

No Arbitrage Condition for Multi-Factor HJM Model under the Fractional Brownian Motion

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Abstract

Fractional Brownian motion (fBm) has properties of behaving tails and exhibiting long memory while remaining Gaussian. In particular, it is well known that interest rates show some long memories and non-Markovian. We present no arbitrage condition for HJM model under the multi-factor fBm reflecting the long range dependence in the interest rate model.

Keywords: Fractional Brownian motion, HJM, Wick Integral, Malliavin calculus, long memory.

1. Introduction

It is well-documented that economic and financial time series such as output growth, asset prices and interest rates have long range dependence. Since Mandelbrot (1971), many authors have studied long memory issues in financial time series. The researches on the applications of fBm to price assets, however, are quite a few compared with the empirical studies. This is because of no preventing arbitrage opportunities in fBm economy. To overcome these difficulties, a new type of asset price modelling or no arbitrage condition under fBm is introduced by Oksendal (2007). This approach adopts Wick integral rather than Ito Integral in defining self-financing portfolio. This is called 'Wick No Arbitrage'. Although Björk and Hult (2005) pointed out some problems of no arbitrage in Wick sense, the modelling by fBm still has strong advantages of explaining the long memory properties of financial time series. This paper studies no arbitrage condition in Wick sense for the multi-factor HJM model reflecting the long range dependence in the interest rate model. It is well known that interest rates show some long memories and non-Markovian; see, for example, Cajueiro and Tabak (2007). Fractional Brownian motion (fBm) is a proper candidate for modelling these empirical phenomena. Following the framework of Oksendal (2007), we investigate multi-factor HJM interest rate theory and obtain the no arbitrage condition.

2. fBm, Wick Integral and Malliavin Calculus

2.1. Some definitions

Before we present the main results, we introduce some properties of fBm, Wick integral and Malliavin calculus for fBm, which are pivotal roles of deriving multi-factor fBm HJM model. The details are referred to Bender (2003), Oksendal (2007), Biagini and Oksendal (2003, 2004) and Duncan *et al.* (2000). The one-factor HJM model is explained in Rhee and Kim (2008).

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We denote by (B_t^H) the one parameter fBm with Hurst parameter $H \in (0, 1)$. fBm is the Gaussian process $B_t^H = B_H(t, \omega)$, $t \in \mathbb{R}$, $\omega \in \Omega$ satisfying

$$B_0^H = E[B_t^H] = 0,$$

for all t and

$$E[B_s^H B_t^H] = \frac{1}{2} [|s|^{2H} + |t|^{2H} - |s - t|^{2H}],$$

where the expectation is taken under the probability measure P and (Ω, \mathcal{F}) is a measurable space. Note that fBm is not a semimartingale except that $H = 1/2$. So we cannot use the theory of stochastic calculus for semimartingale on B_t^H . In this paper, we use Wick-Ito integral as studied by Oksendal (2007). For $F(\cdot, \cdot) : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ such that $\|F\|_{L^2}^2 < \infty$, where

$$\begin{aligned} \|F\|_{L^2}^2 &= E \left[\int_{\mathbb{R}} \int_{\mathbb{R}} F(s)F(t)\phi(s, t)dsdt + \left(\int_{\mathbb{R}} D_t^\phi F(t)dt \right) \right], \\ \phi(s, t) &= H(2H - 1)|s - t|^{2H-2}, \quad \frac{1}{2} < H < 1 \end{aligned}$$

and $D_t^\phi F(t)$ denotes the Malliavin ϕ -derivative of F , the Wick integral is denoted by

$$\int_0^T F(t, \omega)\delta B_t^H.$$

The above integral is so called Skorohod or Wick-Ito integral, and we may denote this by

$$\int_0^T F(t, \omega)dB_t^H \doteq \int_0^T F(t, \omega)\delta B_t^H = \lim_{\Delta t_k \rightarrow 0} \sum_{k=0}^{N-1} F(t_k) \diamond (B_{t_{k+1}} - B_{t_k}),$$

where \diamond denotes the Wick product. More details on Wick product are explained in Bender (2003), Oksendal (2007), Biagini and Oksendal (2003, 2004). Note that

$$E \left[\int_0^T F(t, \omega)dB_t^H \right] = E \left[\int_0^T F(t, \omega)\delta B_t^H \right] = 0,$$

if the integral belongs to $L^2(P)$.

2.2. Malliavin calculus for fBm

Let $B^H = (B_1^{H_1}(t), \dots, B_n^{H_n}(t))$ $t \in \mathbb{R}$, $\omega \in \Omega$ be n -dimensional fBm with Hurst vector $H = (H_1, \dots, H_n) \in (1/2, 1)^n$. Note that $B_k^{H_k}(\cdot)$, $k = 1, \dots, n$, are independent. This implies that we consider Ω as a product $\Omega = \Omega_1 \times \dots \times \Omega_n$ of the copies Ω_k of some $\tilde{\Omega}$. Then \mathcal{F} is the σ -algebra generated by $\{B_k^{H_k}(s); s \in \mathbb{R}^+, k = 1, 2, \dots, n\}$ and \mathcal{F}_t is the σ -algebra generated by $\{B_k^{H_k}(s); 0 \leq s \leq t, k = 1, 2, \dots, n\}$. If $f : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}$ is \mathcal{F} -measurable, $1 \leq k \leq n$, then for some $u > 0$, we set Malliavin calculus for fBm as

$$D_{k,t}^\phi \left(\int_0^u f_l(s)dB_l^{(H)}(s) \right) = \int_0^u D_{k,t}^\phi f_l(s)dB_l^{(H)}(s) + \delta_{kl} \int_0^u f_k(s)\phi_k(t, s)ds,$$

where

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j \end{cases}$$

and

$$\phi_k(t, s) = H_k(2H_k - 1)|t - s|^{2H_k - 2}, \quad k = 1, 2, \dots, n.$$

In the next section, we set up multi-factor fBm HJM model and give main results. For allowing the correlation structures as in Biagini and Oksendal (2003), this paper defines new stochastic processes as

$$dX_i(t) = \sum_{k=1}^n \rho_{ik} dB_k^{H_k}(t), \quad i = 1, 2, \dots, n.$$

Then the covariance structures are given by

$$\begin{aligned} E[X_i(t)X_j(t)] &= \sum_{k=1}^n E[\rho_{ik}\rho_{jk}B_k^{H_k}(t)^2] \\ &= \sum_{k=1}^n \rho_{ik}\rho_{jk}t^{2H_k}. \end{aligned}$$

3. HJM Representation by Multi-Factor fBm

Theorem 1. Define the multi-factor HJM model as

$$f(t, T) = f(0, T) + \int_0^t \alpha(s, T)ds + \sum_{i=1}^n \int_0^t \sigma_i(s, T)dX_i(s),$$

where

$$dX_i(t) = \sum_{k=1}^n \rho_{ik} dB_k^{H_k}(t), \quad i = 1, 2, \dots, n.$$

Here $(B_k^{H_k}(t))$ is n -factor independent fBm, and $\alpha, \sigma_i \in L^2, i = 1, \dots, n$, are all deterministic functions. Then for the multi-factor fBm type HJM no arbitrage condition is

$$\alpha(t, T) = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \frac{\partial \rho'_{ik}(t, T) D_k^\phi Y_j(t)}{\partial T} - \sum_{i=1}^n \sum_{k=1}^n \frac{\partial \rho'_{ik}(t, T)}{\partial T} \gamma_k,$$

and if the equivalent martingale measure with respect to P is Q , then the density process is

$$\frac{dQ}{dP} = \varepsilon' = \exp \left(- \int_0^t \sum_{i=1}^n \sum_{k=1}^n \rho'_{ik}(s, T) dB_k^{H_k}(s) - \int_0^t \int_0^s \sum_{i=1}^n \sum_{k=1}^n \rho'_{ik}(s, T) \rho'_{ik}(u, T) \phi_k(s, u) ds du \right),$$

where

$$\rho'_{ik}(t, T) = \rho_{ik} \sigma'_i(t, T) = \rho_{ik} \int_t^T \sigma_i(s, u) du.$$

Proof: From the following multi-factor forward rate

$$f(t, T) = f(0, T) + \int_0^t \alpha(s, T) ds + \sum_{i=1}^n \int_0^t \sigma_i(s, T) dX_i(s),$$

we can obtain the zero bond as follows :

$$P(t, T) = \exp \left(- \int_t^T f(0, s) ds - \int_0^t \int_t^T \alpha(s, u) dud s - \sum_{i=1}^n \int_0^t \int_t^T \sigma_i(s, u) dud X_i(s) \right).$$

Then the dynamic of the discounted zero bond price is given by

$$\begin{aligned} Z(t, T) &= \exp \left(- \int_0^t f(0, s) ds - \int_0^t \alpha'(s, T) ds - \sum_{i=1}^n \int_0^t \sigma_i'(s, T) dX_i(s) \right) \\ &= \exp \left(- \int_0^t f(0, s) ds - \int_0^t \alpha'(s, T) ds - \sum_{i=1}^n Y_i(t) \right), \end{aligned}$$

where

$$dY_i(t) = \sigma_i'(t, T) dX_i(t) = \sigma_i'(t, T) \sum_{k=1}^n \rho_{ik} dB_k^{H_k}(t).$$

Or we can re-express $Y_i(t)$ as following:

$$\begin{aligned} dY_i(t) &= \sum_{k=1}^n \rho'_{ik}(t, T) dB_k^{H_k}(t), \quad i = 1, 2, \dots, n, \\ \rho'_{ik}(t, T) &= \rho_{ik} \sigma_i'(t, T). \end{aligned}$$

Then by Ito lemma for fBm,

$$\frac{dZ(t, T)}{Z(t, T)} = - \left(\alpha'(t, T) - \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \rho'_{ik}(t, T) D_k^\phi Y_j(t) \right) dt - \sum_{i=1}^n \sum_{k=1}^n \rho'_{ik}(t, T) dB_k^{H_k}(t),$$

where

$$D_k^\phi Y_j(t) = \int_0^t \rho'_{kk}(s, T) \phi_k(s, t) ds, \quad \text{for all } k.$$

We define the risk neutral fBm as

$$dB_k^{H_k}(t) = \gamma_k dt + dB_k^{H_k}(Q)(t), \quad \text{for } k = 1, 2, \dots, n,$$

where Q is the equivalent martingale measure with respect to P . Then the following relationship holds

$$\alpha'(t, T) = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \rho'_{ik}(t, T) D_k^\phi Y_j(t) - \sum_{i=1}^n \sum_{k=1}^n \rho'_{ik}(t, T) \gamma_k,$$

where γ_k , $k = 1, \dots, n$, are the market prices of risk. For the market prices of risk γ_k independent of any maturity T , we obtain the following multi-factor fBm type HJM no arbitrage condition:

$$\alpha(t, T) = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \frac{\partial \rho'_{ik}(t, T) D_k^\phi Y_j(t)}{\partial T} - \sum_{i=1}^n \sum_{k=1}^n \frac{\partial \rho'_{ik}(t, T)}{\partial T} \gamma_k.$$

Then the dynamic of the zero bond price under Q is given by

$$\begin{aligned} \frac{dZ(t, T)}{Z(t, T)} &= - \left(\alpha'(t, T) - \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \rho'_{ik}(t, T) D_k^\phi Y_j(t) \right) dt - \sum_{i=1}^n \sum_{k=1}^n \rho'_{ik}(t, T) (\gamma_k dt + dB_k^{H_k}(Q)(t)) \\ &= - \sum_{i=1}^n \sum_{k=1}^n \rho'_{ik}(t, T) dB_k^{H_k}(Q)(t). \end{aligned}$$

Following the method of Eberlein and Raible (1999), we can express the zero bond as

$$P(t, T) = P(0, T) \exp \left(\int_0^t r_s ds \right) \frac{\exp \left(- \int_0^t \sum_{i=1}^n \sum_{k=1}^n \rho'_{ik}(s, T) dB_k^{H_k}(Q)(s) \right)}{\mathbb{E}^Q \left[\exp \left(- \int_0^t \sum_{i=1}^n \sum_{k=1}^n \rho'_{ik}(s, T) dB_k^{H_k}(Q)(s) \right) \right]},$$

where the expectation $\mathbb{E}^Q[\cdot]$ is taken under the measure Q . From this, the well-known density process under the multi-factor fBm can be obtained in the following form:

$$\begin{aligned} \varepsilon' &= \frac{\exp \left(- \int_0^t \sum_{i=1}^n \sum_{k=1}^n \rho'_{ik}(s, T) dB_k^{H_k}(s) \right)}{\mathbb{E} \left[\exp \left(- \int_0^t \sum_{i=1}^n \sum_{k=1}^n \rho'_{ik}(s, T) dB_k^{H_k}(s) \right) \right]} \\ &= \exp \left(- \int_0^t \sum_{i=1}^n \sum_{k=1}^n \rho'_{ik}(s, T) dB_k^{H_k}(s) - \int_0^t \int_0^t \sum_{i=1}^n \sum_{k=1}^n \rho'_{ik}(s, T) \rho'_{ik}(u, T) \phi_k(s, u) ds du \right). \end{aligned}$$

Note that the Malliavin terms disappear because ρ'_{ik} is the deterministic function. □

4. Two-Factor fBm HJM

This section gives a practical example, which can be applied for pricing bond option or swaption.

Example 1. Define the two-factor forward rate as

$$f(t, T) = f(0, T) + \int_0^t \alpha(s, T) ds + \int_0^t \sigma_1(s, T) dX_1(s) + \int_0^t \sigma_2(s, T) dX_2(s).$$

By the definition of the zero bond and Fübini,

$$P(t, T) = \exp \left(- \int_t^T f(0, s) ds - \int_0^t \int_t^T \alpha(s, u) dud s - \int_0^t \int_t^T \sigma_1(s, u) dud X_1(s) - \int_0^t \int_t^T \sigma_2(s, u) dud X_2(s) \right).$$

The dynamic of the discounted zero bond price is given by

$$\begin{aligned} Z(t, T) &= \exp \left(- \int_0^T f(0, s) ds - \int_0^t \alpha'(s, T) ds - \int_0^t \sigma'_1(s, T) dX_1(s) - \int_0^t \sigma'_2(s, T) dX_2(s) \right) \\ &= \exp \left(- \int_0^T f(0, s) ds - \int_0^t \alpha'(s, T) ds - Y_1(t) - Y_2(t) \right). \end{aligned}$$

Then

$$\begin{aligned} dY_1(t) &= \sigma'_1(t, T) dX_1(t) = \sigma'_1(t, T) (\rho_{11} dB_1^{H_1}(t) + \rho_{12} dB_2^{H_2}(t)) \\ &= \rho'_{11} dB_1^{H_1}(t) + \rho'_{12} dB_2^{H_2}(t) \end{aligned}$$

and

$$\begin{aligned} dY_2(t) &= \sigma'_2(t, T) dX_2(t) = \sigma'_2(t, T) (\rho_{21} dB_1^{H_1}(t) + \rho_{22} dB_2^{H_2}(t)) \\ &= \rho'_{21} dB_1^{H_1}(t) + \rho'_{22} dB_2^{H_2}(t). \end{aligned}$$

We set up

$$dY_i(t) = \sum_{k=1}^2 \rho'_{ik}(t, T) dB_k^{H_k}(t), \quad i = 1, 2,$$

where

$$\rho'_{ik}(t, T) = \rho_{ik} \sigma'_i(t, T).$$

Then by Ito lemma for fBm

$$\frac{dZ(t, T)}{Z(t, T)} = - \left(\alpha'(t, T) - \rho_{11} D_1^\phi Y_1(t) - \rho_{22} D_2^\phi Y_2(t) \right) dt - \sum_{i=1}^2 \sum_{k=1}^2 \rho'_{ik}(t, T) dB_k^{H_k}(t),$$

where

$$D_k^\phi Y_k(t) = \int_0^t \rho'_{kk}(s, T) \phi_k(s, t) ds, \quad k = 1, 2.$$

No arbitrage condition for the two-factor model is

$$\alpha(t, T) = \left(\frac{\partial \rho'_{11} D_1^\phi Y_1(t)}{\partial T} + \frac{\partial \rho'_{22} D_2^\phi Y_2(t)}{\partial T} \right) - \sum_{k=1}^2 \sum_{i=1}^2 \frac{\partial \rho'_{ik}(t, T)}{\partial T} \gamma_k.$$

As in the multi-factor case, we define the change of measure as

$$dB_k^{H_k}(t) = \gamma_k dt + dB_k^{H_k}(Q)(t), \quad \text{for } k = 1, 2.$$

Then the zero bond price is given by

$$P(t, T) = P(0, T) \exp \left(\int_0^t r_s ds \right) \frac{\exp \left(- \int_0^t (\rho'_{11}(s, T) + \rho'_{21}(s, T)) dB_1^{H_1}(Q)(s) - \int_0^t (\rho'_{12}(s, T) + \rho'_{22}(s, T)) dB_2^{H_2}(Q)(s) \right)}{\mathbb{E}^Q \left[\exp \left(\int_0^t \sum_{k=1}^2 \sum_{i=1}^2 \rho'_{ik}(t, T) dB_k^{H_k}(Q)(s) \right) \right]}$$

The density process is given by

$$\varepsilon' = \frac{\exp \left(- \int_0^t (\rho'_{11}(s, T) + \rho'_{21}(s, T)) dB_1^{H_1}(s) - \int_0^t (\rho'_{12}(s, T) + \rho'_{22}(s, T)) dB_2^{H_2}(s) \right)}{\mathbb{E}^Q \left[\exp \left(\int_0^t \sum_{k=1}^2 \sum_{i=1}^2 \rho'_{ik}(s, T) dB_k^{H_k}(s) \right) \right]}$$

5. Conclusion

This short paper demonstrates no arbitrage condition for the HJM under the multi-factor fBm. It is well-known that HJM framework has an advantage of pricing and calibrating zero bonds and related financial products. This paper adopts Wick Integral for keeping Oksendal type of no arbitrage which is called Wick no arbitrage. We also give a two-factor example. The two-factor model can be applied for pricing bond option or swaption, which is left as a further study.

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