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MARK SEQUENCES IN TRIPARTITE MULTIDIGRAPHS

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ABSTRACT. A tripartite r-digraph is an orientation of a tripartite multigraph that is without loops and contains atmost r edges between any pair of vertices from distinct parts. In this paper, we obtain necessary and sufficient conditions for sequences of non-negative integers in non-decreasing order to be the sequences of numbers, called marks (or r-scores), attached to the vertices of a tripartite r-digraph.

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1. Introduction

An r-digraph is an orientation of a multigraph that is without loops and contains atmost r edges between any pair of distinct vertices. So, 1-digraph is an oriented graph, and a complete 1-digraph is a tournament. Let D be an r-digraph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$, and let $d^+_{v_i}$ and $d^-_{v_i}$ denote the outdegree and indegree,respectively, of a vertex v_i. Define p_{v_i} (or simply p_i) $= r(n-1) + d^+_{v_i} - d^-_{v_i}$ as the mark (or r-score) of v_i, so that $0 \leq p_{v_i} \leq 2r(n-1)$. Then the sequence $P = [p_i]^n_1$ in non-decreasing order is called the mark sequence of D.

The following criterion for marks in r-digraphs due to Pirzada and Samee[5] is analogous to a result on scores in tournaments given by Landau [2].

Theorem 1. *A sequence $P = [p_i]^n_1$ of non-negative integers in non-decreasing order is the mark sequence of an r-digraph if and only if for $1 \leq t \leq n$, $\sum_{i=1}^{t} p_i \geq rt(t-1)$ with equality when $t = n$.*

Various results of scores in oriented graphs can be found in[1] while as analogous results for marks in 2-digraphs can be seen in[5,6]. The marks in bipartite 2-digraphs are characterized by Samee et al.[8] and the analogous characterizations of marks in bipartite r-digraphs can be found in[7].

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A tripartite r-digraph is an orientation of a tripartite multigraph that is without loops and contains at most r edges between any pair of vertices, the pair of vertices chosen such that one belongs to one part and the other belongs to some different part. Clearly, a tripartite 1-digraph is an oriented tripartite graph, and a complete tripartite 1-digraph is a tripartite tournament. Let $D(X,Y,Z)$ be a tripartite r-digraph with $X = \{x_1, x_2, \cdots, x_\ell\}$, $Y = \{y_1, y_2, \cdots, y_m\}$ and $Z = \{z_1, z_2, \cdots, z_n\}$. For any vertex v_i in $D(X,Y,Z)$, let $d^+_{v_i}$ and $d^-_{v_i}$ be the outdegree and indegree, respectively, of v_i. Define $p_{x_i} = p_i = r(m+n) + d^+_{x_i} - d^-_{x_i}$, $q_{y_j} = q_j = r(\ell+n) + d^+_{y_j} - d^-_{y_j}$ and $r_{z_k} = r_k = r(\ell+m) + d^+_{z_k} - d^-_{z_k}$ respectively as the marks (or r-scores) of x_i in X, y_j in Y and z_k in Z. We see that $0 \le p_{x_i} \le 2r(m+n)$, $0 \le q_{y_j} \le 2r(\ell+n)$ and $0 \le r_{z_k} \le 2r(\ell+m)$. The sequences $P = [p_i]_1^\ell$, $Q = [q_j]_1^m$ and $R = [r_k]_1^n$ in non-decreasing order are called the mark sequences of tripartite r-digraph $D(X,Y,Z)$.

Let u and v be two vertices from distinct parts X, Y, Z of a tripartite r-digraph $D(X,Y,Z)$. If there are a_1 arcs directed from u to v and a_2 arcs directed from v to u, with $0 \le a_1 \le r$, $0 \le a_2 \le r$ and $0 \le a_1 + a_2 \le r$, we denote this as $u(a_1 - a_2)v$.

The following characterization of mark sequences in tripartite 2-digraphs[3] is analogous to a result on scores in oriented tripartite graphs[4].

Theorem 2. *Let $P = [p_i]_1^\ell$, $Q = [q_j]_1^m$ and $R = [r_k]_1^n$ be the sequences of non-negative integers in non-decreasing order. Then P, Q and R are the mark sequences of some tripartite 2-digraph if and only if $1 \le f \le \ell$, $1 \le g \le m$ and $1 \le h \le n$ $\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j + \sum_{k=1}^{h} r_k \ge 4(fg + gh + hf)$, with equality when $f = \ell$, $g = m$ and $h = n$.*

2. Mark sequences in tripartite r-digraphs

A triple in a tripartite r-digraph is an induced r-subdigraph having three verties with one vertex from each part. That is, of the form $x(a_1 - a_2)y(b_1 - b_2)z(c_1 - c_2)x$, where for $1 \le i \le 2$, $0 \le a_i, b_i, c_i \le r$ and $0 \le a_1 + a_1, b_1 + b_2, c_1 + c_2 \le r$.

In a tripartite r-digraph, an oriented triple is an induced 1-subdigraph having three vertices with one vertex from each part. An oriented triple is transitive if it is of the form $x(1-0)y(1-0)z(0-1)x$, or $x(1-0)y(0-1)z(0-0)x$, or $x(1-0)y(0-0)z(0-1)x$, or $x(1-0)y(0-0)z(0-0)x$, or $x(0-0)y(0-0)z(0-0)x$, otherwise it is intransitive.

We start with the following observation, the proof of which follows easily from Theorem 2.2[1].

Theorem 3. *Let D and D' be two tripartite r-digraphs with the same mark sequences. Then D can be transformed to D' by successively transforming*

(i) appropriate oriented triples in one of the following ways,
either (a) by changing an intransitive oriented triple $x(1-0)y(1-0)z(1-0)x$ to a transitive oriented triple $x(0-0)y(0-0)z(0-0)x$, which has the same mark sequences, or vice versa,

or (b) by changing an intransitive oriented triple $x(1-0)y(1-0)z(0-0)x$ *to a transitive oriented triple* $x(0-0)y(0-0)z(0-1)x$*, which has the same mark sequences, or vice versa, or*

(ii) by changing the symmetric arcs $x(a-a)y$*, where* $1 \leq a \leq \frac{r}{2}$*, to* $x(0-0)y$*, which has the same mark sequences, or vice versa.*

The next result is an immediate consequence of Theorem 3.

Corollary 4. *Among all the tripartite r-digraphs with given mark sequences, those with the fewest arcs are transitive.*

A transmitter is a vertex with zero indegree. We assume without loss of generality that a transitive tripartite r-digraphs has no symmetric arcs. In case there are symmetric arcs $x(a-a)y$, where $1 \leq a \leq \frac{r}{2}$, they can be transformed to $x(0-0)y$ with the same marks. Thus, in a transitive tripartite r-digraph with mark sequences $P = [p_i]_1^\ell$, $Q = [q_j]_1^m$ and $R = [r_k]_1^n$, any of the vertex with mark p_ℓ, or q_m, or r_n can act as a transmitter.

Let $P = [p_i]_1^\ell$, $Q = [q_j]_1^m$ and $R = [r_k]_1^n$ be the sequences of non-negative integers in non-decreasing order with $p_\ell \geq r(m+n)$, $q_m \leq 2r(\ell+n) - r$ and $r_n \leq 2r(\ell+m) - r$. We obtain new sequences P', Q' and R' from P, Q and R by deleting one entry p_ℓ and
either (A) (i) If $p_\ell \geq (2r-1)(m+n)$, then reducing $2r(m+n) - p_\ell$ largest entries of Q and R by one each. (ii) If $p_\ell < (2r-1)(m+n)$, then reducing $(2r-1)(m+n)-p_\ell$ largest entries of Q and R by two each and $p_\ell - 2(r-1)(m+n)$ remaining entries by one each
or (B) (i) If p_ℓ is even, then reducing $\frac{2r(m+n)-p_\ell}{2}$ largest entries of Q and R by two each (ii) If p_ℓ is odd, then reducing $\frac{2r(m+n)-p_\ell-1}{2}$ largest entries of Q and R by two each and reducing remaining largest entries $p_\ell - \{2r(m+n) - p_\ell - 1\}$ of Q and R by one.

The following result provides a useful recursive test whether the sequences of non-negative integers are realizable as marks.

Theorem 5. *The sequences P, Q and R are the mark sequences of some tripartite r-digraph if and only if* P'*,* Q' *and* R' *as given above in A (arranged in non-decreasing order) are the mark sequences of some tripartite r-digraph.*

Proof. Let P', Q' and R' be the mark sequences of some tripartite r-digraph $D'(X', Y', Z')$. First assume Q' and R' are obtained from Q and R as in A(i). Construct a tripartite r-digraph $D(X, Y, Z)$ as follows.

Let $X = X' \cup \{x\}$, $Y = Y'$, $Z = Z'$ with $X' \cap \{x\} = \phi$. Let $x((r-1)-0)v$ for those vertices v of Y' and of Z' whose marks were reduced by one and $x(r-0)v$ for those vertices v of Y' and Z' whose marks are not reduced in going from P, Q and R to P', Q' and R'. Then $D(X, Y, Z)$ is the tripartite r-digraph with mark sequences P, Q and R.

If Q' and R' are obtained from Q and R as in A(ii), construct a tripartite r-digraph $D(X, Y, Z)$ as follows. Let $X = X' \cup \{x\}$, $Y = Y'$, $Z = Z'$ with

$X' \cap \{x\} = \phi$. Let $x((r-1)-0)v$ for those vertices v of Y' and Z' whose marks were reduced by one and $x((r-1)-1)v$ for those vertices v of Y' and Z' whose marks were reduced by two in going from P, Q and R to P', Q' and R'. Then $D(X,Y,Z)$ is the tripartite r-digraph with mark sequences P, Q and R.

Conversely, suppose P, Q and R be the mark sequences of a tripartite r-digraph $D(X,Y,Z)$. By Corollary 4, any of the vertex $x \in X$ or $y \in Y$ or $z \in Z$ with mark p_ℓ or q_m or r_n can act as a transmitter. Let the vertex $x \in X$ with mark p_ℓ be a transmitter. Clearly, $p_\ell \geq r(m+n)$, $q_m \leq 2r(\ell+n)-r$ and $r_n \leq 2r(\ell+m)-r$ because (a) if $p_\ell < r(m+n)$, then by deleting p_ℓ we have to reduce more than $m+n$ entries from Q and R, which is not possile, (b) if $q_m > 2r(\ell+n)-r$ and $r_n > 2r(\ell+m)-r$, then on reduction $q'_m = q_m - 1 > 2r(\ell+n)-r-1 = 2r(\ell+n-1)+r-1$ and $r'_n = r_n - 1 > 2r(\ell+m)-r-1 = 2r(\ell+m-1)+r-1$, which is impossible in both the cases.

(i) If $p_\ell \geq (2r-1)(m+n)$, let V be the set of $2r(m+n)-p_\ell$ vertices of largest marks in Y and Z, and let $W = (Y \cup Z) - V$. Assume $D(X,Y,Z)$ is chosen such that $x((r-1)-0)v$ for all $v \in V$, and $x(r-0)w$ for all $w \in W$. Evidently, $D(X,Y,Z)-x$ realizes P', Q' and R' (arranged in non-decreasing order).
(ii) If $p_\ell < (2r-1)(m+n)$, let V be the set of $(2r-1)(m+n)-p_\ell$ vertices of largest marks in Y and Z, and let $W = (Y \cup Z) - V$. Choose $D(X,Y,Z)$ so that $x((r-1)-1)v$ for all $v \in V$, and $x((r-1)-0)w$ for all $w \in W$. Then, again $D(X,Y,Z)-\{x\}$ realizes P', Q' and R' (arranged in non-decreasing order). □

Theorem 5 provides an algorithm for determining whether or not the sequences P, Q and R of non-negative integers in non-decreasing order are the mark sequences, and for constructing a corresponding tripartite r-digraph. Let P, Q and R with $p_\ell \geq r(m+n)$, $q_m \leq 2r(\ell+n)-r$ and $r_n \leq 2r(\ell+m)-r$, be mark sequences of a tripartite r-digraph $D(X,Y,Z)$ with $X = \{x_1, x_2, \cdots, x_\ell\}$,$Y = \{y_1, y_2, \cdots, y_m\}$ and $Z = \{z_1, z_2, \cdots, z_n\}$. Deleting p_ℓ and performing A(i) or A(ii) according as $p_\ell \geq (2r-1)(m+n)$ or $p_\ell < (2r-1)(m+n)$, we get $Q' = [q'_1, q'_2, \cdots, q'_m]$ and $R' = [r'_1, r'_2, \cdots, r'_n]$. If the marks of the vertices y_j and z_k were decreased by one in this process, then the construction yielded $x_\ell((r-1)-0)y_j$ and $x_\ell((r-1)-0)z_k$, and if these were decreased by two, then the construction yielded $x_\ell((r-1)-1)y_j$ and $x_\ell((r-1)-1)z_k$. For vertices y_s and z_t whose marks remained unchanged, the construction yielded $x_\ell(r-0)y_s$ and $x_\ell(r-0)z_t$. Note that if at least one of the conditions $p_\ell \geq r(m+n)$, or $q_m \leq 2r(\ell+n)-r$, or $r_n \leq 2r(\ell+m)-r$ does not hold, then we delete q_m, or r_n for which these conditions are true and the same argument is used for defining arcs. If this procedure is applied recursively, then a tripartite r-digraph $\Delta(P,Q,R)$ with mark sequences P, Q and R is constructed.

The next recursive critterion follows by using the argument as in Theorem 5.

Theorem 6. *The sequences P, Q and R are mark sequences of some tripartite r-digraph if and only if P', Q' and R' as given in (B)(arranged in non-decreasing order) are the mark sequences of some tripartite r-digraph.*

Theorem 6 also provides an algorithm of checking whether or not the sequences P, Q and R of non-negative integers in non-decreasing order are the mark sequences, and for constructing a corresponding tripartite r-digraph.

The next result is a combinatorial criterion for mark sequences in tripartite r-digraphs.

Theorem 7. *Let $P = [p_i]_1^{\ell}$, $Q = [q_j]_1^m$ and $R = [r_k]_1^n$ be the sequences of non-negative integers in non-decreasing order. Then P, Q and R are the mark sequences of some tripartite r-digraph if and only if for $1 \leq f \leq \ell$, $1 \leq g \leq m$ and $1 \leq h \leq n$,*

$$\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j + \sum_{k=1}^{h} r_k \geq 2r(fg + gh + hf) \tag{1}$$

with equality when $f = \ell$, $g = m$ and $h = n$.

Proof. A sub-tripartite r-digraph induced by f vertices from the first part, g vertices from the second part and h vertices from the third part has a sum of marks $2r(fg + gh + hf)$. This proves the necessity.

For sufficiency, assume that P, Q and R are the sequences of non-negative integers in non-decreasing order satisfying the conditions (1) but are not mark sequences of any tripartite r-digraph. Let these sequences be chosen in such a way that ℓ, m and n are the smallest possible and p_1 is the least with that choice of ℓ, m and n. We have the following two cases.

Case (a). Suppose the equality in (1) holds for some $f < \ell$, $g \leq m$ and $h \leq n$, so that $\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j + \sum_{k=1}^{h} r_k = 2r(fg + gh + hf)$.

By the minimality of ℓ, m and n, $P_1 = [p_i]_1^f$, $Q_1 = [q_j]_1^g$ and $R_1 = [r_k]_1^h$ are the mark sequences of some tripartite r-digraph $D_1(X_1, Y_1, Z_1)$.

Let $P_2 = [p_{f+1} - 2r(g+h), p_{f+2} - 2r(g+h), \cdots, p_\ell - 2r(g+h)]$, $Q_2 = [q_{g+1} - 2r(f+h), q_{g+2} - 2r(f+h), \cdots, q_m - 2r(f+h)]$ and $R_2 = [r_{h+1} - 2r(f+g), r_{h+2} - 2r(f+g), \cdots, r_n - 2r(f+g)]$. Now, $\sum_{i=1}^{s}(p_{f+i} - 2r(g+h)) + \sum_{j=1}^{t}(q_{g+j} - 2r(f+h)) + \sum_{k=1}^{u}(r_{h+k} - 2r(f+g)) = \sum_{i=1}^{f+s} p_i + \sum_{j=1}^{g+t} q_j + \sum_{k=1}^{h+u} r_k - \left(\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j + \sum_{k=1}^{h} r_k\right) - 2rs(g+h) - 2rt(f+h) - 2ru(f+g) \geq 2r((f+s)(g+t) + (g+t)(h+u) + (h+u)(f+s)) - 2r(fg + gh + hf) - 2rs(g+h) - 2rt(f+h) - 2ru(f+g) = 2r(fg + ft + sg + st + gh + gu + th + tu + hf + hs + uf + us - fg - gh - hf - sg - sh - tf - th - uf - ug) = 2r(st + tu + us)$, for $1 \leq s \leq \ell - f$, $1 \leq t \leq m - g$ and $1 \leq u \leq n - h$, with equality when $s = \ell - f$, $t = m - g$ and $u = n - h$.

Therefore, by the minimality of ℓ, m and n, the sequences P_2, Q_2 and R_2 are the mark sequences of some tripartite r-digraph $D_2(X_2, Y_2, Z_2)$. Construct a new tripartite r-digraph $D(X, Y, Z)$ as follows.

Let $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$, $Z = Z_1 \cup Z_2$ with $X_1 \cap X_2 = \phi$, $Y_1 \cap Y_2 = \phi$, $Z_1 \cap Z_2 = \phi$. Let $x_2(r-0)y_1$, $x_2(r-0)z_1$, $y_2(r-0)x_1$, $y_2(r-0)z_1$, $z_2(r-0)x_1$ and $z_2(r-0)y_1$ for all $x_i \in X_i$, $y_i \in Y_i$, $z_i \in Z_i$, where $1 \leq i \leq 2$, so that we get the tripartite r-digraph $D(X,Y,Z)$ with mark sequences P, Q and R, which is a contradiction.

Case (b). Suppose that strict inequality holds in (1) for $f \neq \ell, g \neq m$ and $h \neq n$. Assume that $p_1 > 0$. Let $P_1 = [p_1 - 1, p_2, \cdots, p_{\ell-1}, p_\ell + 1]$, $Q_1 = [q_1, q_2, \cdots, q_m]$ and $R_1 = [r_1, r_2, \cdots, r_n]$,so that P_1, Q_1 and R_1 satisfy the conditions (1). Thus by the minimality of p_1 the sequences P_1, Q_1 and R_1 are the mark sequences of some tripartite r-digraph $D_1(X_1, Y_1, Z_1)$. Let $p_{x_1} = p_1 - 1$ and $p_{x_\ell} = p_\ell + 1$. Since $p_{x_\ell} > p_{x_1} + 1$, there exists a vertex u either in Y_1 or in Z_1 such that $x_\ell(1-0)u(1-0)x_1$, $x_\ell(0-0)u(1-0)x_1$, or $x_\ell(1-0)u(0-0)x_1$, or $x_\ell(0-0)u(0-0)x_1$ is an induced sub-tripartite 1-digraph in $D_1(X_1, Y_1, Z_1)$ and if these are changed to $x_\ell(0-0)u(0-0)x_1$, or $x_\ell(0-1)u(0-0)x_1$, or $x_\ell(0-0)u(0-1)x_1$, or $x_\ell(0-1)u(0-1)x_1$ respectively, the result is a tripartite r-digraph with mark sequences P, Q and R. This is again a contradiction. Hence the result follows. □

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