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A NEW PRIMAL-DUAL INTERIOR POINT METHOD FOR LINEAR OPTIMIZATION

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ABSTRACT. A primal-dual interior point method(IPM) not only is the most efficient method for a computational point of view but also has polynomial complexity. Most of polynomial-time interior point methods(IPMs) are based on the logarithmic barrier functions. Peng et al.([14, 15]) and Roos et al.([3]-[9]) proposed new variants of IPMs based on kernel functions which are called self-regular and eligible functions, respectively. In this paper we define a new kernel function and propose a new IPM based on this kernel function which has $\mathcal{O}(n^{\frac{2}{3}}\ \log \frac{n}{\epsilon})$ and $\mathcal{O}(\sqrt{n}\log \frac{n}{\epsilon})$ iteration bounds for large-update and small-update methods, respectively.

1. INTRODUCTION

In this paper we propose a new primal-dual IPM for the following standard LO problem

$$\min\{c^T x : Ax = b,\ x \geq 0\}, \tag{1}$$

where $A \in \mathbf{R}^{m\times n}$ with $\text{rank}(A) = m$, c, $x \in \mathbf{R}^n$, $b \in \mathbf{R}^m$, and its dual problem

$$\max\{b^T y : A^T y + s = c,\ s \geq 0\}, \tag{2}$$

where $y \in \mathbf{R}^m$ and $s \in \mathbf{R}^n$.

Since Karmarkar's paper [12] in 1984, IPMs have shown their efficiency in solving large-scale linear programming problems with a wide variety of successful applications. In this paper we propose a new primal-dual interior point algorithm based on a new kernel function. Most of polynomial-time interior point algorithms for LO are based on the logarithmic barrier function [2, 10, 11, 18]. Peng et al. [14, 15] proposed a new variant of IPMs based on self-regular barrier functions and achieved so far the best known complexity, i.e. $\mathcal{O}(\sqrt{n}\log n \log \frac{n}{\epsilon})$ for large-update methods with a specific self-regular function. Roos et al. [3, 4, 5, 6, 7, 8, 9] proposed new primal-dual IPMs for LO problems based on eligible barrier functions. They proposed the

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scheme for analyzing the algorithm based on four conditions on the kernel function [4] and obtained $\mathcal{O}(\sqrt{n}\log\frac{n}{\epsilon})$ iteration bound for small-update IPMs in all cases and $\mathcal{O}(\sqrt{n}\log n\log\frac{n}{\epsilon})$ iteration bound for large-update methods with a specific kernel function [3]. Recently, Amini and Haseli [1] proposed a generalized version of the kernel function in [4] and achieved the best known iteration bound for large-update methods by using the general scheme in [4] based on four conditions on the kernel function.

Motivated by their works, we define a new kernel function and propose a new primal-dual interior point algorithm for LO based on this kernel function. And we analyze the complexity for large-update and small-update methods based on the kernel function. Furthermore, the complexity bounds obtained by the algorithm are $\mathcal{O}(n^{\frac{2}{3}}\log\frac{n}{\epsilon})$ and $\mathcal{O}(\sqrt{n}\log\frac{n}{\epsilon})$ for large-update and small-update methods, respectively. For large-update methods the complexity result improves the best known bound since $n^{\frac{2}{3}} < \sqrt{n}\log n$ for $4 \le n \le 2.4128\times 10^7$. For small-update method this is currently the best known complexity result.

The paper is organized as follows. In section 2 we recall the generic IPM and the motivation of the new algorithm. In section 3 we define a new kernel function and give its properties which are essential for complexity analysis. In section 4 we derive the complexity result for the new algorithm. Finally, concluding remarks are given in section 5.

We use the following notations throughout the paper. $\mathbf{R}^{\mathbf{n}}_{+}$ and $\mathbf{R}^{\mathbf{n}}_{++}$ denote the set of n-dimensional nonnegative vectors and positive vectors, respectively. For $x, s \in \mathbf{R}^{\mathbf{n}}$, x_{min} and xs denote the smallest component of the vector x and the componentwise product of the vectors x and s, respectively. We denote X the diagonal matrix from a vector x, i.e. $X = \mathrm{diag}(x)$. $\mathbf{e}$ denotes the n-dimensional vector of ones and $\log n$, the natural logarithm. For $f(x), g(x) : \mathbf{R}_{++} \to \mathbf{R}_{++}$, $f(x) = \mathcal{O}(g(x))$ if $f(x) \le c_1 g(x)$ for some positive constant c_1 and $f(x) = \Theta(g(x))$ if $c_2 g(x) \le f(x) \le c_3 g(x)$ for some positive constants c_2 and c_3.

2. PRELIMINARIES

In this section we recall the generic IPM. Without loss of generality, we assume that both (1) and (2) satisfy the interior point condition (IPC) [16], i.e., there exists (x^0, y^0, s^0) such that

$$Ax^0 = b,\ x^0 > 0,\ A^T y^0 + s^0 = c,\ s^0 > 0.$$

Using the duality theorem (Theorem II.2 in [16]), we can find an optimal solution by solving the following system:

$$\begin{cases} Ax = b, \quad x \ge 0, \\ A^T y + s = c, \quad s \ge 0, \\ xs = 0. \end{cases} \tag{3}$$

The basic idea of primal-dual IPMs is to replace the third equation in (3) by the parameterized equation $xs = \mu\mathbf{e}$ with $\mu > 0$. Now we consider the following system:

$$\begin{cases} Ax = b, \quad x > 0, \\ A^T y + s = c, \quad s > 0, \\ xs = \mu\mathbf{e}. \end{cases} \tag{4}$$

If the IPC holds, then the system (4) has a unique solution for each $\mu > 0$ [13]. We denote this solution as $(x(\mu), y(\mu), s(\mu))$ and call it the μ-center. The set of μ-centers ($\mu > 0$) is called the central path [17]. The limit of the central path (as μ goes to zero) exists and since the limit point satisfies (3), it yields the optimal solution for (1) and (2) [16]. Primal-dual IPMs follow the central path approximately and approach the solution of (1) and (2) as μ goes to zero.
For given $(x, y, s) := (x^0, y^0, s^0)$ by applying Newton method to the system (4) we have the following Newton system

$$\begin{cases} A\Delta x = 0, \\ A^T\Delta y + \Delta s = 0, \\ s\Delta x + x\Delta s = \mu\mathbf{e} - xs. \end{cases} \tag{5}$$

Since A has full row rank, the system (5) has a unique search direction $(\Delta x, \Delta y, \Delta s)$ [16]. By taking a step along the search direction $(\Delta x, \Delta y, \Delta s)$, one constructs a new positive iterate (x_+, y_+, s_+),

$$x_+ = x + \alpha\Delta x,\ y_+ = y + \alpha\Delta y,\ s_+ = s + \alpha\Delta s,$$

for some $\alpha > 0$.

For the motivation of the new algorithm we define the following scaled vectors: for $(x, s) > 0$ and $\mu > 0$,

$$v := \sqrt{\frac{xs}{\mu}},\ d_x := \frac{v\Delta x}{x},\ d_s := \frac{v\Delta s}{s}. \tag{6}$$

Using (6), we can rewrite the system (5) as follows:

$$\begin{cases} \bar{A}d_x = 0, \\ \bar{A}^T\Delta y + d_s = 0, \\ d_x + d_s = v^{-1} - v, \end{cases} \tag{7}$$

where $\bar{A} := \frac{1}{\mu}AV^{-1}X$, $V := \mathrm{diag}(v)$, and $X := \mathrm{diag}(x)$. Note that the right side of the third equation in (7) equals the negative gradient of the logarithmic barrier function $\Psi_l(v)$, i.e.

$$d_x + d_s = -\nabla\Psi_l(v), \tag{8}$$

where

$$\Psi_l(v) := \sum_{i=1}^{n} \psi_l(v_i),\ \psi_l(t) = \sum_{i=1}^{n} \left(\frac{t^2 - 1}{2} - \log t\right).$$

We call ψ_l the kernel function of the logarithmic barrier function $\Psi_l(v)$. In this paper we replace $\psi_l(t)$ with a new kernel function $\psi(t)$ which will be defined in section 3.

The generic interior point algorithm works as follows: Assume that we are given a strictly feasible point (x, y, s) which is in a $\tau-$neighborhood of the given $\mu-$center. Then we decrease μ to $\mu_+ = (1 - \theta)\mu$, for some fixed $\theta \in (0, 1)$ and then we solve the Newton system (5) to obtain the unique search direction. The positivity condition of a new iterate is ensured with the right choice of the step size α which is defined by some line search rule. This procedure is repeated until we find a new iterate (x_+, y_+, s_+) that is in a $\tau-$neighborhood of the μ_+-center and then we let $\mu := \mu_+$ and $(x, y, s) := (x_+, y_+, s_+)$. Then μ is again reduced by the factor

$1-\theta$ and we solve the Newton system targeting at the new μ_+-center, and so on. This process is repeated until μ is small enough, say until $n\mu < \varepsilon$.

Generic Primal-Dual Algorithm for LO

Input:
  a threshold parameter $\tau > 0$;
  an accuracy parameter $\varepsilon > 0$;
  a fixed barrier update parameter $\theta,\ 0 < \theta < 1$;
  (x^0, s^0) and $\mu^0 := 1$ such that $\Psi_l\left(\sqrt{\frac{x^0 s^0}{\mu^0}}\right) \leq \tau$;
begin
  $x := x^0;\ s := s^0;\ \mu := \mu^0;$
  while $n\mu \geq \varepsilon$ do
  begin
    $\mu := (1-\theta)\mu;$
    while $\Psi_l(v) > \tau$ do
    begin
      Solve the system (5) for $\Delta x, \Delta y, \Delta s$;
      Determine a step size α;
      $x := x + \alpha\Delta x;$
      $s := s + \alpha\Delta s;$
      $y := y + \alpha\Delta y;$
      $v := \sqrt{\frac{xs}{\mu}};$
    end
  end
end

Remark 2.1. *If θ is a constant independent of the dimension of the problem n, e.g. $\theta = \frac{1}{2}$, then we call the algorithm a large-update method. If θ depends on n, e.g. $\theta = \frac{1}{\sqrt{n}}$, then the algorithm is called a small-update method.*

3. THE NEW KERNEL FUNCTION

In this section we define a new kernel function and give its properties which are essential to our complexity analysis. We call $\psi : \mathbf{R}_{++} \to \mathbf{R}_+$ a kernel function if ψ is twice differentiable and satisfies the following conditions:

$$\begin{aligned} &\psi'(1) = \psi(1) = 0, \\ &\psi''(t) > 0,\ t > 0, \\ &\lim_{t\to 0^+} \psi(t) = \lim_{t\to\infty} \psi(t) = \infty. \end{aligned} \tag{9}$$

Now we define a new function $\psi(t)$ as follows:

$$\psi(t) := 8t^2 - 12t + 2 + 2t^{-2},\ t > 0. \tag{10}$$

Then we have the following:

$$\psi'(t) = 16t - 12 - 4t^{-3},\ \ \psi''(t) = 16 + 12t^{-4},\ \ \psi'''(t) = -48t^{-5}. \tag{11}$$

From (11), $\psi(t)$ is a kernel function and

$$\psi''(t) > 16,\ t > 0. \tag{12}$$

In this paper, we replace the function $\Psi_l(v)$ in (8) with the function $\Psi(v)$ as follows:

$$d_x + d_s = -\nabla\Psi(v), \tag{13}$$

where $\Psi(v) = \sum_{i=1}^{n} \psi(v_i)$, where $\psi(t)$ is defined in (10). Hence the new search direction $(\Delta x, \Delta y, \Delta s)$ is obtained by solving the following modified Newton system:

$$\begin{cases} A\Delta x = 0, \\ A^T\Delta y + \Delta s = 0, \\ s\Delta x + x\Delta s = -\mu v\nabla\Psi(v). \end{cases} \tag{14}$$

Note that d_x and d_s are orthogonal because the vector d_x belongs to null space and d_s to the row space of the matrix $\bar{A}$. Since d_x and d_s are orthogonal, we have

$$d_x = d_s = 0 \ \Leftrightarrow\ \nabla\Psi(v) = 0 \Leftrightarrow\ v = \mathbf{e}\ \Leftrightarrow\ \Psi(v) = 0\ \Leftrightarrow\ x = x(\mu),\ s = s(\mu).$$

We use $\Psi(v)$ as the proximity function to measure the distance between the current iterate and the μ-center for given $\mu > 0$. We also define the norm-based proximity measure $\delta(v)$ as follows:

$$\delta(v) := \frac{1}{2}||\nabla\Psi(v)|| = \frac{1}{2}||d_x + d_s||. \tag{15}$$

Lemma 3.1. *For $\psi(t)$ we have*
(i) $\psi(t)$ is exponentially convex for all $t > 0$,
(ii) $\psi''(t)$ is monotonically decreasing for all $t > 0$,
(iii) $t\psi''(t) - \psi'(t) > 0$ for all $t > 0$,
(iv) $\psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t) > 0$, $t > 1$, $\beta > 1$.

Proof: For (i), by Lemma 2.1.2 in [15], it suffices to show that the function $\psi(t)$ satisfies $t\psi''(t) + \psi'(t) \geq 0$ for all $t > 0$. Using (11), we have

$$t\psi''(t) + \psi'(t) = t(16 + 12t^{-4}) + (16t - 12 - 4t^{-3}) = 32t - 12 + 8t^{-3}.$$

Let $g(t) = 32t - 12 + 8t^{-3}$. Then $g'(t) = 32 - 24t^{-4}$ and $g''(t) = 96t^{-5} > 0,\ t > 0$. Letting $g'(t) = 0$, we have $t = (\frac{3}{4})^{\frac{1}{4}}$. Since $g(t)$ is strictly convex and has a global minimum $g((\frac{3}{4})^{\frac{1}{4}}) > 27$. Hence we have the result.
For (ii), from (11), we have $\psi'''(t) < 0$.
For (iii), from (11), we have

$$t\psi''(t) - \psi'(t) = t(16 + 12t^{-4}) - (16t - 12 - 4t^{-3}) = 12 + 16t^{-3} > 0.$$

For (iv), using Lemma 2.4 in [4], (ii) and (iii), we have the result. This completes the proof. □

Lemma 3.2. *For* $\psi(t)$ *we have*
(i) $8(t-1)^2 \;\le\; \psi(t) \;\le\; \frac{1}{32}(\psi'(t))^2,\; t>0,$
(ii) $\psi(t) \;\le\; 14(t-1)^2,\; t\ge 1.$

Proof: For (i), using the first condition of (9) and (12), we have

$$\psi(t)=\int_1^t\int_1^\xi \psi''(\zeta)d\zeta d\xi \ge 16\int_1^t\int_1^\xi d\zeta d\xi = 8(t-1)^2.$$

which proves the first inequality. The second inequality is obtained as follows:

$$\begin{aligned}\psi(t)&=\int_1^t\int_1^\xi \psi''(\zeta)d\zeta d\xi \le \frac{1}{16}\int_1^t\int_1^\xi \psi''(\xi)\psi''(\zeta)d\zeta d\xi\\ &=\frac{1}{16}\int_1^t \psi''(\xi)\psi'(\xi)d\xi=\frac{1}{16}\int_1^t \psi'(\xi)d\psi'(\xi)=\frac{1}{32}(\psi'(t))^2.\end{aligned}$$

For (ii), using Taylor's Theorem, $\psi(1)=\psi^{'}(1)=0$, $\psi^{'''}<0$, and $\psi^{''}(1)=40$, we have

$$\begin{aligned}\psi(t)&=\psi(1)+\psi^{'}(1)(t-1)+\frac{1}{2}\psi^{''}(1)(t-1)^2+\frac{1}{3!}\psi^{'''}(\xi)(\xi-1)^3\\ &=\frac{1}{2}\psi^{''}(1)(t-1)^2+\frac{1}{3!}\psi^{'''}(\xi)(\xi-1)^3\\ &<\frac{1}{2}\psi^{''}(1)(t-1)^2=14(t-1)^2,\end{aligned}$$

for some ξ, $1\le\xi\le t$. This completes the proof. □

Let $\varrho:[0,\infty)\to[1,\infty)$ be the inverse function of $\psi(t)$ for $t\ge 1$ and $\rho:[0,\infty)\to(0,1]$ the inverse function of $-\frac{1}{2}\psi^{'}(t)$ for $t\in(0,1]$. Then we have the following lemma.

Lemma 3.3. *For* $\psi(t)$ *we have*
(i) $\sqrt{\frac{s}{8}+1}\le\varrho(s)\le 1+\sqrt{\frac{s}{8}},\; s\ge 0,$
(ii) $\rho(s)\ge(\frac{2}{s+2})^{\frac{1}{3}},\; s\ge 0.$

Proof: For (i), let $s=\psi(t)$, for $t\ge 1$, i.e. $\varrho(s)=t$, $t\ge 1$. By the definition of $\psi(t)$, $s=8t^2-12t+2+2t^{-2}$. This implies that

$$8t^2=s+12t-2-2t^{-2}\ge s+8$$

because $12t-2-2t^{-2}$ is monotone increasing with respect to t and $t\ge 1$. Hence we have

$$t=\varrho(s)\ge\sqrt{\frac{s}{8}+1}.$$

Using Lemma 3.2 (i), we have $s=\psi(t)\ge 8(t-1)^2$. Then we have

$$t=\varrho(s)\le 1+\sqrt{\frac{s}{8}},\; s\ge 0.$$

For (ii), let $z = -\frac{1}{2}\psi'(t)$, for all $t \in (0,1]$. Then by the definition of ρ, $\rho(z) = t$, $t \in (0,1]$ and $2z = -\psi'(t)$. So we have $2z = -16t + 12 + 4t^{-3}$.

$$4t^{-3} = 2z + 16t - 12 \leq 2z + 4,$$

since $16t - 12$ is monotone increasing in t and $0 < t \leq 1$. Hence we have

$$\rho(z) = t \geq (\frac{2}{z+2})^{\frac{1}{3}},\ z \geq 0.$$

□

Using Lemma 3.1 (iv), we have the following. The reader can refer to Theorem 3.2 in [4] for the proof.

Lemma 3.4. *Let $\varrho : [0,\infty) \to [1,\infty)$ be the inverse function of $\psi(t)$, $t \geq 1$. Then we have*

$$\Psi(\beta v) \leq n\psi\left(\beta\varrho\left(\frac{\Psi(v)}{n}\right)\right),\ v \in \mathbf{R}_{++},\ \beta \geq 1.$$

□

In the following theorem we obtain an estimate for the effect of a μ-update on the value of $\Psi(v)$.

Theorem 3.5. *Let $0 \leq \theta < 1$ and $v_+ = \frac{v}{\sqrt{1-\theta}}$. If $\Psi(v) \leq \tau$, then we have*

$$\Psi(v_+) \leq \frac{14}{1-\theta}\left(\sqrt{n}\theta + \sqrt{\frac{\tau}{8}}\right)^2.$$

Proof: Since $\frac{1}{\sqrt{1-\theta}} \geq 1$ and $\varrho\left(\frac{\Psi(v)}{n}\right) \geq 1$, we have $\frac{\varrho\left(\frac{\Psi(v)}{n}\right)}{\sqrt{1-\theta}} \geq 1$. Using Lemma 3.4 with $\beta = \frac{1}{\sqrt{1-\theta}}$, Lemma 3.2 (ii), Lemma 3.3 (i), and $\Psi(v) \leq \tau$, we have

$$\begin{aligned}
\Psi(v_+) &\leq n\psi\left(\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\Psi(v)}{n}\right)\right) \leq 14n\left(\frac{\varrho\left(\frac{\Psi(v)}{n}\right) - \sqrt{1-\theta}}{\sqrt{1-\theta}}\right)^2 \\
&\leq 14n\left(\frac{1 + \sqrt{\frac{\tau}{8n}} - \sqrt{1-\theta}}{\sqrt{1-\theta}}\right)^2 \leq 14n\left(\frac{\theta + \sqrt{\frac{\tau}{8n}}}{\sqrt{1-\theta}}\right)^2 \\
&= \frac{14}{1-\theta}\left(\sqrt{n}\theta + \sqrt{\frac{\tau}{8}}\right)^2,
\end{aligned}$$

where the last inequality holds from $1 - \sqrt{1-\theta} = \frac{\theta}{1+\sqrt{1-\theta}} \leq \theta$, $0 \leq \theta < 1$. This completes the proof. □

Denote

$$\tilde{\Psi}_0 := \frac{14}{1-\theta}\left(\sqrt{n}\theta + \sqrt{\frac{\tau}{8}}\right)^2. \tag{16}$$

Then $\tilde{\Psi}_0$ is an upper bound for $\Psi(v)$ during the process of the algorithm.

Remark 3.6. *For large-update method by taking* $\tau = \mathcal{O}(n)$ *and* $\theta = \Theta(1)$, $\tilde{\Psi}_0 = \mathcal{O}(n)$. *For small-update method with* $\tau = \mathcal{O}(1)$ *and* $\theta = \Theta(\frac{1}{\sqrt{n}})$, $\tilde{\Psi}_0 = \mathcal{O}(1)$.

4. COMPLEXITY ANALYSIS

In this section we compute a proper step size and the decrease of the proximity function during an inner iteration and give the complexity results of the algorithm. For fixed μ, taking a step size α, we have new iterates $x_+ = x + \alpha\Delta x,\ s_+ = s + \alpha\Delta s$. Using (6), we have

$$x_+ = x\left(e + \alpha\frac{\Delta x}{x}\right) = x\left(e + \alpha\frac{d_x}{v}\right) = \frac{x}{v}(v + \alpha d_x)$$

and

$$s_+ = s\left(e + \alpha\frac{\Delta s}{s}\right) = s\left(e + \alpha\frac{d_s}{v}\right) = \frac{s}{v}(v + \alpha d_s).$$

Thus we have

$$v_+ := \sqrt{\frac{x_+ s_+}{\mu}} = \sqrt{(v + \alpha d_x)(v + \alpha d_s)}.$$

Define for $\alpha > 0$

$$f(\alpha) = \Psi(v_+) - \Psi(v).$$

Then $f(\alpha)$ is the difference of proximities between a new iterate and a current iterate for fixed μ. By Lemma 3.1 (i), we have

$$\Psi(v_+) = \Psi(\sqrt{(v + \alpha d_x)(v + \alpha d_s)}) \leq \frac{1}{2}(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)).$$

Hence we have $f(\alpha) \leq f_1(\alpha)$, where

$$f_1(\alpha) := \frac{1}{2}(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)) - \Psi(v). \tag{17}$$

We have

$$f(0) = f_1(0) = 0.$$

Taking the derivative of $f_1(\alpha)$ with respect to α, we have

$$f_1'(\alpha) = \frac{1}{2}\sum_{i=1}^{n}(\psi'(v_i + \alpha[d_x]_i)[d_x]_i + \psi'(v_i + \alpha[d_s]_i)[d_s]_i),$$

where $[d_x]_i$ and $[d_s]_i$ denote the i-th components of the vectors d_x and d_s, respectively. Using (13) and (15), we have

$$f_1'(0) = \frac{1}{2}\nabla\Psi(v)^T(d_x + d_s) = -\frac{1}{2}\nabla\Psi(v)^T\nabla\Psi(v) = -2\delta(v)^2. \tag{18}$$

Differentiating $f_1'(\alpha)$ with respect to α, we have

$$f_1''(\alpha) = \frac{1}{2}\sum_{i=1}^{n}(\psi''(v_i + \alpha[d_x]_i)[d_x]_i^2 + \psi''(v_i + \alpha[d_s]_i)[d_s]_i^2). \tag{19}$$

Since $f_1''(\alpha) > 0$, $f_1(\alpha)$ is strictly convex in α unless $d_x = d_s = 0$.

Lemma 4.1. *Let* $\delta(v)$ *be as defined in* (15)*. Then we have*

$$\delta(v) \geq 2\sqrt{2\Psi(v)}.$$

Proof: Using Lemma 3.2 (i), we have

$$\Psi(v) = \sum_{i=1}^{n} \psi(v_i) \leq \frac{1}{32} \sum_{i=1}^{n} \psi'(v_i)^2 = \frac{1}{32} ||\nabla \Psi(v)||^2 = \frac{1}{8} \delta(v)^2.$$

Hence we have $\delta(v) \geq 2\sqrt{2\Psi(v)}$. □

Remark 4.2. *Throughout the paper we assume that* $\tau \geq 1$*. Using Lemma 4.1 and the assumption* $\Psi(v) \geq \tau$*, we have*

$$\delta(v) \geq 2\sqrt{2\Psi(v)} \geq 2\sqrt{2}. \tag{20}$$

□

For notational convenience we denote $\delta := \delta(v)$ and $\Psi := \Psi(v)$.

Lemma 4.3. (Modification of Lemma 4.1 in [4]) Let $f_1(\alpha)$ be as defined in (17) and δ be as defined in (15). Then we have

$$f_1''(\alpha) \leq 2\delta^2 \psi''(v_{min} - 2\alpha\delta). \tag{21}$$

Lemma 4.4. (Modification of Lemma 4.2 in [4]) If the step size α satisfies the inequality

$$-\psi'(v_{min} - 2\alpha\delta) + \psi'(v_{min}) \leq 2\delta, \tag{22}$$

then we have

$$f_1'(\alpha) \leq 0.$$

Lemma 4.5. (Modification of Lemma 4.3 in [4]) Let $\rho : [0, \infty) \to (0, 1]$ denote the inverse function of $-\frac{1}{2}\psi'(t)$ for all $t \in (0, 1]$. Then, in the worst case, the largest step size $\hat{\alpha}$ satisfying (22) is given by

$$\hat{\alpha} := \frac{1}{2\delta}(\rho(\delta) - \rho(2\delta)).$$

Lemma 4.6. (Modificatiion of Lemma 4.4 in [4]) Let ρ and $\hat{\alpha}$ be as defined in Lemma 4.5. Then we have

$$\hat{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))}.$$

Define

$$\bar{\alpha} := \frac{1}{\psi''(\rho(2\delta))}. \tag{23}$$

Then we have $\bar{\alpha} \leq \hat{\alpha}$.

Lemma 4.7. *Let* $\bar{\alpha}$ *be as defined in* (23)*. If* $\Psi(v) \geq \tau \geq 1$*, then we have*

$$\bar{\alpha} \geq \frac{1}{45\delta^{\frac{4}{3}}}.$$

Proof: Using the definition of $\psi''(t)$, Lemma 3.3 (ii), and (20), we have

$$\begin{aligned}\bar{\alpha} &= \frac{1}{\psi''(\rho(2\delta))} = \frac{1}{16+12\frac{1}{(\rho(2\delta))^4}} \\ &\geq \frac{1}{16+12(\delta+1)^{\frac{4}{3}}} \geq \frac{1}{4\sqrt{2}\delta+12(\delta+1)^{\frac{4}{3}}} \\ &\geq \frac{1}{(12+4\sqrt{2})(\delta+1)^{\frac{4}{3}}} \geq \frac{1}{2^{\frac{4}{3}}(12+4\sqrt{2})\delta^{\frac{4}{3}}} \geq \frac{1}{45\delta^{\frac{4}{3}}}.\end{aligned}$$

□

For notational convenience we denote

$$\tilde{\alpha} = \frac{1}{45\delta^{\frac{4}{3}}}. \tag{24}$$

Note that $\tilde{\alpha} \leq \bar{\alpha}$. We will use $\tilde{\alpha}$ as the default step size.

Lemma 4.8. (Lemma 1.3.3 in [15]) Suppose that $h(t)$ is a twice differentiable convex function with

$$h(0) = 0,\ h'(0) < 0$$

and $h(t)$ attains its (global) minimum at $t^* > 0$ and $h''(t)$ is increasing with respect to t. Then for any $t \in [0, t^*]$, we have

$$h(t) \leq \frac{th'(0)}{2}.$$

Lemma 4.9. (Modification of Lemma 4.5 in [4]) If the step size α is such that $\alpha \leq \bar{\alpha}$, then

$$f(\alpha) \leq -\alpha\delta^2.$$

Proof: Let the univariate function h be such that

$$h(0) = f_1(0) = 0,\ h'(0) = f_1'(0) = -2\delta^2,\ h''(\alpha) = 2\delta^2\psi''(v_{min} - 2\alpha\delta).$$

Then $h(t)$ is twice differentiable, $h(0) = 0$, and $h'(0) < 0$. Since $h''(\alpha) > 0$, $h(t)$ is strictly convex and hence has a global minimum at some $\alpha^* > 0$. From (21), we have $f_1''(\alpha) \leq h''(\alpha)$. As a result, we get $f_1'(\alpha) \leq h'(\alpha)$ and $f_1(\alpha) \leq h(\alpha)$. Taking $\alpha \leq \bar{\alpha}$, we have

$$\begin{aligned}h'(\alpha) &= h'(0) + \int_0^{\alpha} h''(\zeta)d\zeta \\ &= -2\delta^2 + 2\delta^2\int_0^{\alpha}\psi''(v_{min} - 2\xi\delta)d\xi \\ &= -2\delta^2 - \frac{2\delta^2}{2\delta}\int_0^{\alpha}\psi''(v_{min} - 2\xi\delta)d(v_{min} - \xi\delta) \\ &= -2\delta^2 - \delta(\psi'(v_{min} - 2\alpha\delta) - \psi'(v_{min})) \\ &\leq -2\delta^2 + 2\delta^2 = 0,\end{aligned}$$

where the inequality follows from (22). Since $h'''(\alpha) = -4\delta^2\psi'''(v_{min} - 2\alpha\delta) > 0$, $h''(\alpha)$ is monotonically increasing in α. Thus, using Lemma 4.8, we have

$$f_1(\alpha) \leq h(\alpha) \leq \frac{1}{2}\alpha h'(0) = -\alpha\delta^2.$$

Since $f(\alpha) \leq f_1(\alpha)$, the lemma is proved. □

Theorem 4.10. *Let $\tilde{\alpha}$ be as defined in* (24) *and* $\Psi(v) \geq 1$. *Then*

$$f(\tilde{\alpha}) \leq -\frac{2\Psi(v)^{\frac{1}{3}}}{45}.$$

Proof: Using Lemma 4.9, (24), and Lemma 4.1 we have

$$f(\tilde{\alpha}) \quad \leq \quad -\tilde{\alpha}\delta^2 = -\frac{\delta^{\frac{2}{3}}}{45} \leq -\frac{((8\Psi(v))^{\frac{1}{2}})^{\frac{2}{3}}}{45} \quad = -\frac{2\Psi(v)^{\frac{1}{3}}}{45}.$$

This completes the proof. □

Lemma 4.11. (Lemma 1.3.2 in [15]) Let $t_0, t_1, \cdots, t_K$ be a sequence of positive numbers such that

$$t_{k+1} \leq t_k - \gamma t_k^{1-\tilde{\beta}},\ k = 0, 1, \cdots, K-1,$$

where $\gamma > 0$ and $0 < \tilde{\beta} \leq 1$. Then $K \ \leq \ \left\lfloor \frac{t_0^{\tilde{\beta}}}{\gamma\tilde{\beta}} \right\rfloor$.

Denote the value of $\Psi(v)$ after the $\mu-$update as Ψ_0 and the subsequent values in the same outer iteration as Ψ_k, $k = 1, 2, \cdots$. Then we have

$$\Psi_0 \leq \tilde{\Psi}_0, \tag{25}$$

where $\tilde{\Psi}_0$ is defined in (16). Let K denote the total number of inner iterations per outer iteration. Then we have

$$\Psi_{K-1} > \tau,\ 0 \leq \Psi_K \leq \tau.$$

Lemma 4.12. *Let $\tilde{\Psi}_0$ be as defined in (16) and K the total number of inner iterations in the outer iteration. Then we have*

$$K \leq 34\tilde{\Psi}_0^{\frac{2}{3}}.$$

Proof: Using Theorem 4.10 and Lemma 4.11 with $\gamma := \frac{2}{45}$ and $\tilde{\beta} := \frac{2}{3}$, we have

$$K \ \leq \ \left(\frac{45}{2}\right)\left(\frac{3}{2}\right)\tilde{\Psi}_0^{\frac{2}{3}} \ \leq \ 34\tilde{\Psi}_0^{\frac{2}{3}}.$$

This completes the proof. □

Theorem 4.13. *Let a LO problem be given, $\tilde{\Psi}_0$ as defined in (16) and $\tau \geq 1$. Then the total number of iterations to have an approximate solution with $n\mu < \epsilon$ is bounded by*

$$\left\lceil \frac{34}{\theta}\tilde{\Psi}_0^{\frac{2}{3}} \log\frac{n}{\epsilon} \right\rceil.$$

Proof: If the central path parameter μ has the initial value $\mu^0 := 1$ and is updated by multiplying $1-\theta$ with $0 \leq \theta < 1$, then after at most

$$\left\lceil \frac{1}{\theta} \log \frac{n}{\epsilon} \right\rceil$$

iterations we have $n\mu < \epsilon$ [16]. For the total number of iterations, we multiply the number of inner iterations by that of outer iterations. Hence the total number of iterations is bounded by

$$\left\lceil \frac{34}{\theta} \tilde{\Psi}_0^{\frac{2}{3}} \log \frac{n}{\epsilon} \right\rceil.$$

This completes the proof. □

Remark 4.14. *Taking $\tau = \mathcal{O}(n)$ and $\theta = \Theta(1)$, we have $\mathcal{O}(n^{\frac{2}{3}} \log \frac{n}{\epsilon})$ iteration complexity for large-update IPMs which improves the best known complexity result for $4 \leq n \leq 2.4128 \times 10^7$. For small-update methods, we have $\mathcal{O}(\sqrt{n} \log \frac{n}{\epsilon})$ iteration complexity which is the best known complexity result so far.*

5. CONCLUDING REMARKS

Motivated by recent work of Amini and Maseli [1], we propose a new primal-dual interior point algorithm for LO problems based on a new kernel function and analyze the iteration complexity of the algorithm. For large-update methods we have $\mathcal{O}(n^{\frac{2}{3}} \log \frac{n}{\epsilon})$ iteration bound. Since $n^{\frac{2}{3}} < \sqrt{n} \log n$ for $4 \leq n \leq 2.4128 \times 10^7$, this complexity result is better than the one in [1] which is currently the best known complexity, that is $\mathcal{O}(\sqrt{n} \log n \log \frac{n}{\epsilon})$. For small-update methods the iteration bound is $\mathcal{O}(\sqrt{n} \log \frac{n}{\epsilon})$ which is the best known iteration bound.

Future research might focus on the extension to semidefinite optimization and second order cone optimization. Numerical tests will be another topic for future research.

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