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# EXISTENCE OF SOLUTIONS OF NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS FOR 2NTH-ORDER NONLINEAR DIFFERENTIAL EQUATION

YONGXIN GAO* AND RENFEI WANG

ABSTRACT. In This paper we shall study the existence of solutions of nonlinear two point boundary value problems for nonlinear 2nth-order differential equation

$$y^{(2n)} = f(t, y, y', \cdots, y^{(2n-1)})$$

with the boundary conditions

$$g_0(y(a), y'(a), \cdots, y^{(2n-3)}(a)) = 0, g_1(y^{(2n-2)}(a), y^{(2n-1)}(a)) = 0,$$
$$h_0(y(c), y'(c)) = 0, h_i(y^{(i)}(c), y^{(i+1)}(c)) = 0 (i = 2, 3, \cdots, 2n-2).$$

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## 1. Introduction

We assume through out this paper that

$(H_1)$ The function $f(t, y_0, y_1, \cdots, y_{2n-1})$ is continuous on $[a, c] \times R^{2n}$.

$(H_2)$ Every solution of initial value problems for nonlinear 2nth-order differential equation

$$y^{(2n)} = f(t, y, y', \cdots, y^{(2n-1)}) \tag{1}$$

extends to $[a, c]$ or becomes unbounded on its greatest existence interval.

Imitating Ref.[1], give the following definition:

**Definition.** If a function $\varphi(t) \in C^{2n}[a, c]$, satisfying

$$\varphi^{(2n)}(t) \leq f(t, \varphi(t), \varphi'(t), \cdots, \varphi^{(2n-1)}(t)), a \leq t \leq c,$$

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that $\varphi(t)$ is said to be an upper solution of Eq.(1)on $[a,c]$; If a function $\psi(t) \in C^{2n}[a,c]$, satisfying

$$\psi^{(2n)} \geq f(t,\psi(t),\psi'(t),\cdots,\psi^{(2n-1)}(t)), a \leq t \leq c,$$

then $\psi(t)$ is said to be a lower solution of Eq.(1) on $[a,c]$.

By use of $Kamke-theorem$[2] and imitating the proof of lemma 4 in Ref.[3],it is not difficult to prove the following lemma.

**Lemma 1.** *Suppose that* $(H_1)$ *and* $(H_2)$ *hold. If a sequence of functions* $f_m(t, y_0, y_1,\cdots,y_{2n-1})$ $(m=1,2,\cdots)$ *is continuous on* $[a,c]\times R^{2n}$, *and converges uniformly to* $f(t,y_0,y_1,\cdots,y_{2n-1})$ *on any compact subset of* $[a,c]\times R^{2n}$, *besides ,the sequences formed by* $y_m(t)$, *a solution of the equation* $y^{(2n)}=f_m(t,y,y',\cdots, y^{(2n-1)})$ *and its derivatives* $y'_m(t), y''_m(t),\cdots,y_m^{(2n-2)}(t)$ *exist and are uniformly bounded on* $[a,c]$ *then there is a solution* $y(t)$ *of Eq.(1) on* $[a,c]$ *and a subsequence* $\{y_{m_k}(t)\}$ *of* $\{y_m(t)\}$ *such that* $\{y_{m_k}^{(i)}(t)\}$ *converges uniformly to* $y^{(i)}(t)(i=0,1,\cdots,2n-1)$ *on* $[a,c]$.

## 2. Two point boundary value problems

For convenience, give the following conditions first:

$(A_1)$ Function $f(t,y_0,y_1,\cdots,y_{2n-1})$ is nonincreasing in $y_{2i}(i=1,2,\cdots,n-2)$ and nondecreasing in $y_0$ and $y_{2i+1}(i=0,1,\cdots,n-2)$ for fixed t, $y_{2n-2}$ and $y_{2n-1}$.

$(A_2)$ There are upper and lower solutions $\phi(t)$ and $\psi(t)$ of Eq.(1) on $[a,c]$ such that

$$\varphi(t)\leq\psi(t);\psi^{(2i)}(t)\leq\varphi^{(2i)}(t)(i=1,2,\cdots,n-1);$$
$$\varphi^{(2i+1)}(t)\leq\psi^{(2i+1)}(t)(i=0,1,\cdots,n-2).$$

$(A_3)$ Function $g_0(y_0,y_1,y_2,\cdots,y_{2n-3})$ is continuous on $R^{2n-2}$, nondecreasing in $y_{2i+1}(i=0,1,\cdots,n-2)$ and nonincreasing in $y_{2i}$ $(i=1,2,\cdots,n-2)$ for fixed $y_0$, and satisfies

$$g_0(\psi(a),\psi'(a),\cdots,\psi^{(2n-3)}(a))=0=g_0(\varphi(a),\varphi'(a),\cdots,\varphi^{(2n-3)}(a))$$

$(A_4)$ Function $g_1(x,y)$ is continuous on $R^2$, nonincreasing in $y$ for fixed $x$, and satisfies

$$g_1(\psi^{(2n-2)}(a),\psi^{(2n-1)}(a))=0=g_1(\varphi^{(2n-2)}(a),\varphi^{(2n-1)}(a)).$$

$(A_5)$ Function $h_0(x,y)$ is continuous on $R^2$, nondecreasing in $x$ for fixed $y$, and satisfies

$$h_0(\psi(c),\psi'(c))=0=h_0(\varphi(c),\varphi'(c)).$$

$(A_{6_i})$ Functions $h_i(x,y)(i=2,3,\cdots,2n-2)$ are continuous on $R^2$, and nonincreasing in $y$ for fixed $x$, and satisfies

$$h_i(\psi^{(i)}(c),\psi^{(i+1)}(c))=0=h_i(\varphi^{(i)}(c),\varphi^{(i+1)}(c)).$$

The following lemma follows in a routine way from the Shauder fixed-point theorem.

**Lemma 2.** *Suppose that* $(H_1)$ *holds , if* $f$ *is bounded on* $[a,c] \times R^{2n}$ *, then the boundary value problem*

$$y^{(2n)} = f(t, y, y', \cdots, y^{(2n-1)})$$

$$y(a) = a_0, y^{(2n-2)}(a) = a_{2n-2}, y^{(i)}(c) = c_i (i = 1, 2, \cdots, 2n-2) \tag{2}$$

*has a solution.*

**Lemma 3.** *Suppose that* $(H_1)$, $(H_2)$, $(A_1)$ *and* $(A_2)$ *hold. If*

$$\left.\begin{array}{ll} \varphi(a) \le a_0 \le \psi(a), \psi^{(2n-2)}(a) \le a_{2n-2} \le \varphi^{(2n-2)}(a); & \\ \psi^{(2i)}(c) \le c_{2i} \le \phi^{(2i)}(c) & (i = 1, 2, \cdots, n-1); \\ \phi^{(2i+1)}(c) \le c_{2i+1} \le \psi^{(2i+1)}(c) & (i = 0, 1, \cdots, n-2). \end{array}\right\} \tag{3}$$

*then the BVP Eq.(1), (2) has a solution* $y(t)$ *satisfying*

$$\left.\begin{array}{ll} \varphi(t) \le y(t) \le \psi(t); & \\ \psi^{(2i)}(t) \le y^{(2i)}(t) \le \phi^{(2i)}(t) & (i = 1, 2, \cdots, n-1); \\ \phi^{(2i+1)}(t) \le y^{(2i+1)}(t) \le \psi^{(2i+1)}(t) & (i = 0, 1, \cdots, n-2). \end{array}\right\} \tag{4}$$

*on* $[a, c]$.

*Proof.* For $m = 1, 2, \cdots$, and $(t, y_0, y_1, \cdots, y_{2n-1}) \in [a, c] \times R^{2n}$, define the following functions:

$$f_m(t, y_0, y_1, \cdots, y_{2n-1}) = \begin{cases} f(t, y_0, y_1, \cdots, y_{2n-2}, y_{2n-1}), & |y_{2n-1}| \le m \\ f(t, y_0, y_1, \cdots, y_{2n-2}, m \cdot sgn y_{2n-1}), & |y_{2n-1}| > m \end{cases}$$

$$f_{m_{2n-2}}(t, y_0, y_1, \cdots, y_{2n-1}) = \begin{cases} f_m(t, y_0, y_1, \cdots, y_{2n-3}, \varphi^{(2n-2)}(t), y_{2n-1}) \\ \quad + \frac{y_{2n-2} - \varphi^{(2n-2)}(t)}{1 + y_{2n-2} - \varphi^{(2n-2)}(t)}, \quad y_{2n-2} > \varphi^{(2n-2)}(t); \\ f_m(t, y_0, y_1, \cdots, y_{2n-2}, y_{2n-1}), \\ \quad \psi^{(2n-2)}(t) \le y_{2n-2} \le \varphi^{(2n-2)}(t); \\ f_m(t, y_0, y_1, \cdots, y_{2n-3}, \psi^{(2n-2)}(t), y_{2n-1}) \\ \quad - \frac{\psi^{(2n-2)}(t) - y_{2n-2}}{1 + \psi^{(2n-2)}(t) - y_{2n-2}}, \quad y_{2n-2} < \psi^{(2n-2)}(t). \end{cases}$$

$$f_{m_{2n-3}}(t, y_0, y_1, \cdots, y_{2n-1})$$
$$= \begin{cases} f_{m_{2n-2}}(t, y_0, y_1, \cdots, y_{2n-4}, \psi^{(2n-3)}(t), y_{2n-2}, y_{2n-1}), \\ \quad y_{2n-3} > \psi^{(2n-3)}(t); \\ f_{m_{2n-2}}(t, y_0, y_1, \cdots, y_{2n-2}, y_{2n-1}), \\ \quad \varphi^{(2n-3)}(t) \le y_{2n-3} \le \psi^{(2n-3)}(t); \\ f_{m_{2n-2}}(t, y_0, y_1, \cdots, y_{2n-4}, \varphi^{(2n-3)}(t), y_{2n-2}, y_{2n-1}), \\ \quad y_{2n-3} < \varphi^{(2n-3)}(t). \end{cases}$$

$$\cdots\cdots$$

$$f_{m_1}(t, y_0, y_1, \cdots, y_{2n-1}) = \begin{cases} f_{m_2}(t, y_0, \psi'(t), y_2, \cdots, y_{2n-1}), y_1 > \psi'(t) \\ f_{m_2}(t, y_0, y_1, \cdots, y_{2n-1}), \varphi'(t) \le y_1 \le \psi'(t) \\ f_{m_2}(t, y_0, \varphi'(t), y_2, \cdots, y_{2n-1}), y_1 < \varphi'(t) \end{cases}$$

$$F_m(t,y_0,y_1,\cdots,y_{2n-1})=\begin{cases} f_{m_1}(t,\psi(t),y_1,\cdots,y_{2n-1}), y_0>\psi(t)\\ f_{m_1}(t,y_0,y_1,\cdots,y_{2n-1}), \varphi(t)\le y_0\le\psi(t)\\ f_{m_1}(t,\varphi(t),y_1,\cdots,y_{2n-1}), y_0<\varphi(t)\end{cases}$$

Obviously, for any $m\in N$, function $F_m(t,y_0,y_1,\cdots,y_{2n-1})$ is continuous and bounded on $[a,c]\times R^{2n}$. By lemma 2, the BVP

$$y^{(2n)}=F_m(t,y,y',\cdots,y^{(2n-1)})$$
$$y(a)=a_0, y^{(2n-2)}(a)=a_{2n-2}, y^{(i)}(c)=c_i(i=1,2,\cdots,2n-2)$$

has a solution $y_m(t)(m=1,2,\cdots)$.

Let $N=\max\{\max_{[a,c]}|\varphi^{(2n-1)}(t)|,\max_{[a,c]}|\psi^{(2n-1)}(t)|\}$, it is not difficult to prove that if $m\ge N$, then

$$\psi^{(2n-2)}(t)\le y_m^{(2n-2)}(t)\le\varphi^{(2n-2)}(t),\quad t\in[a,c] \tag{5}$$

In fact, if (5) is invalid, there is no harm in setting the right inequality is not true (the case that the left inequality of (5) is not true can be proved in the same way). i.e. there is a $\bar{t}\in[a,c]$ such that $y_m^{(2n-2)}(\bar{t})>\varphi^{(2n-2)}(\bar{t})$, the opposite inequality holds for $t=a,c$, so $y_m^{(2n-2)}(t)-\varphi^{(2n-2)}(t)$ has a positive maximum in $(a,c)$, say at $t_0$, thus

$$y_m^{(2n-2)}(t_0)>\varphi^{(2n-2)}(t_0), y_m^{(2n-1)}(t_0)=\varphi^{(2n-1)}(t_0), y_m^{(2n)}(t_0)\le\varphi^{(2n)}(t_0) \tag{6}$$

On the other hand, according to the definition of $F_m$ and the monotonicity of $f$ and (6), we have

$$\begin{aligned} y_m^{(2n)}(t_0)-\varphi^{(2n)}(t_0) &\ge F_m(t_0,y_m,y_m',\cdots,y_m^{(2n-1)})-f(t_0,\varphi,\varphi',\cdots,\varphi^{(2n-1)})\\ &\ge f_m(t_0,\varphi,\varphi',\cdots,\varphi^{(2n-1)})+\tfrac{y_m^{(2n-2)}(t_0)-\varphi^{(2n-2)}(t_0)}{1+y_m^{(2n-2)}(t_0)-\varphi^{(2n-2)}(t_0)}\\ &-f(t_0,\varphi,\varphi',\cdots,\varphi^{(2n-1)})\\ &\ge\tfrac{y_m^{(2n-2)}(t_0)-\varphi^{(2n-2)}(t_0)}{1+y_m^{(2n-2)}(t_0)-\varphi^{(2n-2)}(t_0)}>0\end{aligned}$$

this is contradicts (6), hence (5) is true . But (3) holds, then it implies that

$$\begin{array}{lllll} \varphi(t) & \le & y_m(t) & \le & \psi(t)\\ \psi^{(2i)}(t) & \le & y_m^{(2i)}(t) & \le & \phi^{(2i)}(t) \quad (i=1,2,\cdots,n-1)\\ \phi^{(2i+1)}(t) & \le & y_m^{(2i+1)}(t) & \le & \psi^{(2i+1)}(t) \quad (i=0,1,\cdots,n-2)\end{array}$$

Consequently, $y=y_m(t)(m\ge N)$ is a solution of $y^{(2n)}=f_m(t,y,y',\cdots,y^{(2n-1)})$ satisfying (2). By lemma 1, the proof of lemma3 can be completed without much difficulty. □

**Theorem 1.** *Suppose that* $(H_1)$,$(H_2)$,$(A_1)$,$(A_2)$ *and* $(A_3)$ *hold. If*

$$\psi^{(2n-2)}(a)\le a_{2n-2}\le\varphi^{(2n-2)}(a);\psi^{(2i)}(c)\le c_{2i}\le\varphi^{(2i)}(c)(i=1,2,\cdots,n-1);$$
$$\varphi^{(2i+1)}(c)\le c_{2i+1}\le\psi^{(2i+1)}(c)(i=0,1,\cdots,n-2),$$

*then the BVP*

$$y^{(2n)}=f(t,y,y',\cdots,y^{(2n-1)}),$$

$$g_0(y(a), y'(a), \cdots, y^{(2n-3)}(a)) = 0, \ \ y^{(2n-2)}(a) = a_{2n-2},$$
$$y^{(i)}(c) = c_i (i = 1, 2, \cdots, 2n-2) \tag{7}$$

*has a solution $y(t)$ satisfying (4) on $[a, c]$.*

*Proof.* For any $s$ such that $\varphi(a) \le s \le \psi(a)$, by lemma 3, the BVP

$$y^{(2n)} = f(t, y, y', \cdots, y^{(2n-1)})$$
$$y(a) = s, y^{(2n-2)}(a) = a_{2n-2}, y^{(i)}(c) = c_i (i = 1, 2, \cdots, 2n-2)$$

has a solution $y_s(t)$ satisfying

$$\begin{array}{lllll} \varphi(t) & \le & y_s(t) & \le & \psi(t) \\ \psi^{(2i)}(t) & \le & y_s^{(2i)}(t) & \le & \phi^{(2i)}(t) \quad (i = 1, 2, \cdots, n-1) \\ \phi^{(2i+1)}(t) & \le & y_s^{(2i+1)}(t) & \le & \psi^{(2i+1)}(t) \ \ (i = 0, 1, \cdots, n-2) \end{array}$$

on $[a, c]$. Let $\pi(s) = \{y_s(t) : \varphi(a) \le s \le \psi(a)\}$. Obviously, $\pi(s)$ is non-empty. There are two cases to consider:

$(I)$ $\varphi(a) = \psi(a)$

As

$$y_s(a) = \varphi(a), \ y_s^{(2i)}(a) \le \varphi^{(2i)}(a) \ (i = 1, 2, \cdots, n-2),$$
$$y_s^{(2i+1)}(a) \ge \varphi^{(2i)}(a) \ (i = 0, 1, \cdots, n-2)$$

and $(A_3)$, it is known that

$$g_0(y_s(a), y_s'(a), \cdots, y_s^{(2n-3)}(a)) \ge g_0(\varphi(a), \varphi'(a), \cdots, \varphi^{(2n-3)}(a)) = 0$$

On the other hand , as

$$y_s(a) = \psi(a), y_s^{(2i)}(a) \ge \psi^{(2i)}(a) \ (i = 1, 2, \cdots, n-2),$$
$$y_s^{(2i+1)}(a) \le \psi^{(2i+1)}(a) \ \ (i = 0, 1, \cdots, n-2)$$

and $(A_3)$, we have

$$g_0(y_s(a), y_s'(a), \cdots, y_s^{(2n-3)}(a)) \le g_0(\psi(a), \psi'(a), \cdots, \psi^{(2n-3)}(a)) = 0$$

Hence

$$g_0(y_s(a), y_s'(a), \cdots, y_s^{(2n-3)}(a)) = 0$$

This implies that if $\varphi(a) = \psi(a)$, the BVP Eq. (1), (7) has a solution $y(t)$ satisfying (4) on $[a, c]$.

$(II)$ $\varphi(a) < \psi(a)$

If the theorem were not true, then for any $y_s(t) \in \pi(s)$,$y_s(t)$ would not be a solution of BVP Eq. (1), (7). Thus

$$g_0(y_s(a), y_s'(a), \cdots, y_s^{(2n-3)}(a)) \ne 0 \tag{8}$$

From what we have proved for case (I) and from (8), we know that

$(i)$ $g_0(y_s(a), y_s'(a), \cdots, y_s^{(2n-3)}(a)) > 0$ if $y_s(t) \in \pi(\varphi(a))$

$(ii)$ $g_0(y_s(a), y_s'(a), \cdots, y_s^{(2n-3)}(a)) < 0$ if $y_s(t) \in \pi(\psi(a))$

Let $E = \{y_s(t) : y_s(t) \in \pi(s)$ and $g_0(y_s(a), y_s'(a), \cdots, y_s^{(2n-3)}(a)) > 0\}$, then $E$ is nonempty, putting $s_0 = sup\{y_s(a) : y_s(t) \in E\}$.

By the definition of $s_0$, there exists $y_m(t) \in E(m = 1, 2, \cdots)$ satisfying $y_m(a) = s_m \to s_0(m \to \infty)$, by lemma 1, the BVP

$$y^{(2n)} = f(t, y, y', \cdots, y^{(2n-1)})$$

$$y(a) = s_0, y^{(2n-2)}(a) = a_{2n-2}, y^{(i)}(c) = c_i \ (i = 1, 2, \cdots, 2n-2)$$

has a solution $y_0(t)$ satisfying

$$\begin{array}{rcccl} \varphi(t) & \leq & y_0(t) & \leq & \psi(t); \\ \psi^{(2i)}(t) & \leq & y_0^{(2i)}(t) & \leq & \phi^{(2i)}(t) \quad (i = 1, 2, \cdots, n-1); \\ \phi^{(2i+1)}(t) & \leq & y_0^{(2i+1)}(t) & \leq & \psi^{(2i+1)}(t) \ \ (i = 0, 1, \cdots, n-2). \end{array}$$

As $g_0(y_m(a), y_m'(a), \cdots, y_m^{(2n-3)}(a)) > 0$ and (8) holds,

$g_0(y_0(a), y_0'(a), \cdots, y_0^{(2n-3)}(a)) > 0$, i.e. $y_0(t) \in E$, so $s_0 < \psi(a)$.

If we use $y_0(t)$ to replace the upper solution $\varphi(t)$ in lemma 3,and the lower solution still use $\psi(t)$. By lemma 3, if

$$y_0(a) \leq s \leq \psi(a); \psi^{(2n-2)}(a) \leq a_{2n-2} \leq y_0^{(2n-2)}(a);$$

$$\psi^{(2i)}(c) \leq c_{2i} \leq y_0^{(2i)}(c)(i = 1, 2, \cdots, n-1);$$

$$y_0^{(2i+1)}(c) \leq c_{2i+1} \leq \psi^{(2i+1)}(c)(i = 0, 1, \cdots, n-2).$$

then the BVP

$$y^{(2n)} = f(t, y, y', \cdots, y^{(2n-1)})$$

$$y(a) = s, y^{(2n-2)}(a) = a_{2n-2}, y^{(i)}(c) = c_i(i = 1, 2, \cdots, 2n-2)$$

has a solution $\bar{y}_s(t)$ satisfying

$$\begin{array}{rcccl} y_0(t) & \leq & \bar{y}_s(t) & \leq & \psi(t) \\ \psi^{(2i)}(t) & \leq & \bar{y}_s^{(2i)}(t) & \leq & y_0^{(2i)}(t)(i = 1, 2, \cdots, n-1) \\ y_0^{(2i+1)}(t) & \leq & \bar{y}_s^{(2i+1)}(t) & \leq & \psi^{(2i+1)}(t) \ (i = 0, 1, \cdots, n-2) \end{array}$$

on $[a, c]$. Let $\bar{\pi}(s) = \{\bar{y}_s(t) : y_0(a) \leq s \leq \psi(a)\}$, Clearly, $\bar{\pi}(s)$ is non-empty.

Owing to $s_0 < \psi(a)$, there exist a positive integer $N$ such that $s_0 + 1/N \leq \psi(a)$, therefore, for $m > N$ we have $s_0 + 1/m < \psi(a)$. By lemma 1, there is a subsequence $\{\bar{y}_{m_k}(t)\}$ of $\{\bar{y}_m(t)\} \subset \bar{\pi}(s_0 + 1/m)$ which converges uniformly to a solution $\bar{y}_0(t)$ of the BVP.

$$y^{(2n)} = f(t, y, y', \cdots, y^{(2n-1)}),$$

$$y(a) = s_0, y^{(2n-2)}(a) = a_{2n-2}, y^{(i)}(c) = c_i(i = 1, 2, \cdots, 2n-2)$$

on $[a, c]$ that satisfies

$$\begin{array}{rcccl} y_0(t) & \leq & \bar{y}_0(t) & \leq & \psi(t); \\ \psi^{(2i)}(t) & \leq & \bar{y}_0^{(2i)}(t) & \leq & y_0^{(2i)}(t) \quad (i = 1, 2, \cdots, n-1); \\ y_0^{(2i+1)}(t) & \leq & \bar{y}_0^{(2i+1)}(t) & \leq & \psi^{(2i+1)}(t) \ \ (i = 0, 1, \cdots, n-2). \end{array}$$

By the definition of $s_0$ and (8), we have

$$g_0(\bar{y}_0(a), \bar{y}_0'(a), \cdots, \bar{y}_0^{(2n-3)}(a)) < 0. \tag{9}$$

On the other hand, $y_0(a) = \bar{y}_0(a)$, $y_0^{(2i)}(a) \geq \bar{y}_0^{(2i)}(a)$ $(i = 1, 2, \cdots, n-2)$, $y_0^{(2i+1)}(a) \leq \bar{y}_0^{(2i+1)}(a)$ $(i = 0, 1, \cdots, n-2)$ and $(A_3)$ hold, then

$$g_0(\bar{y}_0(a), \bar{y}_0'(a), \cdots, \bar{y}_0^{(2n-3)}(a)) \geq g_0(y_0(a), y_0'(a), \cdots, y_0^{(2n-3)}(a)) > 0$$

which contradicts (9), hence, for $\varphi(a) < \psi(a)$, the BVP Eq.(1),(7) has a solution $y(t)$ satisfying (4) on $[a, c]$. □

Imitating the proof of theorem 1, it is not difficult to obtain the following theorems.

**Theorem 2.** *Suppose that* $(H_1)$, $(H_2)$, $(A_1)$, $(A_2)$, $(A_3)$ *and* $(A_4)$ *hold. If*

$$\psi^{(2i)}(c) \leq c_{2i} \leq \phi^{(2i)}(c) \quad (i = 1, 2, \cdots, n-1),$$
$$\phi^{(2i+1)}(c) \leq c_{2i+1} \leq \psi^{(2i+1)}(c) \ (i = 0, 1, \cdots, n-2).$$

*then the BVP*

$$y^{(2n)} = f(t, y, y', \cdots, y^{(2n-1)})$$
$$g_0(y(a), y'(a), \cdots, y^{(2n-3)}(a)) = 0, g_1(y^{(2n-2)}(a), y^{(2n-1)}(a)) = 0,$$
$$y^{(i)}(c) = c_i \quad (i = 1, 2, \cdots, 2n-2)$$

*has a solution* $y(t)$ *satisfying (4) on* $[a, c]$.

**Theorem 3.** *Suppose that* $(H_1)$, $(H_2)$, $(A_1)$- $(A_5)$ *and* $(A_{6_i})(i = 2, 3, \cdots, 2n-2)$ *hold, then the BVP*

$$y^{(2n)} = f(t, y, y', \cdots, y^{(2n-1)})$$
$$g_0(y(a), y'(a), \cdots, y^{(2n-3)}(a)) = 0, g_1(y^{(2n-2)}(a), y^{(2n-1)}(a)) = 0,$$
$$h_0(y(c), y'(c)) = 0, h_i(y^{(i)}(c), y^{(i+1)}(c)) = 0 \quad (i = 2, 3, \cdots, 2n-2).$$

*has a solution* $y(t)$ *satisfying (4) on* $[a, c]$.

**Corollary 1.** *Suppose that* $(H_1)$, $(H_2)$, $(A_1)$ *and* $(A_2)$ *hold. If*

$$\sum_{i=0}^{2n-3} a_i \psi^{(i)}(a) = b_0 = \sum_{i=0}^{2n-3} a_i \varphi^{(i)}(a),$$
$$\sum_{i=2n-2}^{2n-1} a_i \psi^{(i)}(a) = b_{2n-2} = \sum_{i=2n-2}^{2n-1} a_i \varphi^{(i)}(a),$$
$$c_0 \psi(c) + c_1 \psi'(c) = d_0 = c_0 \varphi(c) + c_1 \varphi'(c),$$
$$c_i \psi^{(i)}(c) + c_{i+1} \psi^{(i+1)}(c) = d_i = c_i \varphi^{(i)}(c) + c_{i+1} \varphi^{(i+1)}(c) (i = 2, 3, \cdots, 2n-2),$$

*where* $c_0, a_{2i+1} \geq 0 (i = 0, 1, \cdots, n-2)$; $a_{2n-1}, a_{2i} \leq 0 \quad (i = 1, 2, \cdots, n-2)$; $c_{i+1} \leq 0 \quad (i = 2, 3, \cdots, 2n-2)$; $\sum_{i=0}^{2n-3} |a_i| \neq 0$; $|a_{2n-2}| + |a_{2n-1}| \neq 0$, $|c_0| + |c_1| \neq 0$, $|c_i| + |c_{i+1}| \neq 0, (i = 2, 3, \cdots, 2n-2)$, $|c_{2i}| + |c_{2i+2}| \neq 0$

*and* $|c_{2i+1}| + |c_{2i+3}| \neq 0(i = 1, 2, \cdots, n-2)$, *then Eq.*(1) *with the boundary conditions*

$$a_0 y(a) + a_1 y'(a) + \cdots + a_{2n-3} y^{(2n-3)}(a) = b_0,$$
$$a_{2n-2} y^{(2n-2)}(a) + a_{2n-1} y^{(2n-1)}(a) = b_{2n-2},$$
$$c_0 y(c) + c_1 y'(c) = d_0, c_i y^{(i)}(c) + c_{i+1} y^{(i+1)}(c) = d_i \ (i = 2, 3, \cdots, 2n-2),$$

*has a solution* $y(t)$ *satisfying (4) on* $[a, c]$.

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**Gao Yongxin** received her BS and MS from Dongbei Normal University in 1989 and 1992, respectively. Since 2000 she has been at College of Science of Civil Aviation University of China, and became a professor in 2005.Her research interests focus on boundary value problems for ordinary differential equations.

**Wang Renfei** received his BS from Tianjin University in 2005. Since 2006 he has been at College of Science of Civil Aviation University of China for MS. His research interests focus on boundary value problems for ordinary differential equations.

College of science,Civil Aviation University of China, Tianjin 300300, P.R.China.

e-mail: gao_you@263.net