498588080	1.3497770620	8.17 E-5
0.4	1.4918246980	1.4917081990	1.16 E-4
0.5	1.6487212710	1.6485829960	1.38 E-4

0.6	1.8221188000	1.8219791520	1.39 E-4
0.7	2.0137527070	2.0136354170	1.17E-4
0.8	2.2255409280	2.2254662080	7.47 E-5
0.9	2.4596031110	2.4595771740	2.59 E-5
1.0	2.7182818280	2.7182818280	0.000000

$* \, Error = |Exact solution - Series\, solution|.$

Table 3.2 exhibits the errors obtained by applying the homotopy perturbation method. Higher accuracy can be obtained by using some more terms of the series solution.

FIGURE 3.2



EXAMPLE 3.3 [27, 28, 29] Consider the nonlinear inhomogeneous initial boundary value problem for the integro differential equation related to the Blasius problem

$$y''(x) = \alpha - \frac{1}{2}\int_0^x y(t)y''(t)dt, \qquad -\infty < x < 0$$

with boundary conditions

$$y(0) = 0, \qquad y'(0) = 1.$$

and

$$\lim_{x \to -\infty} y'(x) = 0.$$

It is interesting to point out that the constant α is positive and defined by

$$y''(0) = \alpha \qquad \alpha > 0.$$

Using the transformation $\dfrac{dy}{dx} = q(x),$ the above boundary value problem can be written as the following system of differential equations

$$\begin{cases} \dfrac{dy}{dx} = q(x), \\ \dfrac{dq}{dx} = \alpha - \dfrac{1}{2}\int_0^x y(x)y''(x)dt, \end{cases}$$

with boundary conditions

$$y(0) = 0, \qquad q(0) = 1, \qquad q'(0) = \alpha.$$

The above system of differential equations can be written as the following system of integral equations

$$\begin{cases} y(x) = \int_0^x q(x)dx, \\ q(x) = 1 + \int_0^x \left(\alpha - \dfrac{1}{2}\int_0^x y(x)q'(x)d\,x \right) dx. \end{cases}$$

Applying the convex homotopy method and using He's polynomials

$$\begin{cases} y_0 + p\,y_1 + p^2\,y_2 + \cdots = 1 + p\int_0^x \left(q_0 + p\,q_1 + p^2\,q_2 + \cdots\right)dx, \\ q_0 + p\,q_1 + p^2\,q_2 + \cdots = 1 + p\int_0^x \left(\alpha - \dfrac{1}{2}\int_0^x \left(y_0 + p\,y_1 + \cdots\right)\left(y_0' + p\,y_1' + \cdots\right)dx \right)dx. \end{cases}$$

Comparing the co-efficient of like powers of p

$$p^{(0)} : y_0(x) = 1, \qquad q_0(x) = 1,$$

$$p^{(1)} : \begin{cases} y_1(x) = x, \\ q_1(x) = \alpha\,x, \end{cases}$$

$$p^{(2)} : \begin{cases} y_2(x) = \dfrac{1}{2}\alpha\,x^2, \\ q_2(x) = -\dfrac{1}{12}\alpha\,x^3, \end{cases}$$

$$p^{(3)} : \begin{cases} y_3(x) = -\dfrac{1}{48}\alpha\,x^4, \\ q_3(x) = -\dfrac{1}{12}\alpha\,x^3 - \dfrac{1}{48}\alpha\,x^4 + \dfrac{1}{160}\alpha\,x^5 + \dfrac{1}{480}\alpha^2 x^6, \end{cases}$$

$$\vdots$$

The series solution is given as:

$$\begin{aligned} y(x) &= x + \frac{1}{2}\alpha x^2 - \frac{1}{48}\alpha x^4 - \frac{1}{240}\alpha^2 x^5 + \frac{1}{960}\alpha x^6 + \frac{11}{20160}\alpha^2 x^7 \\ &+ \left(\frac{11}{161280}\alpha^3 + \frac{1}{960}\alpha\right)x^8 - \frac{43}{967680}\alpha^2 x^9 \\ &+ \left(\frac{1}{52960}\alpha - \frac{5}{387072}\alpha^3\right)x^{10} + \left(\frac{587}{212889600}\alpha^2 - \frac{5}{4257792}\alpha^4\right)x^{11} \\ &+ \left(-\frac{1}{16220160}\alpha + \frac{1}{7257792}\alpha^3\right)x^{12} + \cdots, \end{aligned}$$

and consequently

$$\begin{aligned} y'(x) &= 1 + \alpha x - \frac{1}{12}\alpha x^3 - \frac{1}{48}\alpha^2 x^4 + \frac{1}{160}\alpha x^5 \\ &+ \frac{11}{2880}\alpha^2 x^6 + \left(\frac{11}{20160}\alpha^3 - \frac{1}{2688}\alpha\right)x^7 - \frac{43}{107520}\alpha^2 x^8 \\ &+ 10\left(\frac{1}{552960}\alpha - \frac{5}{387072}\alpha^3\right)x^9 + 11\left(\frac{587}{212889600}\alpha^2 - \frac{5}{4257792}\alpha^4\right)x^{10} \\ &+ 12\left(-\frac{1}{16220160}\alpha + \frac{1}{725760}\alpha^3\right)x^{11} + \cdots, \end{aligned}$$

are obtained which is in full agreement with [28] where the same problem was solved by Adomian's decomposition method. Now, we use diagonal pade approximants to determine a numerical value for the constant α by using the given condition [27, 28].

TABLE 3.3 PADE APPROXIMANTS AND NUMERICAL VALUE OF α [27, 28].

Pade approximant	x
[2/2]	0.5778502691
[3/3]	0.5163977793
[4/4]	0.5227030798

4. CONCLUSION

In this paper, we used He's polynomials which are calculated by the homotopy perturbation method for finding the solution of higher order integro differential equations. The method is used in a direct way without using linearization, discritization or restrictive assumption. It may be concluded that the method is very powerful and efficient in finding the analytical solutions for wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. It is concluded that He's polynomials are easier to calculate and are compatible to Adomian's polynomials.

ACKNOWLEDGMENTS

The authors are highly grateful to the referee for his/her very constructive comments. We would like to thank Dr S. M. Junaid Zaidi, Rector CIIT for the provision of excellent research environment and facilities

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