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A JET EMERGING FROM A SLIT AT THE CORNER OF QUARTER PLANE

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ABSTRACT. A numerical solution is provided for a jet produced by a flow emerging from a slit at the bottom corner of a quarter plane. The flow is characterized by the Froude number F, based on the net volume flux and the width of the slit. We perform the free-surface flow for various values of F and another parameter corresponding to the position of the vertical wall. A jet with back-flow near the edge of the vertical wall is obtained, and the limiting case is a jet with a stagnation point.

1. INTRODUCTION

A 2-D steady flow passing through a slit and producing a jet is considered as illustrated in Figure 1. The fluid occupies a quarter domain bounded by two rectangular walls, and a slit is built by shifting one of the walls such that a stream between two free boundaries is formed as the fluid emerges from the slit. In the absence of gravity, Wiryanto [1] solved analytically. The flow domain was firstly mapped conformally; and then an analytic complex function, defined based on the separation points and involving the hodograph variable, was applied to Cauchy's integral formula. If the gravity is present, the analytical function becomes more complicated as the real part of the hodograph variable is not zero anymore. We propose to reconstruct the problem into a boundary integral equation, and to solve numerically.

Referring to Figure 1, a free-surface flow under a sluice gate occurs when the vertical wall is shifted far to the left from the edge of the horizontal wall, before the flow becomes waterfall at the end of the horizontal wall. Wiryanto and Wijaya [2] solved numerically the sluice gate flow, by assuming an infinite length of the horizontal wall. He obtains that the flow tends to uniform stream far from the gate. Some other works for sluice gate flow, but the fluid depth is finite before passing through the gate, can be seen such as Asavanant and Vanden-Broeck [3], Binder and Vanden-Broeck [4, 5], and Vanden-Broeck [6]. Therefore, they can compare between the uniform streams before and after passing the gate. Meanwhile, the waterfall flow can be seen in Clarke [7]. The problem of waterfall explains the changing of the uniform stream to a jet.

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In this paper, we solve the similar problem as in Wiryanto and Wijaya [2], but the vertical wall is shifted close to the edge of the bottom wall, and possible further in front of it. Hence, this problem involves two separation points, each representing the edge of the wall. The difficulty here is to determine the coordinates one of them. In solving this, we follow Tuck [8] who solved the jet emerging from a slit in a vertical wall.

A boundary element method is used in solving the problem in this paper. An integral equation is firstly constructed along the free boundary. The discretisation of the equation then gives a system of equations which can be solved by Newton method. This method has been applied in various free surface flow problems, see for example Wiryanto and Tuck [9, 10], Wiryanto [11] and some works referred above.

As the result, the numerical solution can be compared to the zero gravity case in Wiryanto [1]. Another profile of the solution is the existing of back-flow near the edge of the vertical wall, and a free surface flow with a stagnation point is the limiting case. This is also obtained in Wiryanto and Wijaya [2] and Tuck [8].

2. Problem Formulation

We consider the steady two-dimensional irrotational flow of an inviscid and incompressible fluid in a dam of infinite depth bounded at the right by a vertical wall and a slit width d at the bottom corner, so that a jet is formed as the flow emerges from it smoothly. We choose Cartesian coordinates with the center $x = 0$, $y = 0$ at the edge A of the horizontal wall. The net volume of the flux in the dam is Q per unit distance perpendicular to the plane of flow, see Figure 1.

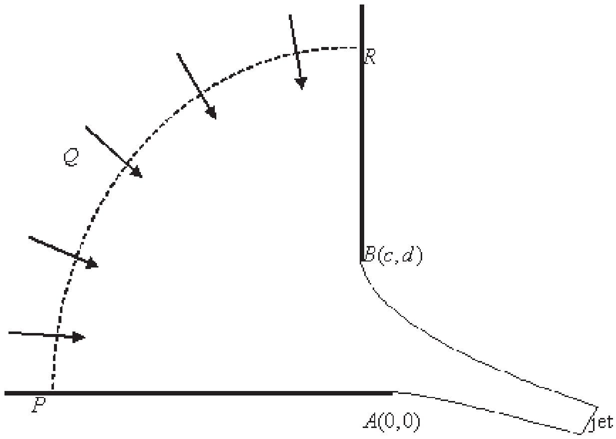


FIGURE 1. Sketch of the flow and the coordinates.

From the assumption of the fluid and the flow, we present the stream in a complex potential $f = \phi + i\psi$ corresponding to the complex velocity $df/dz = u - iv$, where $z = x + iy$. For convenience, we work in non-dimensional variables by taking Q as the unit flux and d as the unit length, and we define $\phi = 0$, $\psi = 0$ at the center coordinates of the physical z-plane. Therefore, the flow domain in f-plane is a strip with width 1, also the width of the slit. Now, our task is to solve the boundary value problem

$$\nabla^2 \phi = 0$$

in the flow domain. The dynamic condition is expressed by Bernoulli equation

$$\frac{1}{2} F^2 \left(\phi_x^2 + \phi_y^2 \right) + y = \text{constant} \tag{2.1}$$

along the free boundaries of the jet. F is Froude number defined as

$$F = \frac{Q}{\sqrt{g d^3}}$$

with acceleration of gravity g. The other condition is kinematic along the solid and free boundaries

$$\frac{\partial \phi}{\partial \bar{n}} = 0 \tag{2.2}$$

where $\bar{n}$ is a normal vector of the boundaries.

In determining ϕ, we first introduce a hodograph variable $\Omega = \tau - i\theta$ having relationship to the velocity vector

$$\frac{df}{dz} = e^{\Omega}. \tag{2.3}$$

Meanwhile, the flow domain in f-plane is mapped into a half lower artificial plane $\zeta = \xi + i\eta$ by

$$f = -\frac{1}{\pi} \log \zeta. \tag{2.4}$$

The downstream jet is mapped to $\zeta = 0$ and the two separation points A and B are mapped to $\zeta = 1$ and $\zeta = -\xi_b$ respectively. The diagram of the flow in f and ζ planes is shown in Figure 2. The bold line corresponds to the solid boundaries, and the thick line corresponds to the free boundary.

Instead of determining ϕ, we solve the hodograph variable Ω with respect to the artificial variable ζ, satisfying

$$\nabla^2 \Omega = 0$$

subject to (the dynamics condition (2.1) becomes)

$$\frac{1}{2} F^2 e^{2\tau} + y = c, \qquad -\xi_b < \xi < 1 \tag{2.5}$$

where c is an unknown constant; and the kinematic condition (2.2) becomes

$$\theta = \begin{cases} -\pi/2, & -\infty < \xi < -\xi_b \\ 0, & 1 < \xi < \infty \end{cases} \tag{2.6}$$

Here θ is unknown for $-\xi_b < \xi < 1$.

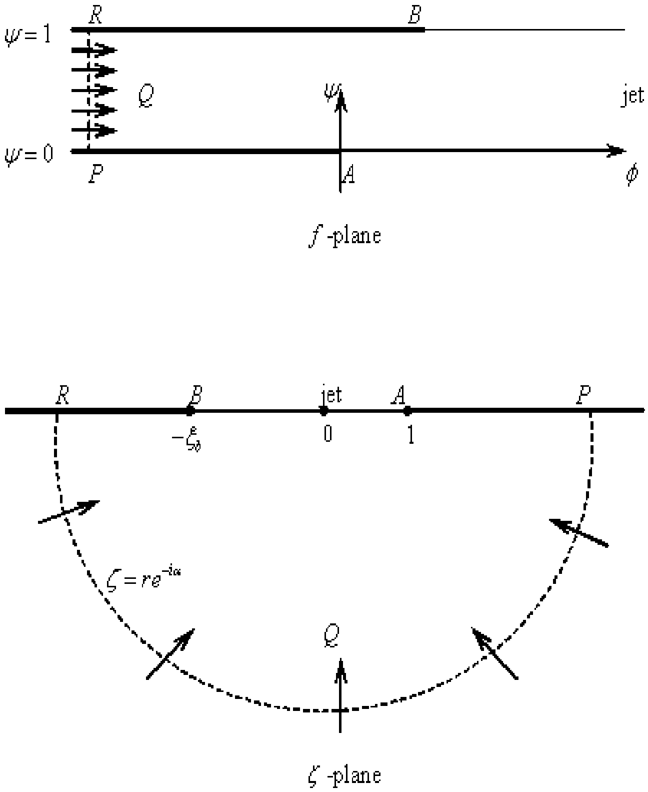


FIGURE 2. The flow domain in f-plane and ζ-plane.

The relation between θ and τ is then required in reducing the unknown variables. This can be obtained by using Cauchy integral theorem. As a complex function, we define

$$\chi(\zeta) = \Omega + \frac{1}{2}\log\zeta. \tag{2.7}$$

This function is analytic and $\chi \to 0$ for $|\zeta| \to \infty$, so that it can be applied to Cauchy integral theorem. We come up to (2.7) since the upstream flow far from the slit is uniform with velocity $df/dz \to 0$ and the streamlines bouncing by horizontal and vertical walls having angle θ as given in (2.6). A logarithm function is the one having character described above, so that

$$\Omega \to -\frac{1}{2}\log\zeta, \qquad \text{for } |\zeta| \to \infty$$

and this is used to construct χ as given in (2.7).

In applying χ to Cauchy integral theorem along closed path covering the flow domain in ζ-plane, it is enough to consider along the real ξ-axis giving

$$\chi(\xi) = -\frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{\xi(s)}{s-\xi} ds \tag{2.8}$$

The function χ is then expressed in τ and θ for both sides in (2.8), and the real part gives

$$\tau(\xi) = -\frac{1}{2}\log|\xi_b + \xi| + \frac{1}{\pi}\mathrm{PV}\int_{-\xi_b}^{1} \frac{\theta}{s-\xi} ds \tag{2.9}$$

for $-\xi_b < \xi < 1$. Here, PV is used to denote Cauchy principal-value for the integration.

The last part, which has to be prepared before obtaining the integral equation, is to determine y along the free surface. We use (2.3) and (2.4) to have

$$\frac{dz}{d\zeta} = -\frac{e^{-\Omega}}{\pi\zeta} \tag{2.10}$$

and the imaginary part along the free surface gives

$$\frac{dy}{d\xi} = -\frac{e^{-\tau}\sin\theta}{\pi\xi} \tag{2.11}$$

Therefore, the value y is obtained by integrating (2.11) giving

$$y(\xi) = 1 - \int_{-\xi_b}^{\xi} \frac{e^{-\tau(s)}}{\pi s}\sin\theta(s) ds \tag{2.12}$$

for $-\xi_b < \xi < 0$ representing the upper free boundary of the jet, and

$$y(\xi) = \int_{\xi}^{1} \frac{e^{-\tau(s)}}{\pi s}\sin\theta(s) ds \tag{2.13}$$

for the lower free boundary of the jet $0 < \xi < 1$. Here, τ is evaluated from (2.9). The formulae y in (2.12) and (2.13), and τ are then substituted to (2.5) giving the integral equation that has to be solved for $\theta(\xi)$ in $(-\xi_b, 1)$.

Now, we analyze the coordinates z corresponding to large $|\zeta|$. This is used to determine $x(-\xi_b)$, the position of the vertical wall from y-axis. From (2.9) we can express τ in

$$\tau(\zeta) = -\frac{1}{2}\log|\zeta| + a\zeta^{-1} + b\zeta^{-2} + O(\zeta^{-3}) \tag{2.14}$$

for large $|\zeta|$. a and b are real constants corresponding to the Taylor series of $\log|\xi_b + \zeta|$ and the denominator of the integrand in (2.9). The series (2.14) is then substituted into (2.10) giving

$$\frac{dz}{d\zeta} = -\frac{e^{i\theta}}{\pi}|\zeta|^{1/2}\left[\zeta^{-1} - a\zeta^{-2} + \frac{1}{2}(a^2 - 2b)\zeta^{-3} + O(\zeta^{-4})\right] \tag{2.15}$$

The abscissa $x(-\xi_b)$ can be obtained by integrating (2.15) along the horizontal wall and then the curve PR shown in Figure 1.

For the horizontal wall, say $\zeta = r > 0$, (2.15) gives

$$z(r) \approx -\frac{1}{\pi}\left[2r^{1/2} + 2ar^{-1/2}\right] + K \tag{2.16}$$

where K is real constant. For the curve PR we suppose $\zeta = re^{-i\alpha}$, $0 \leq \alpha \leq \pi$, and θ is approximated linearly with respect to α, i.e. $\theta = -\alpha/2$. Hence, the complex integral of (2.15) along PR gives

$$z(R) \approx z(P) + \frac{1}{\pi}\left[\left(2r^{1/2} + 2ar^{-1/2}\right) + i\left(2r^{1/2} - 2ar^{-1/2}\right)\right]$$

We denote $z(R)$ as the value of z at the point R, also for $z(P)$. Since $z(P)$ is real, evaluated from (2.16), we obtain $x(-\xi_b) = K$. We can evaluate K from numerical calculation of the real part of (2.10) from $\xi = 1$ to a certain point $\xi = r_0$ for relatively large r_0, namely $x(r_0) = x_0$, and it is substituted in (2.16) giving

$$K = x_0 + \frac{1}{\pi}\left[2r_0^{1/2} + 2ar_0^{-1/2}\right]$$

Meanwhile, the coefficient a is determined from τ evaluated using (2.9) numerically for $\xi = r_0$, namely τ_0, and it is equalized to its approximation (2.14), so that we have

$$a = r_0\left[\tau_0 + \frac{1}{2}\log r_0\right]$$

This value is required in approximating $x(-\xi_b)$ from (2.15) up to order $O(\zeta^{-3})$, and $x(-\xi_b)$ is needed in evaluating the abscissa x for the upper free boundary of the jet.

3. Numerical Procedure

The nonlinear integral equation (2.5) converts to a set of N algebraic equations in N unknowns, if we approximate the integration (2.9) by summation in a suitable manner. The interval of integration $(-\xi_b, 0)$ and (0,1) is first discretized each by defining the end-points of $M-1$ subintervals $\xi_0 = -\xi_b < \xi_1 < \xi_2 \cdots < \xi_{M-1} = -\epsilon$ and $\xi_{M+1} = \epsilon < \xi_{M+2} < \xi_{M+3} \cdots < \xi_{2M} = 1$ with $N = 2M$. Then we let $\theta_j = \theta(\xi_j)$ for $j = 1, 2, \cdots, N$, be $N-2$ unknowns. ϵ is a small value representing the jet relatively far from the slit, and we need this number to truncate the integration (2.9), as it is impossible to know the end of the jet.

In order to evaluate the Cauchy principle-value singular integral in (2.9), we approximate $\theta(\xi)$ as varying linearly on the interval (ξ_{j-1}, ξ_j), and evaluate the integral over each such interval exactly. For any $\xi_j^* \in (\xi_{j-1}, \xi_j)$, $\tau(\xi_j^*)$ is evaluated by

$$\begin{aligned}\tau(\xi_j^*) \approx -&\frac{1}{2}\log|\xi_b + \xi_j^*| \\ &+ \sum_{l=1}^{N-1}(\theta_{l-1} - \theta_l) + \left\{\theta_l + (\theta_{l-1} - \theta_l)\frac{\xi_j^* - \xi_l}{\xi_{l-1} - \xi_l}\right\}\log\left|\frac{\xi_{l-1} - \xi_j^*}{\xi_l - \xi_j^*}\right|\end{aligned}$$

Similarly, the integral (2.12) determining the y-coordinate of the free surface can be evaluated by numerical approximation, such as trapezoidal rule

$$\begin{aligned} y(\xi_j^*) \quad &\approx y(\xi_{j-1}^*) \\ &-\frac{1}{2}\left(\frac{e^{-\tau(\xi_j^*)}}{\pi\xi_j^*}\sin\theta(\xi_j^*) + \frac{e^{-\tau(\xi_{j-1}^*)}}{\pi\xi_{j-1}^*}\sin\theta(\xi_{j-1}^*)\right)(\xi_j^* - \xi_{j-1}^*) \end{aligned}$$

similarly for (2.13).

In obtaining the N algebraic equations, we use N collocation points ξ_j^* as the mid-point in each subinterval (ξ_{j-1}, ξ_j), except $\xi_M = \epsilon/2$ and also $\theta(\xi_j^*)$ defined linearly between θ_{j-1} and θ_j. For each point ξ_j^*, the integral equation (2.5) gives one algebraic equation, so that there are N equations for unknowns $\theta_1, \theta_2, \cdots, \theta_{N-1}$ and the constant c in (2.5). The parameter Froude number F is given, also define $\theta_0 = -\pi/2$, $\theta_N = 0$ at the edge of the vertical and horizontal walls. This closed form is then solved numerically by Newton method. The initial condition for the Newton iteration is given from the analytical solution for the zero gravity case derived by Wiryanto [1].

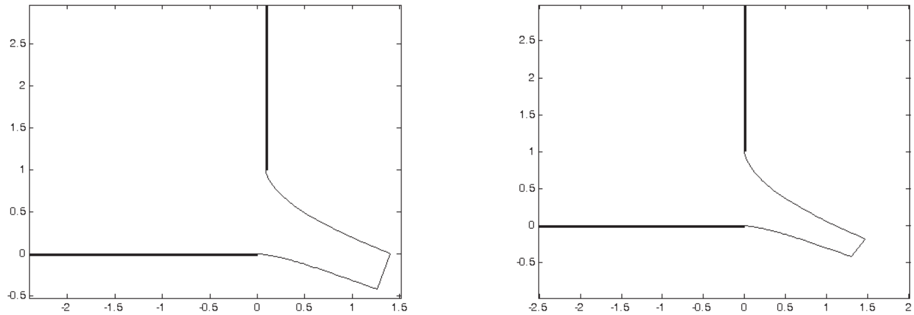
In the post process of solving the integral equation, the abscissa x of the free surface is determined from the real part of (2.10). The integration of (2.10) requires $x(1) = 0$, from the definition of the coordinates, for the lower free surface of the jet and $x(-\xi_b) = K$ for the other free surface. We evaluate K following the procedure described in the previous section. Together with the values of y, plot of (x_j, y_j) is made to get the profile of the jet, for various F and ξ_b.

4. Results

Most calculations of the numerical procedure described above use $N = 200$ and $\epsilon = 0.001$. In evaluating K we truncate the flow in the artificial domain $\epsilon \le |\zeta| \le 1000$ with $x(1000) \approx -20$. Figure 1 is the result for $F = 7.0$ $\xi_b = 9.0$, we perform the plot in the same scale of x and y. The artificial parameter $\zeta = -\xi_b$ corresponds to the position of the vertical wall from y-axis, namely $x_b = x(-\xi_b)$, so that the coordinates of the separation point B are $(x_b, 1)$. In Figure 1, we have the physical quantity $x_b = -0.20$. For smaller ξ_b, the vertical wall shifts forward.

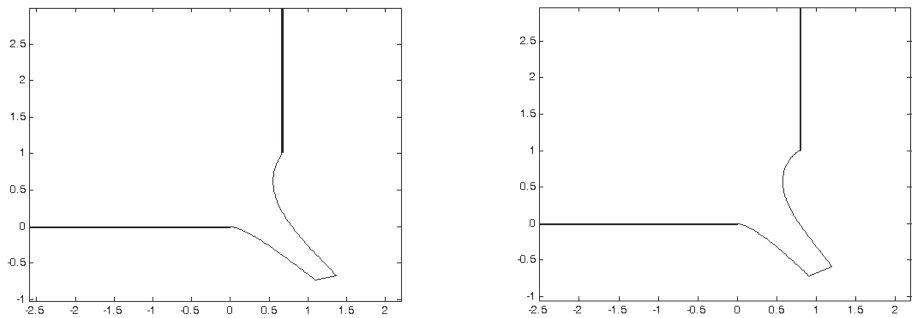
We show in Figure 3a plot of the jet emerging from the slit as the result of our calculation using $\xi_b = 6$ corresponding to $x_b = 0.085$. For the case where the vertical wall is on the y-axis, it occurs at $\xi_b = 6.86$, obtained by calculating trial-error for various ξ_b. For other values F and ξ_b, giving $x_b = 0$, can be determined similarly; $F = 6$ and $F = 4$ correspond to $\xi_b = 6.95$ and 7.41. Plot of the jet with $x_b = 0$ is shown in Figure 3b for $F = 4$. We also calculate for large F, and we found that the solution with $x_b = 0$ is obtained for $\xi_b > 4.54$. The limiting number is obtained by Wiryanto [1] for zero gravity case.

For fixed value $\xi_b = 6$, the Froude number is now decreased. Our calculations give a jet emerging from the slit with increasing x_b, for $F = 3$ we obtain $x_b = 0.167$. Moreover, for smaller F the free surface, separating from the vertical wall, indicates back-flow by existing



(a) (b)

FIGURE 3. Plot of the jet emerging from a slit for (a) $\xi_b = 6$, $F = 7$, (b) $\xi_b = 7.41$, $F = 4$.



(a) (b)

FIGURE 4. Plot of the jet (a) with back-flow for $F = 1.1$, (b) with a stagnation point for $F = 0.97$

$\theta < -\pi/2$ near the point B. Following Wiryanto and Wijaya [2], we determine the minimum value of θ for each Froude number. This value decreases for smaller Froude number, and it corresponds to ξ approaching to $\zeta = -\xi_b$. The limiting flow occurs when $F = 0.96$ with a stagnation point at B and $\theta(-\xi_b) = -5\pi/6$. This agrees to Vanden-Broeck and Tuck [12] who showed that angle value for the stagnation point of the free surface flow separating a wall. In Figure 4a, we perform a jet with back-flow, calculated for $F = 1.1$; and the jet with a stagnation point is performed in Figure 4b.

In case the vertical wall is on the y-axis, i.e. $x_b = 0$, the solution with back-flow is not able to be obtained since the procedure fails to calculate for $F < 3.2$. We indicate that the truncated flow domain is far than enough presenting uniform stream, far different between $\theta(\epsilon)$ and $\theta(-\epsilon)$. Giving smaller ϵ does not help much.

5. Conclusion

We have solved numerically the free surface flow forming a jet emerging from a slit at the bottom corner of a quarter plane by boundary element method. The Froude number and the position of the vertical wall from the vertical axis play an important role in forming the jet. Solutions with back-flow are obtained when the vertical wall is in front of the vertical wall. The limiting case of this problem is flow with a stagnation point at the edge of the vertical wall.

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