J. Appl. Math. & Informatics Vol. 27(2009), No. 5 - 6, pp. 1473 - 1479
Website: http://www.kcam.biz

SOME APPLICATIONS OF RESISTANT LENGTH TO ANALYTIC FUNCTIONS

BO-HYUN CHUNG

ABSTRACT. We introduce the resistant length and examine its properties. We also consider the geometric applications of resistant length to the boundary behavior of analytic functions, conformal mappings and derive the theorem in connection with the fundamental sequences, purely geometric problems. The method of resistant length leads a simple proofs of theorems. So it shows us the usefulness of the method of resistant length.

AMS Mathematics Subject Classification: 30C35, 30C62
Key words and Phrases : Resistant length, fundamental sequence

1. Resistant length

Throughout this paper, $\mathbb{C}$ will denote the finite complex plane, D is a domain in $\mathbb{C}$, ∂D is a boundary of D, and $cl(D)$ is a closure of D. Resistant length was introduced as a conformally invariant measure of curve families.

Let $\{\gamma\}$ be a family whose elements γ are locally rectifiable curves (simply, curves or arcs) in D. We shall introduce a geometric quantity $\lambda(\gamma)$, called the resistant length of $\{\gamma\}$.

Let $\rho(z)$ be a non-negative real-valued function defined on D. We set

$$A_D(\rho) = A(\rho) = \iint_D \rho^2(z)\, dxdy.$$

If there is countable sequence $\{\gamma_i\}$ of disjoint rectifiable arcs, which are parameterized by their arc lengths (i.e., the arc γ_i is given by $z = z_i(s_i)$, where $0 \le s_i \le l_i$, and l_i is the length of γ_i), such that $(\gamma) = \bigcup_i^{\infty}(\gamma_i)$, we set

Received February 2, 2009. Accepted June 23, 2009.
This work was supported by 2007 Hongik University Research Fund

$$l_\rho(\gamma) = \sum_{i=1}^{\infty} \int_{\gamma_i} \rho(z_i(s_i))\, ds_i.$$

We introduce

$$L(\rho) = L_{\{\gamma\}}(\rho) = \inf_{\gamma \in \{\gamma\}} l_\rho(\gamma),$$

where $L_{\{\gamma\}}(\rho) = \infty$, the family $\{\gamma\}$ is empty.

Definition 1.1 ([1]). The function $\rho(z)$ shall be called *admissible* with respect to D and $\{\gamma\}$, if $A(\rho)$ and $L(\rho)$ are not both zero or both infinite.

Definition 1.2 ([1]). The quantity

$$\lambda_D(\gamma) = \lambda(\gamma) = \sup \frac{L^2(\rho)}{A(\rho)},$$

where the supremum is taken over all admissible functions $\rho(z)$, is called the *resistant length* of the family of curves $\{\gamma\}$ with respect to the domain D.

Remark (i) $\lambda_D(\gamma)$ depends only on $\{\gamma\}$ and not on D. Accordingly, we shall simplify the notation to $\lambda(\gamma)$.

(ii) Since almost every curve in $\mathbb{C}$ is rectifiable, the non-rectifiable curves of a family $\{\gamma\}$ have no influence on the resistant length of $\{\gamma\}$. Accordingly, we shall simplify the terminology to curve or arc.

We introduce the following propositions which are frequently used in our paper. The following proposition 1.3 is an immediate consequence of the definition.

Proposition 1.3 ([9]). (Conformal invariance of resistant length) *Let $z^* = f(z)$ be a 1-1 conformal mapping on D upon a domain D^* and $\{\gamma\}$ be a family of curves in D, then*

$$\lambda(\gamma) = \lambda(f(\gamma)).$$

Proposition 1.4 ([7]). (Comparison principle) *Let $\{\gamma\}$ and $\{\gamma'\}$ be two families of curves of a domain D such that each $\gamma \in \{\gamma\}$ contains a $\gamma' \in \{\gamma'\}$. Then*

$$\lambda(\gamma) \geq \lambda(\gamma').$$

Remark There is a physical interpretation of resistant length. Think of the curve family $\{\gamma\}$ as representing a system of homogenous electric wires. Then the resistant length $\lambda(\gamma)$ represents the resistance of $\{\gamma\}$. The above Proposition 1.4 reflect the fact that systems of fewer or longer wires have greater resistance (smaller conductance). Briefly, the set $\{\gamma\}$ of fewer or longer arcs has the larger resistant length.

Proposition 1.5. *In particular, if $\{\gamma\}$ is contained in $\{\gamma'\}$, then*

$$\lambda(\gamma) \geq \lambda(\gamma').$$

Proposition 1.6 ([11]). *Let B be the interior of the annulus formed by two concentric circles of radii r_1 and r_2 ($r_1 < r_2$). The family $\{\gamma\}$ of curves of B which connect the two circles has resistant length*

$$\lambda(\gamma) = \frac{1}{2\pi} \log \frac{r_2}{r_1},$$

while the family $\{\gamma^*\}$ of simple closed curves of B which separate the two boundary components has resistant length

$$\lambda(\gamma^*) = 2\pi \Big(\log \frac{r_2}{r_1} \Big)^{-1}.$$

2. Boundary behavior

The purpose of this paper is to apply the resistant length of a curve family to the boundary behavior of analytic functions.

The following theorem 2.1([5]) applies the resistant length to the analytic function defined on the domain with a number of holes. So it shows us the high usefulness of the method of resistant length in connection with the capacity.

Theorem 2.1 ([5]). *Let $f(z)$ be a bounded single-valued analytic function in the complement of E, where E is a totally disconnected compact set of positive capacity in $\mathbb{C}$. Then it is not the case that for each z in E, except for those z in a set of capacity zero, there exist two curves in the complement of E at z on which $f(z)$ has the limits ω_1 and ω_2, ($\omega_1 \neq \omega_2$).*

Corollary 2.2. *Let E be as in Theorem 2.1, and let D be a Jordan domain such that the Jordan curve bounding D passes every point of E. Let $N(z_0)$ denote a neighborhood of some point z_0 in E, and u a some harmonic function on D. If u is a bounded function in $N(z_0) \cap D$, then it is not the case that z_0 in E, there exist two curves in D at z_0 on which u has the cluster values ω_1 and ω_2, ($\omega_1 \neq \omega_2$).*

Proof. Since u is harmonic on Jordan domain D, there exists a v the harmonic conjugate of u on D. Hence we let $f(z)$ denote a function satisfying

$$f(z) = \exp(u + iv).$$

Then $f(z)$ is a single-valued analytic function on D and $f(z)$ is bounded on $N(z_0) \cap D$. Hence applying Theorem 2.1 to $f(z)$, we obtain the above consequence for $u(x, y) = Re\, f(z)$. □

An important application of resistant length is to the boundary correspondence between two simply connected domains which are mapped conformally on each other.

Definition 2.3 ([3]). *A* crosscut *of D is a Jordan curve γ in D which in both directions tends to a boundary point.*

It is well known that $D-\gamma$ consists of two simply connected components.

Definition 2.4 ([4]). *Choose a fixed $z_0 \in D$ and consider sequences $P=\{p_n\}$ of points in D. With the sequence P, we associate the family $\{\gamma\}_P$ of all clusters of crosscuts of D which separate z_0 from almost all p_n. The sequence P is said to be* fundamental *if $\lambda(\gamma_P)=0$. The definition is independent of the choice of z_0.*

Recall that $\lambda(\gamma_p)=0$ if and only if $L(\rho)=0$ for all ρ with $A(\rho)<\infty$.

Lemma 2.5 ([10]). *Let D and D^* be simply connected domains in the complex plane, and consider sequence $P=\{p_n\}$ of points in D. Let f be a conformal mapping from D to D^*. Then the sequences $\{p_n\}$ and $\{f(p_n)\}$ are simultaneously fundamental.*

Lemma 2.6. *Let $P=\{p_n\}$ be a fundamental sequence in D. Then all accumulation points of the $P=\{p_n\}$ lie on the boundary of D.*

Proof. Let $\alpha \in D$ be an accumulation point of P. Then $\alpha \notin E$, since otherwise $\{\gamma\}$ would be empty, and $\lambda(\gamma)=\infty$, where E is the set of z_0 in definition 2.4. Therefore, the open set $D-E$ contains a closed disk F with center α such that $p_n \in F$ for an infinite number of indices.

If ξ is a curve contained in D which connects α to a point β of E, then every crosscut γ of the family $\{\gamma\}$ intersects the continuum $T=\xi \cup F$. T has positive spherical distance δ from the boundary. Let $\rho^*(z)=\psi(z,\infty)$, where $\psi(z,w)$ is the spherical distance. Then, for each $\gamma \in \{\gamma\}$, $l_{\rho^*}(\gamma) \geq 2\delta$. Hence $L_{\{\gamma\}}(\rho^*) \geq 2\delta$. Since $0<A(\rho^*)<\pi$, ρ^* is admissible. We have

$$\lambda(\gamma) \geq \frac{L^2(\rho^*)}{A(\rho^*)} \geq \frac{4}{\pi} \cdot \delta^2 > 0.$$

This contradiction shows that α cannot be an interior point of D. This completes the proof. □

Lemma 2.7. *A sequence $P=\{p_n\}$ of points of $\Delta=\{z||z|<1\}$ is a fundamental if and only if it converges to a point $\zeta=e^{i\theta}$ of the boundary of Δ.*

Proof. Let $\lim_{n\to\infty} p_n=\zeta$, and set $r_j=|p_j-\zeta|$, $C_j=\{z||z-\zeta|=r_j\}$. Then the sequence $\{r_j\}$ converges to 0. By restriction to a suitable subsequence of $\{p_n\}$, we assume that C_1 separates C_n, $(n>1)$ from z_0. Let $\{\gamma\}$ be the family of crosscuts of Δ which separate z_0 from almost all p_n. For j, every simple closed curve, which separates the C_0 and C_j, contains a crosscut which separates z_0

from almost all p_n and which belongs to the family $\{\gamma\}$. Let $\{\delta_k\}$ be the family of all such simple closed curves by Proposition 1.6 and Proposition 1.4, we have

$$\lambda(\gamma) \leq \lambda(\delta_k) = 2\pi \left(\log \frac{r_0}{r_k}\right)^{-1}.$$

The right side converses to zero as $k \to \infty$, and the result, $\lambda(\gamma) = 0$. Thus, $P = \{p_n\}$ is a fundamental sequence in D.

On the other hand, if $\zeta_1 = e^{i\theta_1}$ and $\zeta_2 = e^{i\theta_2}$ are distinct points of the boundary. Let $\{p_m\}$ and $\{q_n\}$ be sequences converging to ζ_1 and ζ_2, respectively. Let $z_0 = 0$. Each member of the family $\{\gamma\}$ of crosscuts of Δ which separate z_0 from almost all points of the join of our two sequences have Euclidean length at least $|\zeta_1 - \zeta_2|$. Denoting by ρ_0 the function $\rho_0(z) = 1$ for all z of Δ, we have $L_{\{\gamma\}}(\rho_0) \geq |\zeta_1 - \zeta_2|$, while $A(\rho_0) = \pi$; hence,

$$\lambda(\gamma) \geq \frac{1}{\pi}|\zeta_1 - \zeta_2|^2 > 0.$$

A sequence of points of Δ with more than one accumulation point on the boundary cannot be a fundamental sequence. This completes the proof. □

Theorem 2.8. *Let D be a Jordan domain in the complex plane, and let $w = f(z)$ be a conformal mapping on $\Delta = \{z||z| < 1\}$ upon D. Let w_0 be a point on the boundary of D, and consider a sequence $P = \{w_n\}$ of points of D converging to w_0. Let $f^{-1}(P) = \{z_n\}$ be a inverse image of P.*

Then the sequence of points $\{z_n\}$ has no point of accumulation in the interior of Δ, and $\{z_n\}$ has one and only one point of accumulation z_0 on the boundary of Δ.

Proof. Since w_0 is a point on the boundary of D, $P = \{w_n\}$ is a fundamental by Lemma 2.7. Thus, $\{z_n\}$ is a fundamental by Lemma 2.5. Therefore, point of accumulation z_0 of $\{z_n\}$ exist on the boundary of Δ, by Lemma 2.6.

On the other hand, z_0 is one and only one point of accumulation by Lemma 2.7. This completes the proof of the theorem. □

3. Geometric applications

The simplest example of geometric applications of resistant length concerns the ring domain. In our discussion we will need the following.

Lemma 3.1 ([8]). *Let R be a ring domain in $\mathbb{C}$ and let R_0 and R_1 denote the bounded component and unbounded component of R^c the complement of R, respectively. Let ∂R_0 and ∂R_1 denote the two components of the boundary of R, and let $\{\gamma\}_R$ be the family of all curves in R connecting ∂R_0 and ∂R_1. Then*

$$\lambda(\gamma_R) = \infty$$

if and only if R_0 consists of a single point.

Lemma 3.2 ([2]). *Let R, R_0, R_1 and $\{\gamma\}_R$ be as in Lemma 3.1. We say the closed curve γ in R separates R_0 and R_1 if γ has non-zero winding number about the points of R_0. Let $\{\gamma\}_S$ be the family of all closed curves in R which separates R_0 and R_1. Then*

$$\lambda(\gamma_R) \cdot \lambda(\gamma_S) = 1.$$

We say that $\lambda(\gamma_S)$ is the *conjugate resistant length* of $\lambda(\gamma_R)$.

A purely function-theoretic proof of the following theorem 3.3 is difficult. However the use of resistant length makes the proof trivial.

Theorem 3.3. *Let $R, \partial R_0$ and ∂R_1 be as in Lemma 3.1. Let a be the length of the shortest arc in D connecting ∂R_0 and ∂R_1. Let b be the length of the Jordan curve, ∂R_0. Then*

$$a \cdot b \leq X,$$

where X is the area of R.

Proof. Let $\{\gamma\}_R$ and $\{\gamma\}_S$ be as in Lemmas 3.1 and 3.2 respectively. Then by Lemma 3.2,

$$\lambda(\gamma_R) \cdot \lambda(\gamma_S) = 1.$$

On the other hand, we choose the non-negative Borel measurable function $\rho = 1$, then $\lambda(\gamma_R)$ and $\lambda(\gamma_S)$ have the following lower bounds respectively. That is,

$$\begin{aligned} \frac{a^2}{X} \cdot \frac{b^2}{X} &= \frac{[L_{(\gamma_R)}(1)]^2}{A_D(1)} \cdot \frac{[L_{(\gamma_S)}(1)]^2}{A_D(1)} \\ &\leq \lambda(\gamma_R) \cdot \lambda(\gamma_S) \\ &= 1. \end{aligned}$$

and the theorem follows at once. □

References

1. L. V. Ahlfors, *Conformal Invariants. Topics in Geometric Function Theory*, McGraw-Hill, New York (1973)
2. L. V. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand (1987)
3. Bo-Hyun Chung, *Some results for the resistant lengths of curve families* (II), J. Appl. Math. and Computing., 15(No. 1-2) (2004), 495–502
4. Bo-Hyun Chung, *A note on geometric applications of extremal lengths (I)*, J. Appl. Math. and Computing., 18(No. 1-2) (2005), 603–611.
5. Bo-Hyun Chung, *Some applications of extremal length to analytic functions*, Commun. Korean. Math. Soc., 21(No. 1) (2006), 135–143.
6. Bo-Hyun Chung, *Extremal length and geometric inequalities*, J. Chungcheong. Math. Soc., 20(No. 2) (2007), 147–156.
7. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, Springer-Verlag, New York (1973)

8. M. D. O'neill and R. E. Thurman, *Extremal problems for Robin capacity*, Complex Variables Theory and Applications, 41(2000).
9. B. Rodin, *The method of extremal length*, Bull. Amer. Math. Soc., 80(1974), 587–606.
10. Shen Yu-Liang, *Extremal problems for quasiconformal mappings*, Journal of Mathematical Analysis and Applications, 247(2000), 27–44.
11. J. Väisälä, *Lectures on n-Dimensional Quasiconformal Mappings*, Springer-Verlag, New York, (1971)

BO-Hyun Chung received his BS and MS from Korea University and Ph. D from Hongik University. His research interests center on geometric complex function theory.

Mathematics Section, College of Science and Technology, Hongik University, Jochiwon 339–701, Korea
e-mail : bohyun@hongik.ac.kr