# DUALITY AND SUFFICIENCY IN MULTIOBJECTIVE FRACTIONAL PROGRAMMING WITH INVEXITY

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ABSTRACT. In this paper, we introduce generalized multiobjective fractional programming problem with two kinds of inequality constraints. Kuhn-Tucker sufficient and necessary optimality conditions are given. We formulate a generalized multiobjective dual problem and establish weak and strong duality theorems for an efficient solution under generalized convexity conditions.

## 1. INTRODUCTION AND PRELIMINARIES

Multiobjective fractional programming problems arise when more than one ratio objective function is to be optimized over a given feasible region. Efficiency or Pareto optimum is the optimality concept that appears to be the natural extension of the optimization of a single objective to the consideration of multiple objectives.

Recently, there has been an increasing interest in developing programming problems.

In 1961, Wolfe [15] studied the duality theorem of convex programming. Afterward, a number of different duals distinct from the Wolfe dual are proposed for the nonlinear programs by Mond and Weir [11]. Duality relations for objective fractional programming problems with a (generalized) convexity condition, were given by many authors [1, 3, 5, 8, 10, 12, 13, 14].

In 1992, Jeyakumar and Mond [6] introduced V-invexity. After Khan and Hanson [7] established duality theorem for single objective fractional programming problem by using ratio invexity condition, Craven and Mond [3] obtained more generalized duality results by using invexity and V-invexity due to Jeyakumar and Mond [6].

This thesis has two aims. One is an extension of the results in Craven and Mond [3] for two inequality constraint conditions from the single objective to multiobjective cases. And the other is to formulate a generalized multiobjective fractional dual problem.

This paper is organized as follows. In section 1, we introduce generalized multiobjective fractional problems under the concept of efficient solution needed in the proof of a strong

duality relation. In section 2, we establish Kuhn-Tucker sufficient conditions for generalized multiobjective fractional problem. In section 3, we prove weak and strong duality theorems for multiobjective fractional dual problem due to Craven and Mond [3] and generalized multiobjective fractional problem under generalized convexity.

We consider the folllowing generalized multiobjective fractional programming problem:

(MFP)    Minimize    $q(x) = (\dfrac{f_1(x)}{g_1(x)}, \cdots \dfrac{f_p(x)}{g_p(x)})$

subject to   $h_j(x) \le 0, \quad k_j(x) \le 0, \quad j = 1, 2, \cdots, m \quad x \in X,$

where $X$ is an open set of $\mathbb{R}^n$, $f := (f_1, \cdots, f_p) : X \to \mathbb{R}^p$, $g := (g_1, \cdots, g_p) : X \to \mathbb{R}^p$, $h : (h_1, \cdots, h_m) : X \to \mathbb{R}^m$ and $k := (k_1, \cdots, k_m) : X \to \mathbb{R}^m$ are continuously differentiable over $X$. Let $S = \{x \in X : h_j(x) \le 0, k_j(x) \le 0, j = 1, 2, \cdots, m\}$. We assume that $f(x) \ge 0$ for all $x \in X$ and $g(x) > 0$ for all $x \in X$.

The following convention for inequalities will be used in this paper. If $x, u \in \mathbb{R}^n$, then $x \le u$ iff $u - x \in \mathbb{R}^n_+$, $x \le_e u$ iff $u - x \in \mathbb{R}^n_+ \setminus \{0\}$, $x < u$ iff $u - x \in int\mathbb{R}^n_+$, and $x \not\le_e u$ is the negation of $x \le u$. For $x, u \in \mathbb{R}$, $x \le u$ and $x < u$ have the usual meaning.

**Definition 1.1.** *A feasible solution $\bar{x}$ for* (**MFP**) *is a efficient solution for* (**MFP**) *if $q(x) \not\le_e q(\bar{x})$ for all $x \in T$.*

**Definition 1.2.** *A function $F : \mathbb{R}^n \to \mathbb{R}$ is said to be sublinear if $F(\alpha_1 + \alpha_2) \le F(\alpha_1) + F(\alpha_2)$ for any $\alpha_1, \alpha_2 \in \mathbb{R}^n$ and $F(r\alpha) = rF(\alpha)$ for any $r \in \mathbb{R}_+, \alpha \in \mathbb{R}^n$.*

**Definition 1.3.** *Given an open set $X \subset \mathbb{R}^n$, a number $\rho \in \mathbb{R}$, and two functions $\alpha : X \times X \to \mathbb{R}_+ \setminus \{0\}$ and $d : X \times X \to \mathbb{R}$, a differentiable function $f$ over $X$ is said to be $(F, \alpha, \rho, d)$-convex at $\overline{x} \in X$ if for any $x \in X, F(x, \overline{x}; \cdot) : \mathbb{R}^n \to \mathbb{R}$ is sublinear, and $f(x)$ satisfies the following condition:*

$$f(x) - f(\overline{x}) \ge F(x, \overline{x} : \alpha(x, \overline{x})\nabla f(\overline{x})) + \rho d^2(x, \overline{x}).$$

If $\rho = 0$ or $d(x, x_0) = 0$ for all $x, x_0 \in X$ and $F(x, x_0; \alpha(x, x_0)\nabla f(x_0)) = \nabla f(x_0)^T \eta(x, x_0)$ for a certain mapping $\eta : X \times X \to \mathbb{R}^n$, then $(F, \alpha, \rho, d)$-convexity reduces to invexity.

In the proof of a strong duality relation, we will use the following lemma due to Chankong and Haimes[2].

**Lemma 1.1.** *$\overline{x}$ is an efficient solution for* (**MFP**) *if and only if for all $i = 1, 2, \cdots, p$, $\overline{x}$ solves* (**P$_i$**)*

(**P$_i$**)   *Minimize*   $\dfrac{f_i(x)}{g_i(x)}$

*subject to*   $x \in X_i,$

*where $X_i = \{x \in \mathbb{R}^n | \frac{f_j(x)}{g_j(x)} \le \frac{f_j(\overline{x})}{g_j(\overline{x})}$ for all $j \ne i, \ h_j(x) \le 0, \ k_j(x) \le 0\}$.*

## 2. Optimality Conditions

Now, we establish the Kuhn-Tucker type sufficient and necessary optimality theorem for (**MFP**).

**Lemma 2.1.** *Let $X \subset \mathbb{R}^n$ be an open set. Assume that $p, q$ and $r$ are real-valued differentiable functions defined on $X$ and $p(x) + r(x) \geq 0$, $q(x) > 0$ for all $x \in X$. If $p, r$ and $-q$ are $(F, \alpha, \rho, d)$-convex at $\overline{x} \in X$, then $\frac{p+r}{q}$ is $(F, \overline{\alpha}, \overline{\rho}, \overline{d})$-convex at $\overline{x}$, where*

$$\overline{\alpha}(x, \overline{x}) = \frac{\alpha(x, \overline{x}) q(\overline{x})}{q(x)}, \quad \overline{\rho} = \rho\left(1 + \frac{p(\overline{x}) + r(\overline{x})}{q(\overline{x})}\right), \text{ and } \overline{d}(x, \overline{x}) = \frac{d(x, \overline{x})}{q^{\frac{1}{2}}(x)}.$$

*Proof.* For any $x \in X$, we have

$$\frac{p(x) + r(x)}{q(x)} - \frac{p(\overline{x}) + r(\overline{x})}{q(\overline{x})} = \frac{(p(x) + r(x)) - (p(\overline{x}) + r(\overline{x}))}{q(x)} - \frac{(p(\overline{x}) + r(\overline{x}))(q(x) - q(\overline{x}))}{q(x) q(\overline{x})}.$$

By the $(F, \alpha, \rho, d)$-convexity of $p, r$ and $-q$, and $p + r \geq 0, q > 0$, we obtain

$$\frac{p(x) + r(x)}{q(x)} - \frac{p(\overline{x}) + r(\overline{x})}{q(\overline{x})} \geq \frac{1}{q(x)}\left(F(x, \overline{x}; \alpha(x, \overline{x}) \nabla(p(\overline{x}) + r(\overline{x}))) + \rho d^2(x, \overline{x})\right)$$

$$+ \frac{(p(\overline{x}) + r(\overline{x}))}{q(x) q(\overline{x})}\left(F(x, \overline{x}; -\alpha(x, \overline{x}) \nabla q(\overline{x})) + \rho d^2(x, \overline{x})\right).$$

Based on the sublinearity of $F$ and $p + r \geq 0, q > 0$, the following inequalities can be obtained:

$$\frac{p(x) + r(x)}{q(x)} - \frac{p(\overline{x}) + r(\overline{x})}{q(\overline{x})} \geq F\left(x, \overline{x}; \frac{\alpha(x, \overline{x})}{q(x)} \nabla(p(\overline{x}) + r(\overline{x}))\right) + \rho \frac{d^2(x, \overline{x})}{q(x)}$$

$$+ F\left(x, \overline{x}; -\frac{\alpha(x, \overline{x})(p(\overline{x}) + r(\overline{x}))}{q(x) q(\overline{x})} \nabla q(\overline{x})\right) + \rho \frac{d^2(x, \overline{x})(p(\overline{x}) + r(\overline{x}))}{q(x) q(\overline{x})}$$

$$\geq F\left(x, \overline{x}; \frac{\alpha(x, \overline{x})}{q(x)} \cdot \frac{q(\overline{x}) \nabla(p(\overline{x}) + r(\overline{x})) - (p(\overline{x}) + r(\overline{x})) \nabla q(\overline{x})}{q(\overline{x})}\right) + \rho\left(1 + \frac{p(\overline{x}) + r(\overline{x})}{q(\overline{x})}\right) \frac{d^2(x, \overline{x})}{q(x)}$$

$$= F\left(x, \overline{x}; \frac{\alpha(x, \overline{x}) q(\overline{x})}{q(x)} \nabla \frac{p(\overline{x}) + r(\overline{x})}{q(\overline{x})}\right) + \rho\left(1 + \frac{p(\overline{x}) + r(\overline{x})}{q(\overline{x})}\right) \frac{d^2(x, \overline{x})}{q(x)}.$$

Denote $\overline{\alpha}(x, \overline{x}) = \frac{\alpha(x, \overline{x}) q(\overline{x})}{q(x)}$, $\overline{\rho} = \rho\left(1 + \frac{p(\overline{x}) + r(\overline{x})}{q(\overline{x})}\right)$, and $\overline{d}(x, \overline{x}) = \frac{d(x, \overline{x})}{q^{\frac{1}{2}}(x)}$. Then we have

$$\frac{p(x) + r(x)}{q(x)} - \frac{p(\overline{x}) + r(\overline{x})}{q(\overline{x})} \geq F\left(x, \overline{x}; \overline{\alpha}(x, \overline{x}) \nabla \frac{p(\overline{x}) + r(\overline{x})}{q(\overline{x})}\right) + \overline{\rho} \overline{d}^2(x, \overline{x}), \forall x \in X.$$

Therefore, $\frac{p+r}{q}$ is $(F, \overline{\alpha}, \overline{\rho}, \overline{d})$-convex at $\overline{x}$. $\square$

**Theorem 2.2. (Kuhn-Tucker Sufficient Conditions)** *Let $\overline{x}$ be a feasible solution of (**MFP**). Suppose that there exist $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_p)^T > 0$, $v = (v_1, v_2, \cdots, v_m)^T \in \mathbb{R}^m_+$ and $w = (w_1, w_2, \cdots, w_m)^T \in \mathbb{R}^m_+$ such that*

$$\sum_{i=1}^{p} \lambda_i \nabla \frac{f_i(\overline{x}) + v^T h(\overline{x})}{g_i(\overline{x})} + \sum_{j=1}^{m} w_j \nabla k_j(\overline{x}) = 0, \tag{2.1}$$

$$v^T h(\overline{x}) = 0, \quad w^T k(\overline{x}) = 0.$$

*If $f_i, -g_i, i = 1, 2, \cdots, p$, and $v^T h$ are $(F, \alpha_i, \rho_i, d_i)$-convex at $\overline{x}$, $k_j$ for all $j = 1, 2, \cdots, m$ is $(F, \beta_j, \zeta_j, c_j)$-convex at $\overline{x}$, and*

$$\sum_{i=1}^{p} \lambda_i \overline{\rho}_i \frac{\overline{d}_i^2(x, \overline{x})}{\overline{\alpha}_i(x, \overline{x})} + \sum_{j=1}^{m} w_j \zeta_j \frac{c_j^2(x, \overline{x})}{\beta_j(x, \overline{x})} \geq 0, \tag{2.2}$$

where $\overline{\alpha}_i(x,\overline{x}) = \frac{\alpha_i(x,\overline{x})g_i(\overline{x})}{g_i(x)}$, $\overline{\rho}_i = \rho_i(1 + \frac{f_i(\overline{x})+v^T h(\overline{x})}{g_i(\overline{x})})$, and $\overline{d}_i(x,\overline{x}) = \frac{d_i(x,\overline{x})}{g_i^{\frac{1}{2}}(x)}$, then $\overline{x}$ is an efficient solution for (**MFP**).

*Proof.* Suppose that $\overline{x}$ is not a global efficient solution of (**MFP**). Then there exists a feasible solution $x$ such that $\frac{f(x)}{g(x)} \leq_e \frac{f(\overline{x})}{g(\overline{x})}$.

Since $x$ is feasible for (**MFP**) and $v^T h(\overline{x}) = 0$,
$$\frac{f(x)+v^T h(x)e}{g(x)} \leq_e \frac{f(\overline{x})+v^T h(\overline{x})e}{g(\overline{x})}.$$

By Lemma 2.1, for each $i$, $1 \leq i \leq p$, $\frac{f_i + v^T h}{g_i}$ is $(F, \overline{\alpha}_i, \overline{\rho}_i, \overline{d}_i)$-convex, i.e.,
$$\frac{f_i(x)+v^T h(x)}{g_i(x)} - \frac{f_i(\overline{x})+v^T h(\overline{x})}{g_i(\overline{x})} \geq F\left(x,\overline{x}; \overline{\alpha}_i(x,\overline{x})\nabla\frac{f_i(\overline{x})+v^T h(\overline{x})}{g_i(\overline{x})}\right) + \overline{\rho}_i\overline{d}_i^2(x,\overline{x}),$$
where $\overline{\alpha}_i(x,\overline{x}) = \frac{\alpha_i(x,\overline{x})g_i(\overline{x})}{g_i(x)}$, $\overline{\rho}_i = \rho_i\left(1 + \frac{f_i(\overline{x})+v^T h(\overline{x})}{g_i(\overline{x})}\right)$, and $\overline{d}_i(x,\overline{x}) = \frac{d_i(x,\overline{x})}{g_i^{\frac{1}{2}}(x)}$.

Since $\overline{\alpha}_i(x,\overline{x}) > 0$, by the sublinearity of $F$, we have
$$\frac{1}{\overline{\alpha}_i(x,\overline{x})}\left(\frac{f_i(x)+v^T h(x)}{g_i(x)} - \frac{f_i(\overline{x})+v^T h(\overline{x})}{g_i(\overline{x})}\right) \geq F\left(x,\overline{x}; \nabla\frac{f_i(\overline{x})+v^T h(\overline{x})}{g_i(\overline{x})}\right) + \overline{\rho}_i\frac{\overline{d}_i^2(x,\overline{x})}{\overline{\alpha}_i(x,\overline{x})},$$
where the left-hand side of the above inequality is less than or equal to zero.

Hence, we obtain the following inequalities,
$$F\left(x,\overline{x}; \nabla\frac{f_i(\overline{x})+v^T h(\overline{x})}{g_i(\overline{x})}\right) + \overline{\rho}_i\frac{\overline{d}_i^2(x,\overline{x})}{\overline{\alpha}_i(x,\overline{x})} \leq 0, \; i = 1, 2, \cdots, p,$$
and at least one inequality holds strictly.

Multiplying the above inequalities with $\lambda_i > 0$, $i = 1, 2, \cdots, p$, respectively, and then adding them together, we have
$$\sum_{i=1}^p \lambda_i F\left(x,\overline{x}; \nabla\frac{f_i(\overline{x})+v^T h(\overline{x})}{g_i(\overline{x})}\right) + \sum_{i=1}^p \lambda_i\overline{\rho}_i\frac{\overline{d}_i^2(x,\overline{x})}{\overline{\alpha}_i(x,\overline{x})} < 0.$$
By the sublinearity of $F$ and $\lambda_i > 0$, $i = 1, 2, \cdots, p$, we know that
$$\sum_{i=1}^p \lambda_i F\left(x,\overline{x}; \nabla\frac{f_i(\overline{x})+v^T h(\overline{x})}{g_i(\overline{x})}\right) \geq F\left(x,\overline{x}; \sum_{i=1}^p \lambda_i\nabla\frac{f_i(\overline{x})+v^T h(\overline{x})}{g_i(\overline{x})}\right).$$
Hence, we get
$$F\left(x,\overline{x}; \sum_{i=1}^p \lambda_i\nabla\frac{f_i(\overline{x})+v^T h(\overline{x})}{g_i(\overline{x})}\right) + \sum_{i=1}^p \lambda_i\overline{\rho}_i\frac{\overline{d}_i^2(x,\overline{x})}{\overline{\alpha}_i(x,\overline{x})} < 0.$$
Substituting (2.1) into above inequality, we obtain

$$F(x,\overline{x}; -\sum_{j=1}^m w_j\nabla h_j(\overline{x})) + \sum_{i=1}^p \lambda_i\overline{\rho}_i\frac{\overline{d}_i^2(x,\overline{x})}{\overline{\alpha}_i(x,\overline{x})} < 0. \tag{2.3}$$

The sublinearity of $F$ and (2.2) yield
$$F(x,\overline{x}; -\sum_{j=1}^m w_j\nabla k_j(\overline{x})) + \sum_{i=1}^p \lambda_i\overline{\rho}_i\frac{\overline{d}_i^2(x,\overline{x})}{\overline{\alpha}_i(x,\overline{x})} + F(x,\overline{x}; \sum_{j=1}^m w_j\nabla k_j(\overline{x})) + \sum_{j=1}^m w_j\zeta_j\frac{c_j^2(x,\overline{x})}{\beta_j(x,\overline{x})}$$
$$\geq \sum_{i=1}^p \lambda_i\overline{\rho}_i\frac{\overline{d}_i^2(x,\overline{x})}{\overline{\alpha}_i(x,\overline{x})} + \sum_{j=1}^m w_i\zeta_j\frac{c_j^2(x,\overline{x})}{\beta_j(x,\overline{x})} \geq 0.$$
Using (2.3), we obtain

$$F(x,\overline{x}; \sum_{j=1}^m w_j\nabla k_j(\overline{x})) + \sum_{j=1}^m w_j\zeta_j\frac{c_j^2(x,\overline{x})}{\beta_j(x,\overline{x})} > 0. \tag{2.4}$$

On the other hand, for $j = 1, 2, \cdots, m$, by the $(F, \beta_j, \zeta_j, c_j)$-convexity of $k_j$, we have
$$k_j(x) - k_j(\overline{x}) \geq F(x, \overline{x}; \beta_j(x, \overline{x}) \nabla k_j(\overline{x})) + \zeta_j c_j^2(x, \overline{x}).$$
By using $w_j \geq 0$, $\beta_j(x, \overline{x}) > 0$ and the sublinearity of $F$, we have
$$\sum_{j=1}^m w_j \frac{k_j(x) - k_j(\overline{x})}{\beta_j(x, \overline{x})} \geq F(x, \overline{x}; \sum_{j=1}^m w_j \nabla k_j(\overline{x})) + \sum_{j=1}^m w_j \zeta_j \frac{c_j^2(x, \overline{x})}{\beta_j(x, \overline{x})}.$$
Since $x$ is feasible, $w \in \mathbb{R}_+^n$, $w^T k(\overline{x}) = 0$ and $\beta_j(x, x_0) > 0$ implies that
$$\sum_{j=1}^m w_j \frac{k_j(x) - k_j(\overline{x})}{\beta_j(x, \overline{x})} \leq 0.$$

Then, we obtain

$$F(x, \overline{x}; \sum_{j=1}^m w_j \nabla h_j(\overline{x})) + \sum_{j=1}^m w_j \zeta_j \frac{c_j^2(x, \overline{x})}{\beta_j(x, \overline{x})} \leq 0,$$

which contradicts (2.4). Therefore, $\overline{x}$ is an efficient solution for (**MFP**).    $\square$

From Lemma 1.1, we have the following Kuhn-Tucker necessary optimality theorem for (**MFP**).

**Theorem 2.3. (Kuhn-Tucker Nesessary Conditions)** *Let $\overline{x}$ be an efficient solution for (**MFP**). Assume that $\overline{x}$ satisfies a constraint qualification for $(P_i)$, $i = 1, 2, \cdots, p$. Then there exist $\lambda \geq 0$, $v \in \mathbb{R}^m$ and $w \in \mathbb{R}^m$ such that*

$$\sum_{i=1}^p \lambda_i \nabla \frac{f_i(\overline{x}) + v^T h(\overline{x})}{g_i(\overline{x})} + \sum_{j=1}^m w_j \nabla k_j(\overline{x}) = 0,$$

$$v^T h(\overline{x}) = 0, \ v \geq 0, \ w \geq 0.$$

## 3. DUALITY THEOREMS

We propose the following multiobjective fractional dual problem to the primal problem (**MFP**):

(**MFD**)   Maximize   $p(u, v) = (\dfrac{f_1(u) + v^T h(u)}{g_1(u)}, \cdots, \dfrac{f_p(u) + v^T h(u)}{g_p(u)})$

subject to   $\nabla \lambda^T p(u, v) + \nabla w^T k(u) = 0,$    (3.1)

$\quad v \geq 0, \ w \geq 0, \ w^T k(u) = 0, \ \lambda = (\lambda_1, \cdots, \lambda_p)^T > 0,$

where $p : X \times \mathbb{R}^m \to \mathbb{R}^p$ and $e = (1, \cdots, 1)^T \in \mathbb{R}^p$.

**Theorem 3.1. (Weak Duality)** *Assume that $x$ is feasible for (**MFP**) and $(u, v, \lambda, w)$ is feasible for (**MFD**). If $f_i$, $-g_i$, $i = 1, 2, \cdots, p$, and $v^T h$ are $(F, \alpha_i, \rho_i, d_i)$-convex at $u$, $k_j$, $j = 1, 2, \cdots, m$, is $(F, \beta_j, \zeta_j, c_j)$-convex at $u$, and the inequality*

$$\sum_{i=1}^p \lambda_i \overline{\rho}_i \frac{\overline{d}_i^2(x, u)}{\overline{\alpha}_i(x, u)} + \sum_{j=1}^m w_j \zeta_j \frac{c_j^2(x, u)}{\beta_j(x, u)} \geq 0$$    (3.2)

*holds, where* $\overline{\alpha}_i(x,u) = \frac{\alpha_i(x,u)g_i(u)}{g_i(u)}, \overline{\rho}_i = \rho_i(1 + \frac{f_i(u)+v^T h(u)}{g_i(u)})$, *and* $\overline{d}_i(x,u) = \frac{d_i(x,u)}{g_i^{\frac{1}{2}}(x)}$, *then we have* $\frac{f(x)}{g(x)} \not\leq_e \frac{f(u)+v^T h(u)e}{g(u)}$.

*Proof.* Assume to the contrary that

$$\frac{f(x)}{g(x)} \leq_e \frac{f(u) + v^T h(u)e}{g(u)}. \tag{3.3}$$

For each $i$, $1 \leq i \leq p$, by the generalized convexity assumptions and Lemma 2.1, we have

$$\frac{f_i(x)+v^T h(x)}{g_i(x)} - \frac{f_i(u)+v^T h(u)}{g_i(u)} \geq F\left(x,u; \overline{\alpha}_i(x,u)\nabla \frac{f_i(u)+v^T h(u)}{g_i(u)}\right) + \overline{\rho}_i \overline{d}_i^2(x,u),$$

where $\overline{\alpha}_i(x,u) = \frac{\alpha_i(x,u)g_i(u)}{g_i(x)}$, $\overline{\rho}_i = \rho_i\left(1 + \frac{f_i(u)+v^T h(u)}{g_i(u)}\right)$ and $\overline{d}_i(x,\overline{x}) = \frac{d_i(x,u)}{g_i^{\frac{1}{2}}(x)}$. Using $\lambda_i > 0, \overline{\alpha}_i(x,\overline{x}) > 0$ and the sublinearity of $F$, we get, for $i = 1, 2, \cdots, p$,

$$\frac{\lambda_i}{\overline{\alpha}_i(x,u)}\left(\frac{f_i(x)+v^T h(x)}{g_i(x)} - \frac{f_i(u)+v^T h(u)}{g_i(u)}\right) \geq \lambda_i F\left(x,u; \nabla \frac{f_i(u)+v^T h(u)}{g_i(u)}\right) + \lambda_i \overline{\rho}_i \frac{\overline{d}_i^2(x,u)}{\overline{\alpha}_i(x,u)}.$$

Then, by (3.3) and $v^T h(x) \leq 0$, we obtain

$$\lambda_i F\left(x,u; \nabla \frac{f_i(u)+v^T h(u)}{g_i(u)}\right) + \lambda_i \overline{\rho}_i \frac{\overline{d}_i^2(x,u)}{\overline{\alpha}_i(x,u)} \leq 0, \ i = 1, 2, \cdots, p,$$

and at least one of the above inequalities holds strictly.
After adding these inequalities together, we get

$$\sum_{i=1}^p \lambda_i F\left(x,u; \nabla \frac{f_i(u)+v^T h(u)}{g_i(u)}\right) + \sum_{i=1}^p \lambda_i \overline{\rho}_i \frac{\overline{d}_i^2(x,u)}{\overline{\alpha}_i(x,u)} < 0.$$

Hence, it follows from the sublinearity of $F$ that

$$F\left(x,u; \sum_{i=1}^p \lambda_i \nabla \frac{f_i(u)+v^T h(u)}{g_i(u)}\right) + \sum_{i=1}^p \lambda_i \overline{\rho}_i \frac{\overline{d}_i^2(x,u)}{\overline{\alpha}_i(x,u)} < 0. \tag{3.4}$$

By the $(F, \beta_j, \zeta_j, c_j)$-convexity of $k_j$, $j = 1, 2, \cdots, m$, we have

$$k_j(x) - k_j(u) \geq F(x,u; \beta_j(x,u)\nabla k_j(u)) + \zeta_j c_j^2(x,u).$$

Using $w_j \geq 0$ and $\beta_j(x,\overline{x}) > 0$, we get

$$w_j \frac{k_j(x)-k_j(u)}{\beta_j(x,u)} \geq w_j F(x,u; \nabla k_j(u)) + w_j \zeta_j \frac{c_j^2(x,u)}{\beta_j(x,u)}, \ j = 1, 2, \cdots, m.$$

Adding these inequalities together and since $x$ is feasible of (**MFP**) and $w^T k(u) = 0$, we obtain

$$\sum_{j=1}^m w_j F(x,u; \nabla k_j(u)) + \sum_{j=1}^m w_j \zeta_j \frac{c_j^2(x,u)}{\beta_j(x,u)} \leq 0.$$

Using the sublinearity of $F$ again, we have

$$F(x,u; \sum_{j=1}^m w_j \nabla k_j(u)) + \sum_{j=1}^m w_j \zeta_j \frac{c_j^2(x,u)}{\beta_j(x,u)} \leq 0. \tag{3.5}$$

Based on the sublinearity of $F$, (3.1), (3.4), and (3.5), the following contradiction occurs:

$$0 = F(x, u; 0) = F\Big(x, u; \sum_{i=1}^{p} \lambda_i \nabla \frac{f_i(u) + v^T h(u)}{g_i(u)} + \sum_{j=1}^{m} w_j \nabla k_j(u)\Big)$$

$$\leq F\Big(x, u; \sum_{i=1}^{p} \lambda_i \nabla \frac{f_i(u) + v^T h(u)}{g_i(u)}\Big) + F\Big(x, u; \sum_{j=1}^{m} w_j \nabla k_j(u)\Big)$$

$$< -\Big(\sum_{i=1}^{p} \lambda_i \bar{\rho}_i \frac{\bar{d}_i^2(x, u)}{\bar{\alpha}_i(x, u)} + \sum_{j=1}^{m} w_j \zeta_j \frac{c_j^2(x, u)}{\beta_j(x, u)}\Big) \leq 0.$$

Therefore, if follows that $\frac{f(x)}{g(x)} \not\leq_e \frac{f(u) + v^T h(u)e}{g(u)}$. $\qquad \square$

**Theorem 3.2. (Strong Duality)** *Assume that $\overline{x}$ is an efficient solution of (**MFP**) and $\overline{x}$ satisfies a constraint qualification [9] for $(P_i)$. Then there exist $\overline{\lambda}(\in \mathbb{R}^p) > 0$, $\overline{v} \in \mathbb{R}_+^m$, $\overline{w} \in \mathbb{R}_+^m$ such that $(\overline{x}, \overline{v}, \overline{\lambda}, \overline{w})$ is feasible for (**MFD**), and the objective function values of (**MFP**) and (**MFD**) at the corresponding points are equal. If the assumptions about the generalized convexity and the inequality (3.2) in weak duality are also satisfied, then $(\overline{x}, \overline{v}, \overline{\lambda}, \overline{w})$ is an efficient solution of (**MFD**).*

*Proof.* Since $\overline{x}$ is an efficient solution of (**MFP**), from Lemma 1.1, $\overline{x}$ solves $(P_i)$, for all $i = 1, 2, \cdots, p$. Now from the necessary optimality theorem 2.3, there exist $\overline{\lambda} > 0$, $\overline{v} \geq 0$ and $\overline{w} \geq 0$ such that

centerline$\nabla(\overline{\lambda}^T \frac{f(\overline{x}) + \overline{v}^T h(\overline{x})}{g(\overline{x})}) + \nabla \overline{w}^T k(\overline{x}) = 0$ and $\overline{w}^T k(\overline{x}) = 0$.

Thus, $(\overline{x}, \overline{v}, \overline{\lambda}, \overline{w})$ is a feasible solution of (**MFD**). Moreover, by Theorem 3.1, $(\overline{x}, \overline{v}, \overline{\lambda}, \overline{w})$ is an efficient solution. $\qquad \square$

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