THE SEPARABLE QUOTIENT PROBLEM
FOR $(LF)_{tv}$-SPACES

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ABSTRACT. In 1981 S. A. Saxon and P. P. Narayanaswami ([10]) showed that every (LF)-space has an infinite dimensional separable quotient. In this note we prove that this fails for $(LF)_{tv}$-spaces. We construct a wide class of $(LF)_{tv}$-spaces, which have no infinite dimensional separable quotient.

Introduction

One of the famous unsolved problems of functional analysis is whether every infinite dimensional (i.d.) Banach space E has an i.d. separable quotient, i.e., whether it has a closed subspace M such that the quotient space E/M is i.d. and separable. Likely this problem has been posed by S. Mazur in 1932.

It is known that every i.d. reflexive Banach space, or even every i.d. weakly compactly generated Banach space, has an i.d. separable quotient ([14], Section 15-3, Corollary 7). H. P. Rosenthal ([9]) and independently E. Lacey ([5]) proved that the Banach space $C(S)$ of all continuous scalar-valued functions defined on an infinite compact Hausdorff space S has an i.d. separable quotient.

In 1936 M. Eidelheit ([2]) showed that every non-normable Fréchet space (i.e., metrizable complete locally convex space) has a quotient isomorphic to $\mathbb{K}^{\mathbb{N}}$, where $\mathbb{K}$ denotes the field of real or complex scalars, and so it has an i.d. separable quotient. In 1981 S. A. Saxon and P. P. Narayanaswami ([10], Theorem 3) proved that every (LF)-space (i.e., the inductive limit of a strictly increasing sequence of Fréchet spaces) possesses an i.d. separable quotient.

For F-spaces (i.e., metrizable complete topological vector spaces) the separable quotient problem has been solved negatively in 1984 by Popov ([8]). He showed that for every atomless finite measure space (Ω, Σ, μ) with $\dim L^1(\Omega) > \mathfrak{c}$ there exists a subset Ω_1 of Ω with $\mu(\Omega_1) > 0$ such that the space $L^p(\Omega_1)$, $0 < p < 1$, has no i.d. separable quotient. In [12] we proved that for every measure space (Ω, Σ, μ) and every Orlicz function ϕ the Orlicz space $L^\phi(\Omega)$ has an i.d. separable quotient if and only if $\lim_{u \to \infty} \inf \phi(u)u^{-1} = 0$ and $\omega(\Omega) = \aleph_0$

or $\lim_{u\to\infty} \inf \phi(u)u^{-1} > 0$ and $d(\Omega) \geq \aleph_0$ (see [12], Theorems 1.3, 3.2, 3.3 and Corollary 3.1). In particular, the space $L^p([0,1]^{\mathfrak{c}})$, $0 < p < 1$, has no i.d. separable quotient (see [7], Corollary 8.3).

In this paper we prove that there exist metrizable $(LF)_{tv}$-spaces which have no i.d. separable quotient. In order to show this, we prove that every i.d. Orlicz space contains a dense $(LF)_{tv}$-subspace. We also prove that there exist strict $(LF)_{tv}$-spaces which have no i.d. separable quotient.

Preliminaries

Let (E, τ) be a Hausdorff topological vector space (tvs). A *quotient* of (E, τ) is a tvs $(E/M, \tau/M)$, where M is a closed subspace of (E, τ) and τ/M is the quotient topology on E/M. If G is a subspace of (E, τ), $\tau|_G$ denotes the relative topology on G. By $\mathcal{F}(\tau)$ we shall mean the family of all balanced neighbourhoods of zero in (E, τ). A sequence (U_n) of balanced and absorbing subsets of E such that $U_{n+1} + U_{n+1} \subset U_n$ for all $n \in \mathbb{N}$ is called a *string* in (E, τ). A string (U_n) in (E, τ) is *closed* if every U_n is closed in (E, τ) and *topological* if $(U_n) \subset \mathcal{F}(\tau)$.

A tvs (E, τ) is *ultrabarrelled* (*barrelled* in [1]) if every closed string in (E, τ) is topological. A quotient of an ultrabarrelled tvs is ultrabarrelled ([1], Section 6).

We will use the following open mapping theorem ([1], Section 9):

A continuous linear mapping from an F-space onto an ultrabarrelled tvs is open.

The dual of a tvs (E, τ), i.e., the vector space of all continuous linear functionals on (E, τ) will be denoted by $(E, \tau)^*$; it will be called *total* if for every $x \in (E \setminus \{0\})$ there exists $f \in (E, \tau)^*$ with $f(x) \neq 0$. A subspace G of (E, τ) is weakly closed, i.e., closed in the weak topology of (E, τ) if and only if for every $x \in (E \setminus G)$ there exists $f \in (E, \tau)^*$ such that $f(G) = \{0\}$ and $f(x) \neq 0$ (see [3], Section 7.3, Corollary 6).

In [4] (see Proposition 1) we proved the following.

Theorem A. *A tvs (E, τ) has an i.d. separable quotient with the total dual if and only if there exists a strictly increasing sequence (E_n) of weakly closed subspaces of (E, τ) whose union is dense in (E, τ).*

Let (E, τ) be a tvs. If there exists a strictly increasing sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ of F-spaces such that $E = \bigcup_{n=1}^{\infty} E_n$, $\tau_{n+1}|_{E_n} \subset \tau_n$ for each $n \in \mathbb{N}$ and τ is the finest vector topology with $\tau|_{E_n} \subset \tau_n$ for each $n \in \mathbb{N}$, then (E, τ) is said to be an $(LF)_{tv}$-*space* with a *defining sequence* $\{(E_n, \tau_n) : n \in \mathbb{N}\}$. Then we write $(E, \tau) = \varinjlim(E_n, \tau_n)$.

If $(E, \tau) = \varinjlim(E_n, \tau_n)$ is an $(LF)_{tv}$-space, then a string (U_n) in (E, τ) is topological if and only if $U_n \cap E_n \in \mathcal{F}(\tau_n)$ for every $n \in \mathbb{N}$.

F-spaces and $(LF)_{tv}$-spaces are ultrabarrelled ([1], Section 6). By Baire theorem and the open mapping theorem no $(LF)_{tv}$-space is an F-space.

We say that an $(LF)_{tv}$-space $(E, \tau) = \varinjlim(E_n, \tau_n)$ is strict if $\tau_{n+1}|_{E_n} = \tau_n$ for each $n \in \mathbb{N}$; then we have $\tau|_{E_n} = \tau_n$ for any $n \in \mathbb{N}$ ([3], Section 4.6,

Theorem 1). If $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ is a strictly increasing sequence of F-spaces with $\tau_{n+1}|_{E_n} = \tau_n$ for every $n \in \mathbb{N}$, then there exists the finest Hausdorff vector topology τ_0 on $E_0 = \bigcup_{n=1}^{\infty} E_n$ such that $\tau_0|_{E_n} = \tau_n$ for all $n \in \mathbb{N}$ ([3], Section 4.6, Theorem 1); clearly, $(E_0, \tau_0) = \varinjlim(E_n\tau_n)$ and (E_0, τ_0) is a strict $(LF)_{tv}$-space. Strict $(LF)_{tv}$-spaces are complete ([3], Section 4.6, Theorem 1) and non-metrizable.

A tvs (G, γ) is said to be continuously included in a tvs (E, τ) provided $G \subset E$ and $\tau|_G \subset \gamma$. We say that a tvs (E, τ) is dominated by a tvs (E, γ) if $\tau \subset \gamma$.

A sequence (P_n) of projections on a vector space E is *orthogonal* if $P_n(P_m(x)) = 0$ for all $n, m \in \mathbb{N}$ with $n \neq m$, and $x \in E$.

A function $\phi : [0, \infty) \to [0, \infty)$ we will call an Orlicz function if

(01) ϕ is nondecreasing and continuous at $u = 0$;

(02) $\phi(u) = 0$ if and only if $u = 0$;

(03) $\phi(2u) \leq C\phi(u)$ for some constant C and for all $u \geq 0$.

(The condition (03) is known as condition Δ_2.)

By Φ_0 and Φ_1 we denote the set of all Orlicz functions such that

$$\liminf_{u \to \infty} \phi(u)u^{-1} = 0$$

and

$$\liminf_{u \to \infty} \phi(u)u^{-1} > 0,$$

respectively.

Clearly, the set $\Phi = \Phi_0 \cup \Phi_1$ contains all Orlicz functions.

Let (Ω, Σ, μ) be a measure space and let $L^0(\Omega)$ denote the vector space of all (μ-equivalence classes of) scalar valued and measurable functions on Ω. Let $\phi \in \Phi$. Put

$$m_\phi(f) = \int_\Omega \phi(|f(t)|)d\mu \quad \text{for} \quad f \in L^0(\Omega) ,$$

$$L^\phi(\Omega) = \{f \in L^0(\Omega) : m_\phi(f) < \infty\}, \quad \text{and}$$

$$\|f\|_\phi = \inf\{s > 0 : m_\phi(s^{-1}f) < s\} \quad \text{for} \quad f \in L^\phi(\Omega) .$$

The functional $\| \cdot \|_\phi$ on the vector space $L^\phi(\Omega)$ is a complete and absolutely continuous F-norm (i.e., for every $f \in L^\phi(\Omega)$ and every decreasing sequence $(\Omega_n) \subset \Sigma$ such that $\mu(A \cap \Omega_n) \to 0$ for each $A \in \Sigma$ with $\mu(A) < \infty$ we have $\|f\chi_{\Omega_n}\|_\phi \to 0$, where χ_{Ω_n} denotes the characteristic function of the set Ω_n). The F-space $(L^\phi(\Omega), \| \cdot \|_\phi)$ is a symmetric function space (see [12]) and it is called an Orlicz space.

For every $p \in (0, \infty)$ the function $\phi_p : [0, \infty) \to [0, \infty)$, $u \to u^p$ is an Orlicz function; the Orlicz space corresponding with ϕ_p we denote by $L^p(\Omega)$. For $0 < p < 1$ ($1 \leq p < \infty$) the functional

$$\|f\|_p = \int_\Omega |f(t)|^p d\mu \quad \left(\|f\|_p = \left(\int_\Omega |f(t)|^p d\mu\right)^{1/p}\right) \quad \text{for} \ f \in L^p(\Omega),$$

is an F-norm on $L^p(\Omega)$ which is equivalent to $\|\cdot\|_{\phi_p}$. Clearly, $\phi_p \in \Phi_0$ for $0 < p < 1$, and $\phi_p \in \Phi_1$ for $1 \le p < \infty$.

Let $A \in \Sigma$. Put $\Sigma_A = \{B \in \Sigma : B \subset A\}$, $\Sigma_A^* = \{B \in \Sigma_A : \mu(B) < \infty\}$, $\Sigma_A^+ = \{B \in \Sigma_A : 0 < \mu(B) < \infty\}$, $\mu_A = \mu|_{\Sigma_A}$ and $\Sigma^+ = \Sigma_\Omega^+$. Consider the following equivalence relation $\sim^A$ on Σ_A^*: $B \sim^A C$ if and only if $\mu(B \triangle C) = 0$.

The function

$$m_A : (\Sigma_A^*/\sim^A) \times (\Sigma_A^*/\sim^A) \to [0, \infty), \quad ([B], [C]) \to \mu(B \triangle C)$$

is a metric in $(\Sigma_A^*/\sim^A)$. Put $d(A) = \mathrm{dens}((\Sigma_A^*/\sim^A), m_A)$. Let

$$w(A) = \inf\{d(B) : B \in \Sigma_A^+ \quad \text{and} \quad d(B) \ge \aleph_0\} \quad \text{if} \quad d(A) \ge \aleph_0.$$

It is easy to see that for every measure space (Ω, Σ, μ) with $\{0, 1\} \subsetneq \mu(\Sigma) \subset [0, 1]$ and every cardinal number $\alpha > \aleph_0$ we have $w(\Omega^\alpha) \ge \alpha$, where $(\Omega^\alpha, \Sigma^\alpha, \mu^\alpha)$ is the product measure space $(\Omega, \Sigma, \mu)^\alpha$.

Let $L_\Omega^\phi(A) = \{f \in L^\phi(\Omega) : f = f \cdot \chi_A\}$ for $A \in \Sigma$. The set $\{\chi_B : B \in \Sigma_A^*\}$ is linearly dense in $L_\Omega^\phi(A)$ for $A \in \Sigma$. If $d(A) \ge \aleph_0$, then $\mathrm{dens}L_\Omega^\phi = d(A)$.

In [12] (see Theorems 1.3, 1.6, and 1.10) we proved the following.

Theorem B. *Let (Ω, Σ, μ) be a measure space and let $\phi \in \Phi_0$. Then $L^\phi(\Omega)$ has an i.d. separable quotient if and only if there exists $A \in \Sigma$ such that L_Ω^ϕ is i.d. and separable.*

Theorem C. *Let (Ω, Σ, μ) be a measure space with a σ-finite measure μ. Then there exists $A \in \Sigma$ such that for every $\phi \in \Phi_0$ the subspace $L_\Omega^\phi(A)$ is separable and the subspace $L_\Omega^\phi(\Omega \setminus A)$ has no i.d. separable quotient.*

Theorem D. *Let (Ω, Σ, μ) be a measure space with a σ-finite measure μ and let $\phi \in \Phi_0$. Then for every closed subspace X of $L^\phi(\Omega)$ there exists $A \in \Sigma$ with $\dim(L^\phi(\Omega)/X) = \dim L_\Omega^\phi(A)$.*

1. Results

First we prove that there exist metrizable $(LF)_{tv}$-spaces, which have no i.d. separable quotient. We shall need the following non locally convex variant of Theorem 4 of [11].

Lemma 1. *Assume that an F-space (E, γ) possesses an orthogonal sequence of continuous projections (P_n) such that for every $n \in \mathbb{N}$ the space $(P_n(E), \gamma|_{P_n(E)})$ contains a proper dense subspace $(G_n, \gamma|_{G_n})$ dominated by an F-space (G_n, τ_n). Then $E_0 = \bigcup_{k=1}^\infty \bigcap_{n=k}^\infty P_n^{-1}(G_n)$ is a dense subspace of (E, γ) and $(E_0, \gamma|_{E_0})$ is an $(LF)_{tv}$-space. If the series $\Sigma_{n=1}^\infty P_n x$ is convergent in (E, γ) for every $x \in E_0$ and the space (G_n, τ_n) has the total dual for every $n \in \mathbb{N}$, then the $(LF)_{tv}$-space $(E_0, \gamma|_{E_0})$ has a defining sequence each of whose members has an i.d. separable quotient with the total dual.*

Proof. Put $E_k = \bigcap_{n=k}^{\infty} P_n^{-1}(G_n)$, $k \in \mathbb{N}$. Then $\bigcup_{k=1}^{\infty} E_k = E_0$ and $E_k \subsetneq E_{k+1}$, $k \in \mathbb{N}$. We show that E_1 is a dense subspace of (E, γ). Let (U_k) be a base of neighbourhoods of zero in (E, γ) such that $U_{k+1} + U_{k+1} \subset U_k$, $k \in \mathbb{N}$. Let $x \in E$, $m \in \mathbb{N}$ and $x_k \in G_k \cap (P_k(x) + P_k(E) \cap U_{m+k+1})$ for every $k \in \mathbb{N}$. Then the series $\Sigma_{k=1}^{\infty}(x_k - P_k x)$ is convergent in (E, γ) and $(x + \Sigma_{k=1}^{\infty}(x_k - P_k x)) \in (x + U_m) \cap E_1$. Hence $(x + U_m) \cap E_1 \neq \emptyset$. Thus E_1 is dense in (E, γ).

Let $k \in \mathbb{N}$, $F_k = E \times \prod_{n=k}^{\infty} G_n$, $\alpha_k = \gamma \times \prod_{n=k}^{\infty} \tau_n$. Then (F_k, α_k) is an F-space and the linear map $Q_k : E_k \to F_k$, $x \to (x, P_k x, P_{k+1} x, \dots)$ is injective. Put $\gamma_k = \{Q_k^{-1}(U) : U \in \alpha_k\}$. (E_k, γ_k) is an F-space, since $Q_k(E_k)$ is a closed subspace of (F_k, α_k). The sets in the form $E_k \cap U \cap \bigcap_{n=k}^{m} P_n^{-1}(V_n)$, where $m \in \mathbb{N}$, $m \geq k$, $U \in \mathcal{F}(\gamma)$ and $V_n \in \mathcal{F}(\tau_n)$, compose a base of neighbourhoods of zero in (E_k, γ_k). It is obvious that $\gamma_{k+1}|_{E_k} \subset \gamma_k$ and $\gamma|_{E_k} \subset \gamma_k$. Let γ_0 be the finest vector topology on E_0 such that $\gamma_0|_{E_k} \subset \gamma_k$ for each $k \in \mathbb{N}$. Clearly, $\gamma|_{E_0} \subset \gamma_0$. We prove that $\gamma_0 \subset \gamma|_{E_0}$.

First we show that $\gamma_0|_{E_k} \subset \gamma|_{E_k}$ for $k \in \mathbb{N}$. Let $k \in \mathbb{N}$ and $V \in \mathcal{F}(\gamma_0)$. Let $V_0 \in \mathcal{F}(\gamma_0)$ with $V_0 + V_0 \subset V$. Then $E_k \cap V_0 \in \mathcal{F}(\gamma_k)$ and there exist an integer $m \geq k$ and sets $U_0 \in \mathcal{F}(\gamma)$, $V_n \in \mathcal{F}(\tau_n)$ for $k \leq n \leq m$ such that

$$E_k \cap U_0 \cap \bigcap_{n=k}^{m} P_n^{-1}(V_n) \subset E_k \cap V_0 .$$

Let $l = m + 1$. Then $E_l \cap V_0 \in \mathcal{F}(\gamma_l)$ and there exist an integer $t \geq l$ and sets $U_1 \in \mathcal{F}(\gamma)$, $V_n \in \mathcal{F}(\tau_n)$ for $l \leq n \leq t$ such that

$$E_k \cap U_1 \cap \bigcap_{n=l}^{t} P_n^{-1}(V_n) \subset E_l \cap V_0 .$$

The map $P = (\Sigma_{n=k}^{m} P_n)|_{E_k}$ is a continuous projection in $(E_k, \gamma|_{E_k})$, since $P_n(E_k) \subset E_k$ for $n \geq k$. Thus

$$(E_k, \gamma|_{E_k}) = (\ker P, \gamma|_{\ker P}) \oplus (\mathrm{Im} P, \gamma|_{\mathrm{Im} P}) .$$

Since

$$U_0 \cap \ker P \subset E_k \cap U_0 \cap \bigcap_{n=k}^{m} P_n^{-1}(V_n) \subset E_k \cap V_0$$

and

$$U_1 \cap \mathrm{Im} P \subset E_k \cap U_1 \cap \bigcap_{n=l}^{t} P_n^{-1}(V_n) \subset E_k \cap V_0$$

then

$$U_0 \cap \ker P + U_1 \cap \mathrm{Im} P \subset E_k \cap V .$$

This follows that $\gamma_0|_{E_k} \subset \gamma|_{E_k}$ for each $k \in \mathbb{N}$.

Now we prove that $\gamma_0 \subset \gamma|_{E_0}$. Let $W \in \mathcal{F}(\gamma_0)$ and $V \in \mathcal{F}(\gamma_0)$ with $V + V \subset W$. Then there exists an open set $U \in \mathcal{F}(\gamma)$ with $U \cap E_1 \subset V \cap E_1$. For $k \in \mathbb{N}$

we have

$$U \cap E_k \subset \mathrm{cl}_{(E_k, \gamma|_{E_k})} U \cap E_k \cap E_1 \subset \mathrm{cl}_{(E_k, \gamma_0|_{E_k})} V \cap E_k \cap E_1 \subset W .$$

Thus $U \cap E_k \subset W$ for $k \in \mathbb{N}$, so $U \cap E_0 \subset W$. This follows that $\gamma_0 \subset \gamma|_{E_0}$. We have shown that $\gamma_0 = \gamma|_{E_0}$. Thus $(E_0, \gamma|_{E_0}) = \varinjlim (E_k, \gamma_k)$.

Now assume that the series $\Sigma_{n=1}^{\infty} P_n x$ is convergent in (E, γ) for every $x \in E_0$ and the space (G_n, τ_n) has the total dual for every $n \in \mathbb{N}$. Let $k \in \mathbb{N}$. Put $E_k^m = E_k \cap \bigcap_{n=m}^{\infty} P_n^{-1}(0)$ for $m \in \mathbb{N}$, $m \geq k$. Let $x \in E_k$ and $x_m = x - \Sigma_{n=m}^{\infty} P_n x$ for $m \geq k$. Then $x_m \in E_k^m$ for $m \geq k$ and $Q_k(x_m) \to_{m \to \infty} Q_k x$ in (F_k, α_k). Hence $x_m \to x$ in (E_k, γ_k). Thus $\bigcup_{m=k}^{\infty} E_k^m$ is a dense subspace of (E_k, γ_k).

Let $m \in \mathbb{N}$, $m \geq k$. Let $x \in E_k \setminus P_m^{-1}(0)$. Then $x_{m+1} \in (E_k^{m+1} \setminus E_k^m)$, so $E_k^m \subsetneq E_k^{m+1}$.

Let $x \in (E_k \setminus E_k^m)$. Then $P_n x \in G_n$ for every $n \geq k$ and $P_t x \neq 0$ for some $t \geq m$, so there exists $f_t \in (G_t, \tau_t)^*$ with $f_t(P_t x) \neq 0$.

Put $S_t : F_k \to G_t$, $(x, x_k, x_{k+1}, \dots) \to x_t$ and $f = f_t \circ S_t \circ Q_k$.

Clearly, $f \in (E_k, \gamma_k)^*$, $f(E_k^m) = \{0\}$ and $f(x) \neq 0$. Thus E_k^m is a weakly closed subspace of (E_k, γ_k). By Theorem A, (E_k, γ_k) has an i.d. separable quotient with the total dual. $\square$

Remark. The ideas of the construction in Lemma 1 were obtained independently by M. Valdivia and P. Pérez Carreras in their paper [13], in which they proved that every non-normable Fréchet space contains a proper dense subspace which is an (LF)-space. More details, examples and comments can be seen in Sections 8.7, 8.8, and 8.9 of the book [6].

Now we can prove our main result.

Theorem 2. *Let (Ω, Σ, μ) be a measure space and let $\phi \in \Phi$. Assume that $\dim L^{\phi}(\Omega) = \infty$. Then $L^{\phi}(\Omega)$ contains a proper dense subspace X such that*

  (a) *X is an $(LF)_{tv}$-space with a defining sequence each of whose members has an i.d. separable quotient with the total dual;*
  (b) *X has an i.d. separable quotient if and only if $\phi \in \Phi_1$ or $\omega(\Omega) = \aleph_0$.*

Proof. We consider three cases.

Case 1: $\phi \in \Phi_1$. Since $d(\Omega) \geq \aleph_0$ then there exists a sequence $(\Omega_n) \subset \Sigma$ of pairwise disjoint sets such that $d(\Omega_n) \geq \aleph_0$ for $n \in \mathbb{N}$.

Let $P_n : L^{\phi}(\Omega) \to L^{\phi}(\Omega)$, $f \to f \cdot \chi_{\Omega_n}$ for $n \in \mathbb{N}$. Then (P_n) is an orthogonal sequence of continuous projections on $L^{\phi}(\Omega)$. Let $X_n = L_{\Omega}^{\phi}(\Omega_n) \cap L^{\infty}(\Omega)$ and $|||f|||_n = \|f\|_{\phi} + \|f\|_{\infty}$ for $f \in X_n$, $n \in \mathbb{N}$. Then X_n is a dense proper subspace of $P_n(L^{\phi}(\Omega)) = L_{\Omega}^{\phi}(\Omega_n)$, dominated by the F-space $(X_n, ||| \cdot |||_n)$ with the total dual. By Lemma 1 the space $X = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} P_n^{-1}(X_n)$ is a dense subspace of $L^{\phi}(\Omega)$ and it is an $(LF)_{tv}$-space with a defining sequence each of whose members has an i.d. separable quotient with the total dual, since the series $\Sigma_{n=1}^{\infty} P_n f$ is convergent in $L^{\phi}(\Omega)$ for every $f \in L^{\phi}(\Omega)$. Let $k \in \mathbb{N}$. Put $A_k = (\Omega \setminus \bigcup_{n=k+1}^{\infty} \Omega_n)$, $B_k = (\Omega \setminus A_k)$. Then $L_{\Omega}^{\phi}(A_k) \subset \bigcap_{n=k}^{\infty} P_n^{-1}(0) \subsetneq X$.

The quotient space $L^\phi(\Omega)/L_\Omega^\phi(A_k)$ is isomorphic to $L_\Omega^\phi(B_k)$, so it has the total dual, by Lemma 3.4, [12]. Therefore $L_\Omega^\phi(A_k)$ is a weakly closed subspace of X. Clearly, $\bigcup_{k=1}^\infty L_\Omega^\phi(A_k)$ is a dense subspace of X. By Theorem A the space X has an i.d. separable quotient with the total dual.

Case 2: $\phi \in \Phi_0$ and $\omega(\Omega) = \aleph_0$. Then there exists $A \in \Sigma$ with $d(A) = \aleph_0$. Put $B = (\Omega \backslash A)$. As in Case 1 we construct a dense subspace X of $L^\phi(\Omega)$, which is an $(LF)_{tv}$-space; since $d(A) = \aleph_0$ we can assume that $(\Omega_n) \subset \Sigma_A$. Then $L_\Omega^\phi(B) \subset X$ and $X/L_\Omega^\phi(B)$ is a dense subspace of an i.d. separable metrizable space $L^\phi(\Omega)/L_\Omega^\phi(B)$. Thus $X/L_\Omega^\phi(B)$ is an i.d. separable quotient of X.

Case 3: $\phi \in \Phi_0$ and $\omega(\Omega) > \aleph_0$. By Theorem B the space $L^\phi(\Omega)$ has no i.d. separable quotient. As in Case 1, we construct a dense subspace X of $L^\phi(\Omega)$, which is an $(LF)_{tv}$-space. X has no i.d. separable quotient. Indeed, suppose, on the contrary, that X/Z is an i.d. separable quotient of X. Then $L^\phi(\Omega)/\mathrm{cl}_{L^\phi(\Omega)}Z$ is an i.d. separable quotient of $L^\phi(\Omega)$; a contradiction. $\square$

Corollary 3. *Let (Ω, Σ, μ) be a measure space with $\omega(\Omega) > \aleph_0$ and $\phi \in \Phi_0$. Then $L^\phi(\Omega)$ contains a proper dense subspace X such that*

  (a) *X is a metrizable $(LF)_{tv}$-space with a defining sequence each of whose members has an i.d. separable quotient with the total dual;*
  (b) *X has no i.d. separable quotient.*

Corollary 4. *Let $0 < p < 1$. The Orlicz space $L_p([0,1]^\mathfrak{c})$ contains a proper dense subspace X such that*

  (a) *X is a metrizable $(LF)_{tv}$-space with a defining sequence each of whose members has an i.d. separable quotient with the total dual;*
  (b) *X has no i.d. separable quotient.*

By Theorems 2 and C we obtain

Corollary 5. *Let (Ω, Σ, μ) be a measure space with a σ-finite measure μ and let $\phi \in \Phi_0$. If $L^\phi(\Omega)$ is non-separable, then it contains a subspace X such that:*

  (a) *X is an $(LF)_{tv}$-space with a defining sequence each of whose members has an i.d. separable quotient with the total dual;*
  (b) *X has no i.d. separable quotient;*
  (c) *the quotient space $L^\phi(\Omega)/\mathrm{cl}_{L^\phi(\Omega)}X$ is separable.*

Now we show that there exist strict $(LF)_{tv}$-spaces which have no i.d. separable quotient.

Proposition 6. *Let (Ω, Σ, μ) be a measure space with $\omega(\Omega) > 2^\mathfrak{c}$ and $\phi \in \Phi_0$. Assume that $(\Omega_n) \subset \Sigma^+$ is an increasing sequence of atomless subsets of Ω such that $L_\Omega^\phi(\Omega_n) \subsetneq L_\Omega^\phi(\Omega_{n+1})$ for $n \in \mathbb{N}$. Then the strictly $(LF)_{tv}$-space $(E, \tau) = \varinjlim L_\Omega^\phi(\Omega_n)$ has no i.d. separable quotient.*

Proof. Put $E_n = L_\Omega^\phi((\Omega_n))$ for $n \in \mathbb{N}$. Let $(E/M, \tau/M)$ be an i.d. quotient of (E, τ). Then $\dim(E/M) \geq \dim(E_k/E_k \cap M) \geq 1$ for some $k \in \mathbb{N}$. By Theorem D, $\dim(E_k/E_k \cap M) = \dim L_\Omega^\phi(A)$ for some $A \in \Sigma_{\Omega_k}^+$. Then $\dim L_\Omega^\phi(A) \geq$

$d(A) \geq \omega(\Omega)$, so $\dim(E/M) > 2^{\mathfrak{c}}$. Hence the quotient $(E/M, \tau/M)$ is non-separable. Thus (E, τ) has no i.d. separable quotient. $\qquad\square$

Proposition 7. *Let (Ω, Σ, μ) be a measure space and let $\phi \in \Phi_0$. Assume that $(\Omega_n) \subset \Sigma$ is an increasing sequence of sets such that $L_\Omega^\phi(\Omega_n) \subsetneq L_\Omega^\phi(\Omega_{n+1})$ for every $n \in \mathbb{N}$ and the space $L_\Omega^\phi(\Omega_m)$ has an i.d. separable quotient for some $m \in \mathbb{N}$. Then the strict $(LF)_{tv}$-space $(E, \tau) = \varinjlim L_\Omega^\phi(\Omega_n)$ has an i.d. separable quotient, which is an F-space.*

Proof. By Theorem B there exists $A \in \Sigma_{\Omega_m}$ such that the space $L_\Omega^\phi(A)$ is i.d and separable. Let γ be the topology of $L^\phi(\Omega)$. Put $X = \bigcup_{k=1}^\infty L_\Omega^\phi(\Omega_k \setminus A)$ and $Y = L_\Omega^\phi(A)$. Clearly, $X \cap Y = \{0\}$, $X + Y = E$, $X = L_\Omega^\phi(\bigcup_{k=1}^\infty \Omega_k \setminus A) \cap E$, $\gamma|_Y = \tau|_Y$ and $\gamma|_E \subset \tau$. Hence X is closed in (E, τ). The linear map

$$T : (Y, \gamma|_Y) \to (E/X, \tau/X), \quad y \to y + X,$$

is injective, surjective and continuous. Since $(Y, \gamma|_Y)$ is an F-space and $(E/X, \tau/X)$ is ultrabarrelled, T is open by the open mapping theorem. Thus $(E/X, \tau/X)$ is isomorphic to $(Y, \gamma|_Y)$, so (E, τ) has an i.d. separable quotient which is an F-space. $\qquad\square$

Proposition 8. *Let (Ω, Σ, μ) be a measure space and $\phi \in \Phi_1$. Assume that $(\Omega_n) \subset \Sigma$ is an increasing sequence of sets such that $L_\Omega^\phi(\Omega_n) \subsetneq L_\Omega^\phi(\Omega_{n+1})$ for $n \in \mathbb{N}$. Then the strict $(LF)_{tv}$-space $(E, \tau) = \varinjlim L_\Omega^\phi(\Omega_n)$ has an i.d. separable quotient with the total dual.*

Proof. Let γ be the topology of $L^\phi(\Omega)$. Let $k \in \mathbb{N}$. $L_\Omega^\phi(\Omega_k)$ is a weakly closed subspace of $L^\phi(\Omega)$ (see the proof of Theorem 2). Since $\gamma|_E \subset \tau$, then $L_\Omega^\phi(\Omega_k)$ is weakly closed in (E, τ). Clearly, $\bigcup_{k=1}^\infty L_\Omega^\phi(\Omega_k) = E$. By Theorem A the space (E, τ) has an i.d. separable quotient with the total dual. $\qquad\square$

In 1989 S. A. Saxon and P. P. Narayanaswami proved that a quotient of an (LF)-space is an (LF)-space or a Fréchet space ([11], Theorem 2). Using similar arguments we obtain an analogical results for $(LF)_{tv}$-spaces.

Proposition 9. *Let $(E, \gamma) = \varinjlim(E_n, \gamma_n)$ be an $(LF)_{tv}$-space and let M be a closed subspace of (E, γ). Then the quotient $(E/M, \gamma/M)$ is*

(a) *an F-space if $E_n + M = E$ for some $n \in \mathbb{N}$;*
(b) *an $(LF)_{tv}$-space if $E_n + M \neq E$ for every $n \in \mathbb{N}$.*

Proof. The quotient $(H, \tau) = (E/M, \gamma/M)$ is ultrabarrelled. Let $Q : E \to H$ be the quotient map. Put $Q_n = Q|_{E_n}$ and $H_n = Q_n(E_n)$, $n \in \mathbb{N}$. Clearly, $Q_n^{-1}(0) = E_n \cap M$, $n \in \mathbb{N}$.

If $E_n + M = E$ for some $n \in \mathbb{N}$, then $H_n = H$. By the open mapping theorem the continuous map $Q_n : (E_n, \gamma_n) \to (H, \tau)$ is open. Thus (H, τ) is an F-space (isomorphic to $(E_n/E_n \cap M, \gamma_n/E_n \cap M)$).

Now suppose that $E_n + M \neq E$ for any $n \in \mathbb{N}$. Then $H = \bigcup_{n=1}^{\infty} H_n$ and $H \neq H_n \subset H_{n+1}$, $n \in \mathbb{N}$. Without loss of generality we can assume that $H_n \neq H_{n+1}$ for each $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. Then $\tau_n = \{U \subset H_n : Q_n^{-1}(U) \in \gamma_n\}$ is a vector topology on H_n such that the map $Q_n : (E_n, \gamma_n) \to (H_n, \tau_n)$ is continuous and open. Since the map $Q_n : (E_n, \gamma|_{E_n}) \to (H_n, \tau|_{H_n})$ is continuous and $\gamma|_{E_n} \subset \gamma_n$, then $\tau|_{H_n} \subset \tau_n$. Hence τ_n is Hausdorff and (H_n, τ_n) is an F-space isomorphic to $(E_n/E_n \cap M, \gamma_n/E_n \cap M)$ by the open mapping theorem.

Let $n \in \mathbb{N}$ and $U \in \tau_{n+1}$. Since $Q_n^{-1}(U \cap H_n) = Q_{n+1}^{-1}(U) \cap E_n \in \gamma_{n+1}|_{E_n}$ and $\gamma_{n+1}|_{E_n} \subset \gamma_n$, then $U \cap H_n \in \tau_n$. Thus $\tau_{n+1}|_{H_n} \subset \tau_n$, $n \in \mathbb{N}$.

Let τ_0 be the finest Hausdorff vector topology on H such that $\tau_0|_{H_n} \subset \tau_n$ for all $n \in \mathbb{N}$. Since $\tau|_{H_n} \subset \tau_n$ for $n \in \mathbb{N}$, then $\tau \subset \tau_0$.

Now we prove that $\tau_0 \subset \tau$. Let (W_n) be a topological string in (H, τ_0). Then $(Q^{-1}(W_n))$ is a string in (E, γ) and $W_n \cap H_n \in \mathcal{F}(\tau_n)$ for $n \in \mathbb{N}$. Hence $Q^{-1}(W_n) \cap E_n = Q_n^{-1}(W_n \cap H_n) \in \mathcal{F}(\gamma_n)$ for $n \in \mathbb{N}$. Thus $(Q^{-1}(W_n))$ is a topological string in (E, γ) and the map $Q : (E, \gamma) \to (H, \tau_0)$ is continuous. So $\tau_0 \subset \tau$, since $\tau = \{U \subset H : Q^{-1}(U) \in \gamma\}$. Hence $(H, \tau) = \varinjlim(H_n, \tau_n)$, thus (H, τ) is an $(LF)_{tv}$-space. $\qquad\square$

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