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DISTRIBUTED FEEDBACK CONTROL OF THE BURGERS EQUATION BY A REDUCED-ORDER APPROACH USING WEIGHTED CENTROIDAL VORONOI TESSELLATION

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ABSTRACT. In this article, we study a reduced-order modelling for distributed feedback control problem of the Burgers equations. Brief review of the centroidal Voronoi tessellation (CVT) are provided. A weighted (nonuniform density) CVT is introduced and low-order approximate solution and compensator-based control design of Burgers equation is discussed. Through weighted CVT (or CVT-nonuniform) method, obtained low-order basis is applied to low-order functional gains to design a low-order controller, and by using the low-order basis order of control modelling was reduced. Numerical experiments show that a solution of reduced-order controlled Burgers equation performs well in comparison with a solution of full order controlled Burgers equation.

1. INTRODUCTION

Recently the application of reduced-order models to the computational simulation for (nonlinear) complex systems, optimal control problems or feedback control problems has received increasing amount of attention. The Proper Orthogonal Decomposition (POD) technique has been widely discussed in literatures of the past fifteen years as a tool for model reduction. However, currently the CVT as reduced order modelling technique became an active research field. CVT-based reduced-order modelling of fluid flows was developed by [9, 10].

So far, the CVT reduced-order modelling problems have been studied in uniform density ($\rho(\mathbf{y}) = 1$); see [9, 10, 11]. We call this case as the "CVT-uniform". However, sometimes the generators obtained by CVT-uniform do not lead to satisfactory results in the reduced-order

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modelling (or controller) problems. Therefore, to overcome this disadvantage, we extend the uniform density to more general nonuniform densities (variable densities). We call this case as the "CVT-nonuniform". In the reduced-order model, we have used the "CVT-nonuniform" method successfully; see [12, 15].

In general, the approach for low order control design has two types, "reduce-then-design" and "design-then-reduce"; for detailed discussions, see [1, 2]. In this paper we use the latter approach to obtain reduced order compensator-based feedback controllers for systems described by PDEs, since this approach holds much more information of the solution of the original equation than the former does. In this paper, we apply the technique of CVT-nonuniform methods to obtain a low order basis for controller approximation.

The rest of paper goes as follows. In section 2, we give some definitions and properties of CVT's, and introduce a CVT-nonuniform algorithm. Section 3 is devoted to applying CVT (CVT-nonuniform) to solve the time-dependent Burgers equation. We present numerical results with CVT-nonuniform-based reduced-order basis technique to a distributed feedback control problem for Burgers equation in Section 4. Some numerical results for a distributed feedback control problem are also given in Section 5.

2. Centroidal Voronoi Tessellation

The concept of the *centroidal Voronoi Tessellations* (CVTs) has been studied in [6]. CVTs have been successfully used in several data compression settings, for example in image processing and the clustering of data. Reduced-order modelling of complex systems is another data compression setting, that is one replaces high-dimensional approximations with low-dimensional ones. CVTs can be used for this purpose as well.

2.1. **Definition of CVTs for discrete data sets.**

The definition of CVT for discrete data sets begins with a set $\mathcal{S} = \{\mathbf{y}_k\}_{k=1}^{m}$ consisting of m vectors belonging to $\mathbb{R}^n$. Of course, $\mathcal{S}$ can also be viewed as a set of m points in $\mathbb{R}^n$ or a possibly complex-valued $n \times m$ matrix. In the context of CVT, it will be useful to think of the columns $\{S_{\cdot,k}\}_{k=1}^{m}$ of $\mathcal{S}$ as the spatial coordinate vectors of a dynamical system at time t_k. Similarly, we consider the rows $\{S_{i,\cdot}\}_{i=1}^{n}$ of $\mathcal{S}$ as the time trajectories of the dynamical system evaluated at the locations x_i.

Given a discrete set $\mathcal{S}$ belonging to $\mathbb{R}^n$, the set $\{V_i\}_{i=1}^{\ell}$ is called a *clustering* or a *tessellation* of the set $\mathcal{S}$ if $V_i \cap V_j = \emptyset$ for $i \neq j$ and $\cup_{i=1}^{\ell} V_i = \mathcal{S}$. Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^n$. Given a set of points $\{\mathbf{z}_i\}_{i=1}^{\ell}$ belonging to $\mathbb{R}^n$ (but not necessarily to $\mathcal{S}$), the *Voronoi region* $\hat{V}_i$ corresponding to the point $\mathbf{z}_i$ is defined by

$$\hat{V}_i = \{\mathbf{y} \in \mathcal{S} : |\,\mathbf{y} - \mathbf{z}_i\,| \leq |\,\mathbf{y} - \mathbf{z}_j\,| \quad \text{for } j = 1, \cdots, \ell, \quad j \neq i\},$$

where equality holds only for $i < j$. The points $\{\mathbf{z}_i\}_{i=1}^{\ell}$ are called *generating points* or *(cluster) generators*. Such a set $\{\hat{V}_i\}_{i=1}^{\ell}$ is known as a *Voronoi tessellation* or *Voronoi clustering* of $\mathcal{S}$ and each $\hat{V}_i$ is referred to as the *Voronoi region* or *cluster* corresponding to $\mathbf{z}_i$.

Given a density function $\rho(\mathbf{y})$ defined on $\mathcal{S}$, for each cluster $\hat{V}_i$, we can define its *cluster centroid* $\mathbf{z}_i^*$ by

$$\mathbf{z}_i^* = \frac{\sum_{\mathbf{y}\in\hat{V}_i} \mathbf{y}\rho(\mathbf{y})}{\sum_{\mathbf{y}\in\hat{V}_i} \rho(\mathbf{y})} \qquad i = 1, \cdots, \ell.$$

Given a set $\mathcal{S}$ of m vectors in $\mathbb{R}^n$ and a positive integer $\ell \leq m$, a *centroidal Voronoi tessellation* (CVT) or *centroidal Voronoi clustering* of $\mathcal{S}$ is a special Voronoi tessellation satisfying

$$\mathbf{z}_i = \mathbf{z}_i^* \qquad i = 1, \cdots, \ell \tag{2.1}$$

i.e., the generators of the Voronoi tessellation coincide with the centroids of the corresponding Voronoi clusters. It is important to note that general Voronoi tessellations do not satisfy the CVT property (2.1) so that, for given a set $\mathcal{S}$ and positive integer ℓ, a CVT must be constructed. Algorithms for this purpose are discussed in Subsection 2.2.

Centroidal Voronoi tessellations are closely related to minimizers of an "*energy*". Specifically, let

$$\mathcal{E}(\{\mathbf{z}_i\}_{i=1}^{\ell}, \{\hat{V}_i\}_{i=1}^{\ell}) = \sum_{i=1}^{\ell} \sum_{\mathbf{y}\in\hat{V}_i} |\mathbf{y} - \mathbf{z}_i|^2 \rho(\mathbf{y}),$$

where $\{\hat{V}_i\}_{i=1}^{\ell}$ is a tessellation of $\mathcal{S}$ and $\{\mathbf{z}_i\}_{i=1}^{\ell}$ are points in $\mathbb{R}^n$. No a priori relation is assumed between the $\hat{V}_i$'s and the $\mathbf{z}_i$'s. We refer to $\mathcal{E}$ as the "*cluster energy*"; in the statistics literature, it is called the *variance* or *cost*. It is easy to prove that a necessary condition for $\mathcal{E}$ to be minimized is that $\{\mathbf{z}_i, \hat{V}_i\}_{i=1}^{\ell}$ is a centroidal Voronoi tessellation of $\mathcal{S}$.

The connection between CVTs and reduced-order bases is now easily made. The set $\mathcal{S}$ is obviously the snapshot set. Then the CVT reduced basis set is the set of generators $\mathbf{z} = \{\mathbf{z}_i\}_{i=1}^{\ell}$ of a CVT of the snapshot set $\mathcal{S}$.

2.2. **A new algorithm for constructing discrete CVTs.** There are many known methods for constructing centroidal Voronoi tessellations such as two typical methods *Lloyd's method* [13] and *McQueen's method* [14]. Lloyd's method and its convergence properties have been analyzed; see [6, 7] and also the references cited therein.

The density function has been usually chosen by $\rho(\cdot) = 1$ (uniform density) in CVT (or maybe POD) reduced-order modelling. From the results based on CVT reduced-order modelling (see [9, 10], etc.), we can observe that the ℓ_2-norm errors (or ℓ_2 relative errors) of the difference between the full finite element solution and the reduced-order solution is strongly varying in time. The error at intimidate nodes is usually larger that the both end sides (starting time and final time), however, errors at the both ends may become larger than that of interior as the number of basis decreases. Since the reduced-order basis are not constructed with time evolution, we use the variable density (or nonuniform density) capability of CVT-based reduced-order modelling to obtain quasioptimal reduced-order basis without paying serious computing cost. In this paper, we choose a density (or weight) as follow

(1) Derive the reduced-order basis(centroids) by the Lloyd's algorithm with constant density ($\rho^{(0)}(\mathbf{y}_k) = 1$) from the set of snapshots $\mathcal{S} = \{\mathbf{y}_k(x) : k = 1, ..., m\}$;
(2) Compute the approximation solution $\mathbf{y}^{cvt}$ of the Burgers equation using reduced-order basis (centroids);
(3) Compute the relative error at each time step,

$$e_k = \frac{\| \mathbf{y}_k(x) - \mathbf{y}_k^{cvt}(x) \|_{L^2(\Omega)}}{\| \mathbf{y}_k(x) \|_{L^2(\Omega)}}, \qquad k = 1, ..., m; \tag{2.2}$$

(4) Then, derive the density from the following formula:

$$\rho^{(q)}(\mathbf{y}_k) = \rho^{(q-1)}(\mathbf{y}_k) + \exp(e_k - \frac{1}{m} \textstyle\sum_{k=1}^m e_k) \tag{2.3}$$

and then normalize;
(5) Derive the reduced-order basis (centroids) by the Lloyd's algorithm with the density $\rho(\mathbf{y}_k)$ from the set of snapshots $\mathcal{S}$;
(6) Compute the relative error (2.2). Stop if the stopping criterion is satisfied, or go back to step 4.

Here, we want to find the density function which make the distribution of error as uniform as possible in time. The $\mathbf{y}_k$ and $\mathbf{y}_k^{cvt}$ are full order finite element approximate solution and CVT-ROM approximate solution at time t_k, respectively. Although the weight function (2.3) is not optimal for finding desired reduced-order basis, we use a feedback control idea to obtain a quasi-optimal reduced-order basis. The above iteration can be done within $q = 3$ or 4. In this paper, we mainly study using CVT-nonuniform-based basis to reduce order of Burgers equation.

3. Model reduction for the Burgers equation

3.1. **Generating snapshot sets.** We now turn our attention to the computations. In order to generate a snapshot, we wish to numerically solve Burgers equation with homogeneous Dirichlet boundary conditions on Ω. Consider Burgers equation

$$\frac{\partial y}{\partial t}(t,x) = \nu \frac{\partial^2 y}{\partial x^2}(t,x) - y(t,x)\frac{\partial y}{\partial x}(t,x) + f(t,x) \quad \text{for} \quad x \in \Omega, \quad t > 0, \tag{3.1}$$

$$y(t,0) = y(t,L) = 0, \quad t > 0, \tag{3.2}$$

$$y(0,x) = y_0(x) \quad \text{for} \quad x \in \Omega, \tag{3.3}$$

where Ω is the finite interval $[0, L]$.

Accurate Galerkin method finite element approximations of the solutions of (3.1)-(3.3) are obtained using the linear finite element ("hat" funtion) on a n nodes; see [3]. Finite element solutions are used for the generation of snapshots and later for comparison with CVT based reduced-order solutions.

We assume the approximate solution of $y(t,x)$ is defined by

$$y^h(t,x) = \sum_{i=1}^{n} c_i(t)\phi_i(x), \tag{3.4}$$

where $\phi_i(x)(i = 1, 2, \cdots, n)$ is a linear basis function on the Ω, h is a discretization parameter and the coefficients $c_i(t)$ remain to be computed.

We use a variational formulation to define a finite element method to approximate (3.1). Integrating by parts and using homogeneous Dirichlet boundary conditions yield the weak formulation of the problem,

$$\begin{aligned}\int_\Omega \frac{\partial y^h}{\partial t}(t,x)w(x)dx = & - \int_\Omega y^h(t,x)\frac{\partial y^h}{\partial x}(t,x)w(x)dx \\ & - \nu\int_\Omega \frac{\partial y^h}{\partial x}(t,x)\frac{\partial w}{\partial x}(x)dx + \int_\Omega f(t,x)w(x)dx, \end{aligned}\tag{3.5}$$

where test function w are in a finite-dimensional subspace V_h of the Sobolev space $V = H_0^1(\Omega)$. Using (3.4), it is easy to see that (3.5) is equivalent to the system of nonlinear ordinary differential equations

$$\sum_{i=1}^{n} \dot{c}_i(t)(\phi_i,\phi_j) + \nu\sum_{i=1}^{n} c_i(t)(\phi_i^{'},\phi_j^{'}) + \left(\sum_{i=1}^{n} c_i(t)\phi_i^{'},\ \phi_j\sum_{i=1}^{n} c_i(t)\phi_i\right) = (f,\phi_j), \tag{3.6}$$

for $j = 1, \ldots, n$ along with the initial conditions

$$\sum_{i=1}^{n} c_i(0)(\phi_i,\phi_j) = (y_0,\phi_j), \tag{3.7}$$

where $\phi^{'}$ denotes the derivative of ϕ in space and $(\cdot,\cdot)$ means the $L^2(\Omega)$ inner product. The set of ordinary differential equations (3.6)-(3.7) is solved by using the Adams-Bashforth-Moulton method.

The m snapshot vectors

$$\mathbf{c}_k = [c_1(t_k) \quad c_2(t_k) \quad \cdots \quad c_n(t_k)]^T, \qquad k = 1, ..., m \tag{3.8}$$

are determined by evaluating the solution of equation (3.6)-(3.7) at m equally spaced time values $t_k, k = 1, \cdots, m$, ranging from $t_1 = 0$ to $t_m = T$.

3.2. **Determining reduced-order approximation.** We next apply the algorithms introduced in Section 2.2 to determine generators of the CVT from already know the snapshot sets $\{\mathbf{c}_k\}_{k=1}^m$ in Subsection 3.1; a set of generators is to be used as a reduced order basis. Note that each basis function satisfies a zero Dirichlet boundary condition. In the interior of the region, each basis function satisfies the (discretized) continuity equation.

We assume that

$$y^{cvt}(t,x) = \sum_{i=1}^{\ell} d_i(t) z_i(x),$$

where z_i denotes the i-th CVT basis function, $d_i(t)$ is the corresponding coefficient, and ℓ is the total number of CVT basis functions.

We consider the approximation Burgers equation

$$\begin{aligned} \int_\Omega \frac{\partial y^{cvt}}{\partial t}(t,x) z_j(x) dx = & - \int_\Omega y^{cvt}(t,x) \frac{\partial y^{cvt}}{\partial x}(t,x) z_j(x) dx \\ & + \nu \int_\Omega \frac{\partial^2 y^{cvt}}{\partial x^2}(t,x) z_j(x) dx + \int_\Omega f(t,x) z_j(x) dx \end{aligned} \tag{3.9}$$

for $j = 1, \cdots, \ell$, where test function z_j's are in the subspace of V_h. Integrating by parts and using homogeneous Dirichlet boundary conditions yield the weak formulation of the problem,

$$\sum_{i=1}^{\ell} \dot{d}_i(t)(z_i, z_j) + \nu \sum_{i=1}^{\ell} d_i(t)(z_i^{'}, z_j^{'}) + \left(\sum_{i=1}^{\ell} d_i(t) z_i^{'},\ z_j \sum_{i=1}^{\ell} d_i(t) z_i \right) = (f, z_j) \tag{3.10}$$

for $j = 1, \ldots, \ell$ along with the initial conditions

$$\sum_{i=1}^{\ell} d_i(0)(z_i, z_j) = (y_0, z_j). \tag{3.11}$$

Each basis, generator $\mathbf{z}_j \in \mathbb{R}^n$ of a CVT, defined a finite function, that is, if

$$\mathbf{z}_j = [Z_{1,j}\ Z_{2,j}\ \cdots\ Z_{n,j}]^T \qquad \text{for} \quad j = 1, \cdots, \ell,$$

we then have the corresponding finite element functions

$$z_j(x) = \sum_{i=1}^{n} Z_{i,j} \phi_i(x) \qquad \text{for} \quad j = 1, \cdots, \ell.$$

From definition for the reduced-order basis, we theoretically could be know the finite element approximation (3.5) agrees with the reduced-order approximation (3.9); see [9]. To implement the computational code, this scheme is implemented in MATLAB and the resulting ODE (3.10)-(3.11) is solved using the Adams-Bashforth-Moulton method.

4. Feedback Control Design

We apply the CVT-based reduced-order modelling method to a feedback control problem for the Burgers equation. The optimal control problem is to stabilize the solution to (3.1)-(3.3). The forcing term $f(t,x)$ is used to describe a distributed control. For an uncontrolled problem, $f(t,x) = 0$. For the controlled problem, the control term is assumed to have the special form $f(t,x) = b(x)u(t)$, where $u(t)$ is the control input and $b(x)$ is a given function used to distribute the control over the domain.

Now we describe our control problem. Find an optimal control $u^*(t)$ which minimizes the cost functional

$$J(u) = \int_0^\infty ||y(t,\cdot)||^2_{L^2(\Omega)} + |u(t)|^2 dt$$

subject to the constraint equations

$$\frac{\partial y}{\partial t}(t,x) = \nu \frac{\partial^2 y}{\partial x^2}(t,x) - y(t,x)\frac{\partial y}{\partial x}(t,x) + u(t)b(x) \quad \text{for} \quad x \in \Omega, \quad t > 0, \tag{4.1}$$

$$y(t,0) = y(t,L) = 0, \quad t > 0, \tag{4.2}$$

$$y(0,x) = y_0(x) \quad \text{for} \quad x \in \Omega, \tag{4.3}$$

Implementation of distributed parameter control theory requires the abstract form of the PDE which be obtained as follows. Let $y(t) = y(t,\cdot)$ be the state in state space $L^2(\Omega)$. Define the linear operator $\mathcal{A}_\nu$ as $\mathcal{A}_\nu y = \nu y''$, for all $y \in \mathcal{D}(\mathcal{A}_\nu) = H^1_0(\Omega) \cap H^2(\Omega)$. The abstract form of the controlled model problem of (4.1)-(4.3) can be written as the initial value problem

$$\dot{y}(t) = \mathcal{A}_\nu y(t) + G(y(t)) + Bu(t), \quad y(0) = y_0, \quad \text{for} \quad t > 0$$

on the space $L^2(\Omega)$, where $G(y) = -yy'$ is defined on $H^1_0(\Omega)$. It is known that $\mathcal{A}_\nu$ is the infinitesimal generator of an analytic semigroup on $L^2(\Omega)$. Additionally, mild solutions of the system exist.

4.1. **Linear Quadratic Regulator Design.** Assuming the nonlinear term in the Burgers equation is small, a suboptimal feedback control u^* can be obtained by using the well-known linear quadratic regulator theory; see [5]. That is, a full state feedback control is to find an optimal control $u^* \in L^2([0,T), L^2(\Omega))$ by minimizing the cost functional

$$J(u) = \int_0^\infty (Qy(t,\cdot), y(t,\cdot))_{L^2(\Omega)} + (Ru(t), u(t)))dt$$

subject to the constraint equations

$$\dot{y}(t) = Ay(t) + Bu(t), \quad y(0) = y_0, \quad \text{for} \quad t > 0$$

where $Q : L^2(\Omega) \to L^2(\Omega)$ is a nonnegative definite self-adjoint weighting operator for state and $R : L^2(\Omega) \to L^2(\Omega)$ is a positive definite weighting operator for the control. The optimal control $u^*(t)$ can be found as

$$u^*(t) = -\frac{1}{2}R^{-1}B^T\Pi y(t) = -Ky(t),$$

where K is called the feedback operator and Π is symmetric positive definite solution of the algebraic Riccati equation

$$\Pi A + A^T\Pi - \Pi BR^{-1}B^T\Pi + Q = 0. \tag{4.4}$$

4.2. **Linear feedback controllers with state estimate feedback.** A simple, classical feedback control design, linear quadratic regulator (LQR), assumes the full state is "feed back" into the system by the control. However, knowledge of the full state is not possible for many complicated physical systems. As a realistic alternative, a compensator design provides a state estimate based on state measurements to be used in the feedback control law.

We do not assume that we have knowledge of the full state. Instead, we assume a state measurement of the form

$$z(t) = Cy(t), \tag{4.5}$$

where $C \in \mathcal{L}(L^2(\Omega), \mathbb{R}^m)$. We can apply the theory and results to show that a stabilizing compensator based controller can be applied to the system; see [4]. Recently Atwell and King investigate reduced order controllers for spatially distributed systems using proper orthogonal decomposition theory.

The observer design is mainly needed in order to provide the feedback control law with estimated state variables. Therefore, the control law and observer are combined together into a complete system. The combined system is called compensator. This technique assumes the availability of a limited measurement of the state. Assume we have a system in the abstract form

$$\dot{y}(t) = Ay(t) + G(y(t)) + Bu(t), \qquad y(0) = y_0, \tag{4.6}$$

where $y(t)$ is in a state space $L^2(\Omega)$ and $u(t)$ is in a control space U.

According to the Given state measurement (4.5), a state estimate, $\tilde{y}(t)$, is computed by solving the observer equation

$$\dot{\tilde{y}}(t) = A\tilde{y}(t) + G(\tilde{y}(t)) + Bu(t) + L[z(t) - C\tilde{y}(t)], \qquad \tilde{y}(0) = \tilde{y}_0. \tag{4.7}$$

The feedback control law is given by

$$u(t) = -K\tilde{y}(t), \tag{4.8}$$

where K is called the feedback operator. Where functional gain operator K and estimator gain operator L are determined by linear quadratic regulator (LQR) and Kalman estimator (LQE), respectively, in usual manner. According to the result of the above, we already know

$$K = R^{-1}B^T\Pi. \tag{4.9}$$

Next, P is found as the non-negative definite solution to

$$AP + PA^T - PC^TCP + \bar{Q} = 0,$$

where $\bar{Q}$ is a non-negative definite weighting operator. If the solution P exists, we can define

$$L = PC^T. \tag{4.10}$$

Use (4.5)-(4.10), we obtained the closed loop compensator as

$$\begin{bmatrix} \dot{y}(t) \\ \dot{\tilde{y}}(t) \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - LC - BK \end{bmatrix} \begin{bmatrix} y(t) \\ \tilde{y}(t) \end{bmatrix} + \begin{bmatrix} G(y(t)) \\ G(\tilde{y}(t)) \end{bmatrix},$$

$$\begin{bmatrix} y(0) \\ \tilde{y}(0) \end{bmatrix} = \begin{bmatrix} y_0 \\ \tilde{y}_0 \end{bmatrix}.$$

4.3. **Reduced Order Compensators.** Implementation of the controller for a PDE system requires a numerical discretization. For example, use of a finite element method provides finite dimensional approximations of (4.5)-(4.6) of order N (where order refers to the freedom of finite element), given by

$$\dot{y}^N(t) = A^N y^N(t) + G^N(y^N(t)) + B^N u^N(t), \qquad y^N(0) = y_0^N,$$
$$z^N(t) = C^N y^N(t).$$

In a full order compensator design, the order N approximations are used to compute K^N and L^N. Then finite dimensional approximations of the compensator equation (4.7) and control law (4.8) are given by

$$\dot{\tilde{y}}^N(t) = A^N \tilde{y}^N(t) + G^N(\tilde{y}^N(t)) + B^N u^N(t) + L^N[z^N(t) - C^N \tilde{y}^N(t)], \quad \tilde{y}^N(0) = \tilde{y}_0^N,$$
$$u^N(t) = -K^N \tilde{y}^N(t),$$

respectively. The approximation to the closed-loop compensator system (which will henceforth be refereed to as full order) is given by

$$\begin{bmatrix} \dot{y}^N(t) \\ \dot{\tilde{y}}^N(t) \end{bmatrix} = \begin{bmatrix} A^N & -B^N K^N \\ L^N C^N & A^N - L^N C^N - B^N K^N \end{bmatrix} \begin{bmatrix} y^N(t) \\ \tilde{y}^N(t) \end{bmatrix} + \begin{bmatrix} G^N(y^N(t)) \\ G^N(\tilde{y}^N(t)) \end{bmatrix}, \tag{4.11}$$
$$\begin{bmatrix} y^N(0) \\ \tilde{y}^N(0) \end{bmatrix} = \begin{bmatrix} y_0^N \\ \tilde{y}_0^N \end{bmatrix}.$$

Real-time control using the full order compensator may be impossible for many physical problems in that they may require large discretized systems for adequate approximation. Therefore, a reduced order compensator is required. A "reduce-then-design" approach has a potential drawback that important physics or information contained in the model can be lost before obtaining the controller; see [8]. Hence, in this paper, we adopt a "design-then-reduce" approach. In other words, a controller is designed based on the high order model, and then reduced.

$$\dot{\tilde{y}}^l(t) = A^l \tilde{y}^l(t) + G^l(\tilde{y}^l(t)) + B^l u^l(t) + L^l[z^l(t) - C^l \tilde{y}^l(t)], \qquad \tilde{y}^l(0) = \tilde{y}_0^l, \tag{4.12}$$
$$u^l(t) = -K^l \tilde{y}^l(t), \tag{4.13}$$
$$\dot{y}^l(t) = A^l y^l(t) + G^l(y^l(t)) + B^l u^l(t), \qquad y^l(0) = y_0^l. \tag{4.14}$$

The suggested control law (4.13) is substituted into equations (4.12) and (4.14) producing

$$\begin{bmatrix} \dot{y}^l(t) \\ \dot{\tilde{y}}^l(t) \end{bmatrix} = \begin{bmatrix} A^l & -B^l K^l \\ L^l C^l & A^l - L^l C^l - B^l K^l \end{bmatrix} \begin{bmatrix} y^l(t) \\ \tilde{y}^l(t) \end{bmatrix} + \begin{bmatrix} G^l(y^l(t)) \\ G^l(\tilde{y}^l(t)) \end{bmatrix}, \tag{4.15}$$
$$\begin{bmatrix} y^l(0) \\ \tilde{y}^l(0) \end{bmatrix} = \begin{bmatrix} y_0^l \\ \tilde{y}_0^l \end{bmatrix}.$$

In this work, reduced bases are formed using the CVT process as described in Section 2. The reduced bases are used to compute the compensator equation, feedback control law and model problem in (4.12)-(4.13). Then the reduced systems given by (4.15) are compared with the full order compensator system in (4.11).

5. NUMERICAL RESULTS AND CONCLUSION.

For numerical computations, the viscosity coefficient was taken to be $\nu = 0.01$ and the spatial interval is taken to be $\Omega = [0, 1]$. The time interval is $[0, T]$ where $T = 2$ or 10 and $N = 120$. The control input operator is $B = \int_0^1 b(x)\phi(x)dx$, where $b(x) = x$ and $\phi(x)$ is a test function. The state weighting operator (used in Riccati equation calculations) Q is taken to be mass matrix. We set the control weighting operator $R(1, 1) = 0.2$ and the weighting operator $\bar{Q}$ is also chosen as the mass matrix. Finally, we create the measurement matrix C with $Cy(t, x) = 8\int_{3/4}^{5/6} y(t, x)dx$ for the state estimate feedback controller. An initial condition of $y_0(x) = \sin\pi x$ is applied. We obtain a standard finite element approximation solution of the full order PDE and shown in Figure 1.

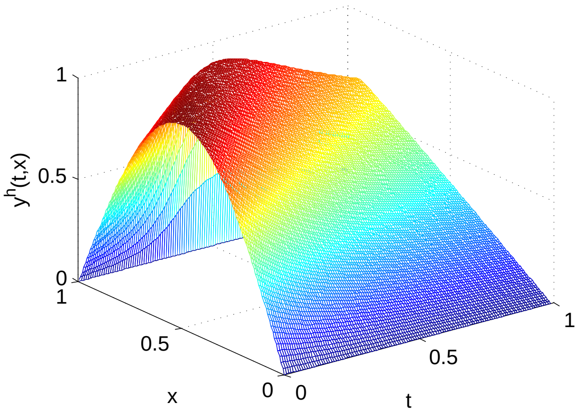
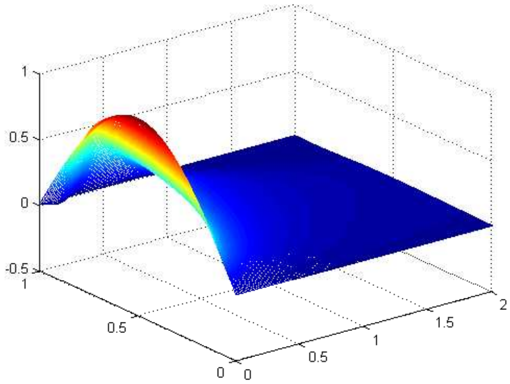


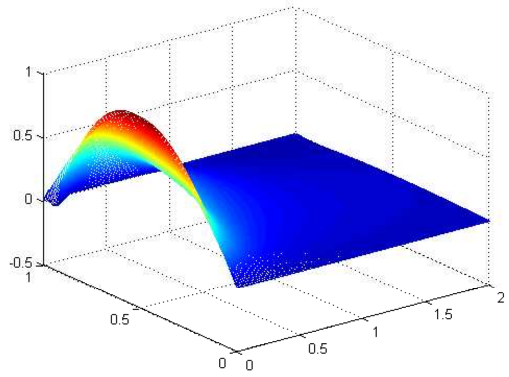
FIGURE 1. Full finite element solution of the Burgers equation.

Then we generate snapshots and CVT bases according to Section 2. Simulations of the full order PDE and the reduced order compensator are compared. Figure 2 shows the solutions of the controlled Burgers equation for full FEM and reduced order where compensator feedback law is used. Figure 3 shows that reduced-order feedback control methods are quite effective for full order feedback control. In Tables 1 and 2, we report the CPU times and L^2 norms of the controlled solution at $T = 2$ and 10 for state estimate control cases. One can see that the CPU time for reduced-order model is about 200 times less than that of full FEM with a relative error of 10^{-4}.

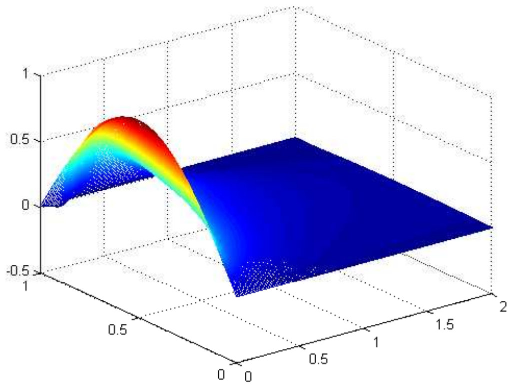
We have introduced and discussed a weighted (nonuniform density) centroidal Voronoi tessellation for low order approximate solution and compensator-based control design of Burgers equation. First, we used a low order basis obtained through CVT-nonuniform as applied to finite element approximate solution and functional gains to design a low order controller, and by using the low order basis the order of control modelling was reduced. Numerical experiments show that a solution of reduced-order controlled Burgers equation performs well in comparison with a solution of full order controlled Burgers equation. Future efforts involve application of



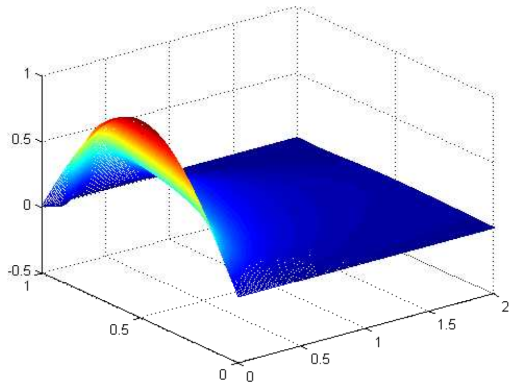
(a) Full order control: $N = 120$



(b) Reduced order control: $l = 3$



(c) Reduced order control: $l = 6$



(d) Reduced order control: $l = 9$

FIGURE 2. Controlled Burgers equation for state estimate feedback: $\nu = 1/100, R = 0.2, T = 2$.

TABLE 1. Full order control vs. Reduced order control, state estimate feedback: $\nu = 1/100, T = 2$

number of generators	$l = 3$	$l = 6$	$l = 9$	N=120
CPU Time	5.8438	7.0625	8.9063	1.7445e+003
$\|\|y(T, \cdot)\|\|_2$	0.1112	0.0967	0.0934	0.0899

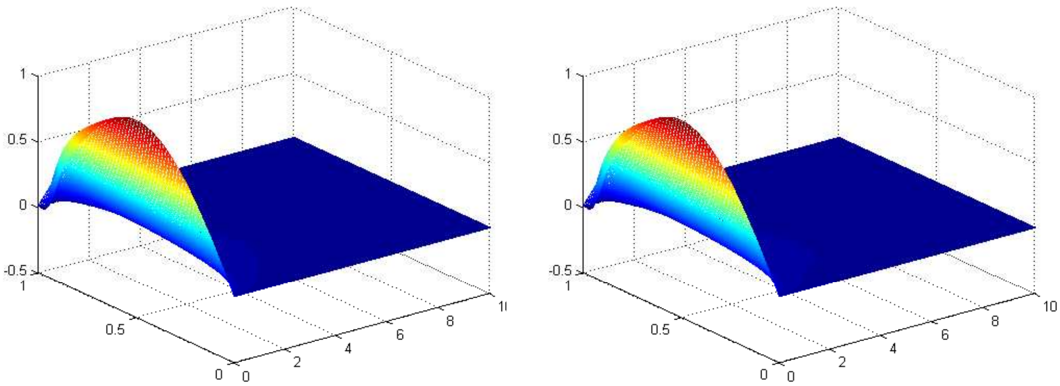


FIGURE 3. Full order(left) and reduced order(using 9 basis) controlled Burgers equation: $\nu = 1/100, R = 0.2, T = 10$.

TABLE 2. Full order control *vs.* Reduced order control, state estimate feedback: $\nu = 1/100, T = 10$

number of generators	$l = 9$	N=120
CPU Time	42.2969	1.0473e+004
$\|\|y(T,\cdot)\|\|_2$	2.6738e-004	2.4946e-004

the reduced bases framework to more complex physical problems, such as those in fluid flows and materials processing, and more systematically interpreting how to choose the nonuniform density argument (nonconstant weight function).

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