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ON NAGATA-HIGMAN THEOREM

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ABSTRACT. Nagata[3] and Higman[1] showed that nil-algebra of the nil-index n is nilpotent of finite index. In this paper we show that the bounded degree of the nilpotency is less than or equal to $2^n - 1$. Our proof needs only some elementary fact about Vandermonde determinant, which is much simpler than Nagata's or Higman's proof.

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1. Introduction.

Let K be a field of characteristic 0 and A be an K-algebra. If there exists $n \in \mathbb{N}$ such that $a^n = 0$ for all $a \in A$, then A is called a *nil-algebra* and the natural number n is called the *nil-index* of A. And A is *nilpotent of index* N or A has *nilpotency* N if $A^N = 0$, but $A^{N-1} \neq 0$.

Theorem 1.1. [1, 3] (Nagata-Higman Theorem) *Any nil-algebra of finite nil-index is nilpotent of finite index.*

We denote by $N(n)$ or simply N the nilpotency of a nil-algebra of nil-index n. This theorem was proved by Nagata [3] in 1952 and Higman [1] showed $N(n) \leq 2^n - 1$ and $\frac{n^2}{e^2} < N(n)$ for sufficiently large n. In 1975, Kuzmin [2] improved the lower bound of $N(n)$ to $\frac{n(n+1)}{2}$ and conjectured $N(n) = \frac{n(n+1)}{2}$. Meanwhile, using the fact that a Young diagram with $n^2 + 1$ boxes has either $n + 1$ boxes or more in the first row, or $n + 1$ boxes or more in the first column, Razmyslov [4] proved that $N(n) \leq n^2$.

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Let $K\langle X\rangle = K\langle x_1, x_2, \ldots\rangle$ be a free associative algebra in countably many variables. If a_1, a_2, $\ldots$, $a_n \in K\langle X\rangle$, we denote by $S_n(a_1, a_2, \ldots, a_n)$ or simply S_n the sum of the $n!$ products of $a_1, a_2, \ldots, a_n$ in every possible order, so called the *symmetric polynomial* of $a_1, a_2, \ldots, a_n$, i.e.

$$S_n(a_1, a_2, \ldots, a_n) = \sum_{\sigma\in \mathrm{Sym}(n)} a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}\,,$$

where $\mathrm{Sym}(n)$ is the symmetric group on n letters.

An ideal I of $K\langle X\rangle$ is called a *T-ideal* if $\phi(I) \subseteq I$ for every algebra endomorphism ϕ of $K\langle X\rangle$. If we let I_n be the T-ideal generated by x^n for all $x \in X$. then the Nagata-Higman Theorem can be rephrased as following; there exists N such that $x_1x_2\cdots x_N \in I_n$ for all x_1, x_2, $\ldots$, $x_N \in X$ if $x_i^n \in I_n$, $1 \le i \le N$.

Lemma 1.2. *The ideal I_n is generated by S_n.*

2. Vandermonde determinant.

In this section, we introduce Vandermonde determinant and prove the basic properties of Vandermonde determinant.

Definition 2.1. Given elements $b_1, \ldots, b_t$ of a commutative ring K, let $|b_1, \ldots, b_t|$ denote the determinant of $\sum_{i,j=1}^{t} b_i^{j-1}e_{ij} \in M_t(K)$, where e_{ij} is the $t\times t$ matrix whose (i,j)-entry is 1, and all other entries are 0.

For example,

$$|b_1, b_2| = \begin{vmatrix} 1 & b_1 \\ 1 & b_2 \end{vmatrix} = b_2 - b_1.$$

Lemma 2.2. (Vandermonde determinant)

$$|b_1, \ldots, b_t| = \prod_{1\le i<j\le t} (b_j - b_i)$$

Proof. $|b_1, \ldots, b_t|$ is a polynomial of degree $0+1+2+\cdots+(t-1) = \dfrac{t(t-1)}{2}$ in b_1, $\ldots$, b_t. If we take $b_i = b_j$, then we have two identical rows and its determinant is 0. Therefore $b_j - b_i$ is a factor of $|b_1, \ldots, b_t|$ for all $j > i$, which implies $\prod_{1\le i<j\le t} (b_j - b_i)$ divides $|b_1, \ldots, b_t|$. But it is clear that

$$\deg\left(\prod_{1\le i<j\le t} (b_j - b_i)\right) = \binom{t}{2} = \frac{t(t-1)}{2}.$$

So $|b_1, \ldots, b_t| = m \prod_{1 \le i < j \le t} (b_j - b_i)$ for some $m \in \mathbb{Z}$. Comparing the coefficient of any one term, say $1 b_2 b_3^2 \cdots b_t^{t-1}$, we can conclude $m = 1$. □

We are now ready to prove an easy consequence of Vandermonde determinant.

Lemma 2.3. (Vandermonde argument) *Suppose that we have $b_1, \ldots, b_{l+1}$ in $Z(A)$ and $a_1, \ldots, a_{l+1}$ in A such that for each i, $1 \le i \le l+1$, we have $\sum_{j=1}^{l+1} b_i^{j-1} a_j = 0$. Then $|b_1, \ldots, b_{l+1}|\, a_j = 0$ for all j. Particularly, if $\mathrm{Ann}_A\, |b_1, \ldots, b_{l+1}| = 0$, then each $a_j = 0$ for all j.*

Proof. Let $V = (a_1, \ldots, a_{l+1})^T$ and

$$B = \begin{pmatrix} 1 & b_1 & \cdots & b_1^l \\ 1 & b_2 & \cdots & b_2^l \\ & & \vdots & \\ 1 & b_{l+1} & \cdots & b_{l+1}^l \end{pmatrix}.$$

Then $BV = 0$ in matrix form. But $B \in M_{l+1}(Z(A))$, so if we multiply the adjoint matrix of B to the left, we have $\det(B)V = |b_1, \ldots, b_{l+1}|\, V = 0$. □

3. Nagata-Higman Theorem.

We will begin with some basic facts about polynomial identity rings.

Definition 3.1. A ring R is a polynomial identity ring (or, R satisfies a polynomial identity) if there exists a nonzero polynomial $f(x_1, \ldots, x_m) \in K\langle X \rangle$ such that $f(r_1, \ldots, r_m) = 0$ for all $r_1, \ldots, r_m \in R$.

By this definition, a nil-algebra A is a ring with a polynomial identity $f(x) = x^n$ for all $x \in A$.

Theorem 3.2. *If A is a K-algebra without 1, satisfying the identity x^n, then $x_1 \cdots x_N$ is an identity of A, where $N = 2^n - 1$.*

Proof. Consider the expansion of $(x_1 + kx_2)^n$ for k various values, say α, β, $\ldots$, γ. Define $p_i^j(f) = \sum$ monomials of degree j for variable x_i in f, and

$g(x_1, x_2) = \sum_{i=0}^{n-1} x_1^i x_2 x_1^{n-i-1}$. Since $(x_1 + kx_2)^n = \sum_{j=0}^{n} k^j p_2^j((x_1 + x_2)^n)$, we have the following equation

$$\begin{pmatrix} 1 & \alpha & \cdots & \alpha^n \\ 1 & \beta & \cdots & \beta^n \\ & & \vdots & \\ 1 & \gamma & \cdots & \gamma^n \end{pmatrix} \begin{pmatrix} p_2^0((x_1 + x_2)^n) = x_1^n \\ p_2^1((x_1 + x_2)^n) = g(x_1, x_2) \\ \vdots \\ p_2^n((x_1 + x_2)^n) = x_2^n \end{pmatrix} = 0.$$

By the Lemma 2.3, $p_2^j((x_1 + x_2)^n)$ is an identity for $0 \le j \le n$ if $\alpha, \beta, \ldots, \gamma$ are all distinct to each others. Then we have the identity

$$\begin{aligned} \sum_{j=0}^{n-1} g(x_1, x_3 x_2^j) x_2^{n-j-1} &= \sum_{i,j=0}^{n-1} x_1^i x_3 x_2^j x_1^{n-i-1} x_2^{n-j-1} \\ &= n x_1^{n-1} x_3 x_2^{n-1} + \sum_{i=0}^{n-2} x_1^i x_3 g(x_2, x_1^{n-i-1}) \end{aligned}$$

Hence $x_1^{n-1} x_3 x_2^{n-1}$ is an identity. The conclusion follows by induction on n. □

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