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MIXED TYPE SECOND-ORDER DUALITY WITH SUPPORT FUNCTION

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ABSTRACT. Mixed type second order dual to the non-differentiable problem containing support functions is formulated and duality theorems are proved under generalized second order convexity conditions. It is pointed out that the mixed type duality results already reported in the literature are the special cases of our results.

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1. Introduction

Many authors have studied duality for a class of nonlinear programming problems in which the objective function contains a differentiable convex function along with either a positive homogenous function or the sum of positive homogenous functions, e.g., Sinha [25], Zhang and Mond [27], Mond [9,12,13], Chandra and Gulati [5] and Mond and Schechter [19,20]. These authors have introduced the square root of positive semidefinite quadratic form $(x^T Bx)^{1/2}$ or a norm term of the type $\|Pxt\|$ as a positive homogenous function. The popularity of this kind of problem stems from the fact that, even though the objective function and/or constraint functions are nondifferentiable, the dual problem comes out to be a differentiable problem and hence is more amenable to handle from the computational point of view. Also as demonstrated by Sinha [25], these problems have applications in the modelling of certain stochastic programming problem. While most of these studies have considered only the Wolf type dual. Chandra, et al [4] studied duality for such problems in the spirit of Mond and Weir [21] in order to relax convexity conditions assumed in the fore cited references.

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Mangasarian [12] was the first to identify a second order dual formulation for non-linear programs under the assumptions that are complicated and somewhat difficult to verify. Mond [16] introduced the concept of second order convex functions (named as bonvex functions by Bector and Chandra [2] studied second order duality for nonlinear programs.

Mond and Schechter [20], studied symmetric duality for nondifferentiable problems containing support functions of certain compact convex sets instead of the usual term of the type $(x^T Bx)^{1/2}$ or $\|Px\|$. Further Husain, Abha and Jabeen [7] studied the duality for nondifferentiable nonlinear programming problem in which the objective as well as the constraint functions contains a term of a support function.Subsequently Husain and Jabeen [8] studied its fractional case.

Recently Husain et al [9] formulated Wolfe and Mond-Weir type second order dual for nonlinear programming problem, whose objective and constraint functions contain support functions. The purpose of this paper is to present a mixed type second order dual to the non differentiable program which combines Wolfe and Mond-Weir second order duals considered in Husain et al [9].It is also pointed out that first order mixed type duality results proved in [9] are special cases of our results.It is also indicated that the duality results studied by Zhang and Mond [28] becomes special cases of our results if the support function is the objective is replaced by square root of positive semi definite quadratic form and the support functions that appear in the constraints are suppressed.

2. Notations and preliminaries

In this section we mention some notations to be used in the analysis of our exposition and recourse some preliminaries for easy references.

Definitions. (*i*) *Support function:* Let C be compact convex set in R^n. The function $S(x/C)$ given by $S(x/C) = \text{Max}\{z^T x : z \in C\}$, is called a support function of C.

It may be noted that the support function $S(x/C)$ is a non differentiable convex function and has sub-differential given by $\partial S(x/C) = \{z \in C : z^T x = S(x/C)\}$.

(*ii*) *Normal cone*: For any set $X \subseteq R^n$, the normal cone to X at a point $x \in X$ is defined by $N_X(x) = \{y : y^T(z - x) \leq 0, \quad \text{for all} \ \ z \in X\}$

It can be easily seen that for a compact convex set C, $y \in N_C(x)$ iff $S(y/C) = x^T y$, or equivalently x is sub differential of $S(y/C)$.

(*iii*) *Second order invex* (*Binvex*): Let f be a real valued twice differentiable function defined on an open set $X \subseteq R^n$, then f is said to be second order invex, if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x, u \in X$

$$f(x) - f(u) \geq \eta^T(x, u)[\nabla f(u) + \nabla^2 f(u)p] - \frac{1}{2}p^T \nabla^2 f(u)p.$$

(*iv*) *Second order incave* (*Bincave*): Let f be a real valued twice differentiable function defined on an open set $X \subseteq R^n$, then f is said to be second order invex, if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x, u \in X$

$$f(x) - f(u) \leq \eta^T(x,u)[\nabla f(u) + \nabla^2 f(u)p] - \frac{1}{2}p^T\nabla^2 f(u)p.$$

(v) *Second order pseudoinvex* (*Pseudobinvex*): Let f be a real valued twice differentiable function defined on an open set $X \subseteq R^n$, then f is said to be second order pseudoinvex, if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x, u \in X$

$$\eta^T(x,u)[\nabla f(u) + \nabla^2 f(u)p] \geq 0 \Rightarrow f(x) \geq f(u) - \frac{1}{2}p^T\nabla^2 f(u)p.$$

(vi) *Second order pseudoincave* (*Pseudobincave*): Let f be a real valued twice differentiable function defined on an open set $X \subseteq R^n$, then f is said to be second order pseudoincave, if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x, u \in X$

$$\eta^T(x,u)[\nabla f(u) + \nabla^2 f(u)p] \leq 0 \Rightarrow f(x) \leq f(u) - \frac{1}{2}p^T\nabla^2 f(u)p.$$

(vii) *Second order quasi-invex* (*Quasibinvex*): Let f be a real valued twice differentiable function defined on an open set $X \subseteq R^n$, then f is said to be second order quasi-invex, if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x, u \in X$

$$f(x) - f(u) + \frac{1}{2}p^T\nabla^2 f(u)p \leq 0 \Rightarrow \eta^T(x,u)[\nabla f(u) + \nabla^2 f(u)p] \leq 0.$$

$(viii)$ *Second order quasi-incave* (*Quasibincave*): Let f be a real valued twice differentiable function defined on an open set $X \subseteq R^n$, then f is said to be second order quasi-incave, if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x, u \in X$

$$f(x) - f(u) + \frac{1}{2}p^T\nabla^2 f(u)p \geq 0 \Rightarrow \eta^T(x,u)[\nabla f(u) + \nabla^2 f(u)p] \geq 0.$$

Let $f : R^n \to R$ and $g_i : R^n \to R$ $(i = 1, 2, \ldots, m)$ be subdifferentiable Lipchitz functions. Let C be a compact convex set in R^n. Then consider the following nonlinear programming problem:

(P) Min $f(x)$

Subject to,

$$g_i(x) \leq 0 \quad (i = 1, 2, \ldots, m), \qquad x \in C.$$

The following lemmas relating to (P) results will be used here:

Lemma 1 ([24]). *If $\bar{x}$ is an optimal solution for (P), then there exists $\lambda \in R_+$ and $\mu \in R^m_+$, such that*

$$0 \in \lambda\partial f(\bar{x}) + \sum_{i=1}^{m} \mu_i \partial g_i(\bar{x}) + N_C(\bar{x})$$

$$\mu_i g_i(\bar{x}) = 0, \quad i = 1, 2, \ldots, m$$

$$(\lambda, \mu) \geq 0, \qquad (\lambda, \mu) \neq 0$$

Lemma 2 ([24]). *If $\bar{x}$ is an optimal solution for (P), and a Slater's suitable constraint qualification [11] holds for (P), then there exist non negative constants μ_j $(j = 1, 2, \ldots, m)$, such that*

$$0 \in \partial f(x) + \sum_{i=1}^{m} \mu_i \partial g_i(\bar{x}) + N_C(\bar{x}), \qquad \mu_i g_i(\bar{x}) = 0, \quad i = 1, 2, \ldots, m.$$

It is to be noted that under the above stated conditions of convexity on the functions f and g_i, $(i = 1, 2, \ldots, m)$, these necessary conditions are also sufficient for the optimality of $\bar{x}$ for (P).

3. Non-differentiable programming problem containing support functions and duality

Let $f : R^n \to R$ and $g_i : R^n \to R$ $(i = 1, 2, \ldots, m)$ be twice differentiable functions. Let C and D_i $(i = 1, 2, \ldots, m)$ be compact convex sets in R^n. We consider the following nondifferentiable nonlinear programming problem:

(NP) Min $f(x) + S(x/C)$

Subject to,

$$g_i(x) + S(x/D_i) \leq 0, \quad (i = 1, 2, \ldots, m) \tag{1}$$

In studying duality for (NP) certain optimality conditions in the non-smooth setting will be required. These conditions which can be derived from [24] along with the application of Lemma 1 and Lemma 2 are as follow:

Theorem 1. *If $\bar{x}$ is an optimal solution for (NP), then there exists $\bar{\alpha} \in R$, $\bar{z} \in C$, $\bar{y} \in R^m$ and $\bar{w}_i \in D_i$, $(i = 1, 2, \ldots, m)$ such that*

$$\bar{\alpha}(\nabla f(\bar{x}) + \bar{z}) + \sum_{i=1}^{m} \bar{y}_i(\nabla g_i(\bar{x}) + \bar{w}_i) = 0,$$

$$\sum_{i=1}^{m} \bar{y}_i(\nabla g_i(\bar{x}) + \bar{w}_i^T(\bar{x})) = 0,$$

$$\bar{z}^T(\bar{x}) = S(\bar{x}/C), \quad \textit{and} \quad \bar{w}_i^T(\bar{x}) = S(\bar{x}/D_i), \quad \textit{for all} \ \ i = 1, 2, \ldots, m$$

$$(\bar{\alpha}, \bar{y}) \geq 0, \ (\bar{\alpha}, \bar{y}) \neq 0.$$

When a suitable constraint qualification holds for (NP) the above Fritz John optimality conditions reduces to the Karush-Kuhn-Tucker optimality conditions, as this asserts positiveness of the multiplier $\bar{\alpha}$ associated with the objective function.

4. Mixed second order type duality

We propose the following mixed type second order dual type to the problem (NP) which combines both Wolfe and Mond -Weir type dual models, considered in [9].Second order has tighter bound and enjoy computational advantage over first order dual to any non-linear programming problem [16].

$$\text{(Mix SD)}: \text{Maximize} f(u) + u^T z + \sum_{i\in I_0} y_i(g_i(u) + u^T w_i)$$
$$-\frac{1}{2}\nabla^2 p^T\left[f(u) + \sum_{i\in I_0} y_i g_i(u)\right]p$$

Subject to

$$\nabla f(u) + z + \sum_{i=1}^{m} y_i(\nabla g_i(u) + w_i) + \nabla^2(f(u) + y^T g(u))p = 0 \quad (2)$$

$$\sum_{i\in I_\alpha} y_i(g_i(u) + u^T w_i) - \frac{1}{2}p^T\nabla^2\Big(\sum_{i\in I_\alpha} y_i g_i(u)\Big)p \geq 0, \alpha = 1, 2, \ldots, r. \quad (3)$$

$$y \geq 0 \quad (4)$$

$$z\in C,\ w_i\in D_i, \quad i = 1, 2, \ldots, m. \quad (5)$$

where

1. $I_\alpha\subseteq M = \{1, 2, \ldots, m\}$, $\alpha = 0, 1, 2, \ldots, r$ with $\bigcup_{i=0}^{r} I_\alpha = M$ and $I_\alpha \bigcap I_\beta = \phi$ if $\alpha\neq\beta$.
2. $u \in R^n$, $p \in R^n$ and $y \in R^m$.

Theorem 2 (Weak Duality). *Let x be feasible for (NP) and $(u, y, z, p, w_1, \ldots, w_m)$ feasible for (MixSD). If for all feasible $(x, u, y, z, w_i, \ldots, w_m)$, $f(\cdot) + (\cdot)^T z + \sum_{i\in I_0} y_i(g_i(\cdot) + (\cdot)^T w_i)$ is second order pseudoinvex and $\sum_{i\in I_\alpha} y_i(g_i(\cdot) + (\cdot)^T w_i)$, $\alpha = 1, 2, \ldots, r$ is second order quasi-invex with respect to the same η, then* $\inf(NP)\geq \sup(Mix\ SD)$.

Proof. Since x is feasible for (NP) and $(x, y, z, w, \ldots, w_m)$ feasible for (MixSD), we have, in view of $x^T w_i\leq S(x \mid D_i)$ where $w_i\in D_i$, $i = 1, 2, \ldots, m$ and for $\alpha = 1, 2, \ldots, r$.

$$\begin{aligned}\sum_{i\in I_\alpha} y_i(g_i(x) + S(x|Di)) &\leq \sum_{i\in I_\alpha} y_i(g_i(x) + x^T w_i)\\ &\leq 0 \leq \sum_{i\in I_\alpha} y_i(g_i(u) + u^T w_i) - \frac{1}{2}p^T\nabla^2\Big(\sum_{i\in I_\alpha} y_i g_i(u)\Big)p\end{aligned}$$

By second order quasi-invexity of $\sum_{i\in I_\alpha} y_i(g_i(\cdot) + (\cdot)^T w_i)$, $\alpha = 1, 2, \ldots, r$, it follows that

$$\eta^T(x, u)\Big(\nabla\Big(\sum_{i\in I_\alpha} y_i(g_i(u) + u^T w_i)\Big) + \nabla^2\Big(\sum_{i\in I_\alpha} y_i g_i(u)\Big)p\Big) \leq 0, \quad \alpha = 1, 2, \ldots, r$$

Hence $\eta^T(x,u)\Big(\nabla\Big(\sum_{i\in M-I_0} y_i(g_i(u)+u^T w_i)\Big)+\nabla^2\Big(\sum_{i\in M-I_0} y_i g_i(u)\Big)p\Big) \leq 0.$
Thus from (2), this yields

$$\eta^T(x,u)\Big(\nabla\Big(f(u)+u^T z)+\sum_{i\in I_0} y_i\nabla(g_i(u)+u^T w_i\Big)+\nabla^2\Big(f(u)+\sum_{i\in I_0} y_i g_i(u)\Big)p\Big)\geq 0$$

Since $f(\cdot)+(\cdot)^T z+\sum_{i\in I_0} y_i(g_i(\cdot)+(\cdot)^T w_i)$ is second order pseudoinvex, this implies

$$\begin{aligned} f(x)+x^T z+\sum_{i\in I_\alpha} y_i(g_i(x)+x^T w_i) \quad &\geq \quad f(u)+u^T z+\sum_{i\in I_\alpha} y_i(g_i(x)+u^T w_i)\\ &\quad -\frac{1}{2}p^T\nabla^2\Big(f(u)+\sum_{i\in I_0} y_i g_i(u)\Big)p \end{aligned}$$

Since $x^T z\leq s(xjC)$, $x^T w_i\leq S(xjD_i)$, $i\in I_0$ and $g_i(x)+S(xjD_i)\leq 0$, together with $y\leq 0$, for $i\in I_0$, the above inequality gives

$$f(x)+S(x/C)\geq f(u)+u^T z+\sum_{i=I_0} y_i(g_i(u)+u^T w_i)-\frac{1}{2}p^T\nabla^2\Big(f(u)+\sum_{i=I_0} y_i g_i(u)\Big)p$$

That is, infimum (NP)≥supremum (MixSD). □

Theorem 3 (Strong duality). *If $\bar{x}$ is an optimal solution (NP) and Slater's constraint qualification [11] is satisfied at $\bar{x}$, then there exists $\bar{y}\in R^m$ with $\bar{y}=(\bar{y}_1,\bar{y}_2,\ldots,\bar{y}_m)$, $\bar{z}\in C$ and $\bar{w}_i\in D_i$, $i=1,2,\ldots,m$ such that $(\bar{x},\bar{y},\bar{z},\bar{w}_1,\bar{w}_2,\ldots,\bar{w}_m,p=0)$ is feasible for (MixSD) and the corresponding values of (NP) and (MixSD) are equal.*

If also, $f(\cdot)+(\cdot)^T z+\sum_{i\in I_0} y_i(g_i(\cdot)+(\cdot)^T w_i)$ is second order pseudo-invex for $z\in C$ and $w_i\in D_i$, $i\in I_0$ and $\sum_{i\in I_\alpha} y_i(g_i(\cdot)+(\cdot)^T w_i)$ for $w_i\in D_i$, $i\in I_\alpha$, $\alpha=1,2,\ldots,r$ is second order quasi-invex with respect to the same η, then $(\bar{x},\bar{y},\bar{z},\bar{w}_1,\ldots,\bar{w}_m,p=0)$ is an optimal solution of (Mix SD).

Proof. Since $\bar{x}$ is an optimal solution to the problem (NP) and the Slater's constraint qualification is satisfied at $\bar{x}$, then from Theorem 1, there exist $\bar{y}\in R^m$, $\bar{z}\in C$ and $\bar{w}_i\in D_i$, $i=1,2,\ldots,m$ such that

$$\begin{aligned} &\nabla(f(\bar{x})+\bar{x}^T\bar{z})+\sum_{i\in 1} y_i\nabla(g_i(\bar{x})+\bar{x}_i^T\bar{w}_i)=0,\\ &\sum_{i\in 1} y_i(g_i(\bar{x})+\bar{x}_i^T w_i)=0, \quad \bar{x}^T\bar{z}=S(\bar{x}/C),\\ &\bar{x}_i^T\bar{w}_i=S(\bar{x}/D_i), \quad i=1,2,\ldots,m,\\ &\bar{z}\in C, \ =w_i\in D_i, \qquad i=1,2,\ldots,m, \ \ \bar{y}\geq 0 \end{aligned}$$

The relation $\sum_{i\in I} y_i(g_i(\bar{x}) + \bar{x}_i^T\bar{w}_i) = 0$ implies $\sum_{i\in I_0} \bar{y}_i(g_i(\bar{x}) + \bar{x}_i^T\bar{w}_i) = 0$ and $\sum_{i\in I_\alpha} \bar{y}_i(g_i(\bar{x})+\bar{x}_i^T\bar{w}_i) = 0$, $\alpha = 1,2,\ldots,r$. Consequently, it implies that $(\bar{x}, \bar{y}, \bar{z}, \bar{w}_1, \ldots, \bar{w}_m, \bar{p} = 0)$ is feasible for (Mix SD) and the corresponding values of (NP) and (MixSD) are equal. If $f(\cdot) + (\cdot)^T z + \sum_{i\in I_0} y_i(g_i(\cdot) + (\cdot)^T w_i)$ is pseudoinvex, for all $z\in C$ and $w_i \in D_i$, $i = 1,2,\ldots,m$ and $\sum_{i\in I_\alpha} y_i(g_i(\cdot) + (\cdot)^T w_i)$ is second order quasi-convex for $i\in I_\alpha$, $\alpha = 1,2,\ldots,r$, then from Theorem 1 $(\bar{x}, \bar{y}, \bar{z}, \bar{w}_1, \ldots, \bar{w}_m, \bar{p} = 0)$ must be an optimal solution of (MixSD).

We shall prove a Mangasarian type [13] strict converse duality theorem for (MixSD) to (NP). □

Theorem 4 (Strict Converse duality). *Let $\bar{x}$ be an optimal solution of (NP) at which Slater's constraint qualification is satisfied. If $(\hat{x}, \hat{y}, \hat{p}, \hat{z}, \hat{w})$ is an optimal solution of (MixSD), where $\hat{w} = (\hat{w}_1, \ldots, \hat{w}_m)$ and $f(\cdot) + (\cdot)^T\hat{z} + \sum_{i\in I_0} \hat{y}_i(g_i(\cdot) + (\cdot)^T\hat{w}_i)$ is second order pseudoinvex at $\hat{x}$ and $i\in I_0$, $\sum_{i\in I_\alpha} \hat{y}_i(g_i(\cdot) + (\cdot)^T\hat{w}_i)$, $\alpha = 1,2,\ldots,r$ is second order quasi invex at $\hat{x}$ with respect to the same η, then $\bar{x} = \hat{x}$, i.e. $\hat{x}$ is an optimal solution of (NP).*

Proof. We shall assume that $\hat{x} \neq \bar{x}$ and exhibit a contradiction. Since $\bar{x}$ is an optimal solution of (NP) at which Slater's qualification is satisfied, it follows from Theorem 2 that there exists $\bar{y} \in R^m$, $\hat{z} \in C$ and $\hat{w}_i \in D_i$, $i = 1,2,\ldots,m$ such that $(\bar{x}, \bar{y}, \hat{z}, \hat{w}_1, ..., \hat{w}_m, \hat{p} = 0)$ is optimal for (MixSD). Hence

$$\begin{aligned} f(\bar{x}) + S(\bar{x}/C) &= f(\bar{x}) + \bar{x}^T\hat{z} + \sum_{i\in I_0} \bar{y}_i(g_i(\bar{x}) + \bar{x}^T\hat{w}_i) \\ &\quad -\frac{1}{2}\hat{p}\nabla^2\Big(f(\hat{x}) + \sum_{i\in I_0} \bar{y}_i(g_i(\hat{x}))\Big)^p \\ &= f(\hat{x}) + \hat{x}^T\hat{z} + \sum_{i\in I_0} \hat{y}_i(g_i(\hat{x}) + \hat{x}^T\hat{w}_i) \\ &\quad -\frac{1}{2}\hat{p}\nabla^2\Big(f(\hat{x}) + \sum_{i\in I_0} \hat{y}_i g_i(\hat{x})\Big)^p \end{aligned}$$

Since $\bar{x}$ is feasible for (NP) and $(\hat{x}, \hat{y}, \hat{z}, \hat{w}_1, ..., \hat{w}_m, \hat{p})$, $i \in I_\alpha$ is feasible for (MixSD), we have

$$\sum_{i\in I_\alpha} \hat{y}_i(g_i(\bar{x}) + \bar{x}\hat{w}_i) \;\leq\; \sum_{i\in I_\alpha} \hat{y}_i(g_i(\hat{x}) + \hat{x}\hat{w}_i) - \frac{1}{2}\hat{p}\nabla^2\Big(\sum_{i\in I_\alpha} \hat{y}_i g_i(\hat{x})\Big)^p$$

By second order quasi-invexity of $\sum_{i\in I_\alpha} \hat{y}_i(g_i(\cdot) + (\cdot)^T\hat{w}_i)$, this yields,

$$\eta^T(\bar{x},\hat{x})\left[\sum_{i\in I_\alpha}\nabla\hat{y}_i(g_i(\hat{x})+\hat{x}\hat{w}_i)+\nabla^2\sum_{i\in I_\alpha}\hat{y}_i g_i(\hat{x})\hat{p}\right]\le 0$$

Because $(\hat{x},\hat{y},\hat{p},\hat{z},\hat{w})$ is feasible, we have

$$\nabla(f(\hat{x})+\hat{x}^T\hat{z})+\sum_{i=1}^{m}\hat{y}_i\nabla(g_i(\hat{x})+\hat{x}\hat{w}_i)+\nabla^2\Big(\sum_{i=1}^{m}\hat{y}_i g_i(\hat{x})\Big)p=0$$

From this equation, it implies

$$\sum_{i\in I_\alpha}\hat{y}_i\nabla(g_i(\hat{x})+\hat{x}\hat{w}_i)+\nabla^2\Big(\sum_{i\in I_\alpha}\hat{y}_i(g_i(\hat{x}))\Big)\hat{p}$$
$$-\left[\nabla(f(\hat{x})+\hat{x}^T\hat{z})+\sum_{i\in I_0}\hat{y}_i\nabla(g_i(\hat{x})+\hat{x}\hat{w}_i)+\nabla^2\Big(\sum_{i\in I_0}\hat{y}_i g_i(\hat{x})\Big)\hat{p}\right]=0$$

Using this in (7), we obtain

$$\eta^T(\bar{x},\hat{x})\left[\nabla(f(\hat{x})+\hat{x}^T\hat{z})+\sum_{i\in I_0}\hat{y}_i\nabla(g_i(\hat{x})+\hat{x}\hat{w}_i)+\nabla^2\Big(\sum_{i\in I_\alpha}\hat{y}_i g_i(\hat{x})\Big)\hat{p}\right]\ge 0$$

This, because of second order pseudo-invexity of $\sum_{i\in I_\alpha} y_i(g_i(\cdot)+(\cdot)\hat{w}_i)$ implies

$$\begin{aligned} f(\bar{x})+\bar{x}^T\hat{z}+\sum_{i\in I_0}\hat{y}_i(g_i(\bar{x})+\bar{x}^T\hat{w}_i) \quad &\ge \quad f(\hat{x})+\hat{x}^T\hat{z}+\sum_{i\in I_0}\hat{y}_i(g_i(\hat{x})+\hat{x}^T\hat{w}_i)\\ &\quad -\frac{1}{2}\hat{p}^T\nabla^2\Big(f(\hat{x})+\sum_{i\in I_0}\hat{y}_i g_i(\hat{x})\Big)\hat{p}\end{aligned}$$

Since $\bar{x}^T\hat{z}=S(\bar{x}/C)$ and $\bar{x}^T\hat{w}_i=S(\bar{x}/D_i)$, $i=1,2,\ldots,m$, this implies

$$\begin{aligned}&f(\bar{x})+S(\bar{x}/C)+\sum_{i\in I_0}\hat{y}_i(g_i(\bar{x})+S(\bar{x}/D_i))\\ &\quad\ge\quad f(\hat{x})+\hat{x}^T\hat{z}+\sum_{i\in I_0}\hat{y}_i(g_i(\hat{x})+\hat{x}^T\hat{w}_i)-\frac{1}{2}\hat{p}^T\nabla^2\Big(f(\hat{x})+\sum_{i\in I_0}\hat{y}_i g_i(\hat{x})\Big)\hat{p}\end{aligned}$$

Since $\hat{y}_i\ge 0$ and $g_i(\bar{x})+S(\bar{x}/D_i)\le 0$ for all $i\in\{1,2,\ldots,m\}$, hence $\hat{y}_i$ $(g_i(\bar{x})+S(\bar{x}/D_i))\le 0$, $\forall i\in I_0$. Thus from the inequality (8), we have

$$\begin{aligned} f(\bar{x})+S(\bar{x}/C) \quad &\ge \quad f(\hat{x})+\hat{x}^T\hat{z}+\sum_{i\in I_0}\hat{y}_i(g_i(\hat{x})+\hat{x}^T\hat{w}_i)\\ &\quad -\frac{1}{2}\hat{p}^T\nabla^2\Big(f(\hat{x})+\sum_{i\in I_0}\hat{y}_i g_i(\hat{x})\Big)\hat{p}.\end{aligned}$$

This ensues a contradiction to (6). Hence $\hat{x}=\bar{x}$, i.e., $\hat{x}$ is an optimal solution of (NP). This completes the proof of the theorem. □

Theorem 5 (Converse duality). *Let $(\bar{x}, \bar{y}, \bar{w}, \bar{p})$ be an optimal solution to (MixSD) at which*
(A_1): *for all $\alpha = 1, 2, \ldots, r$, either*

(a) *The $n \times n$ Hessian matrix $\nabla^2\left(\sum_{i\in I_\alpha} \bar{y}_i g_i(\bar{x})\right)$ is positive definite and $\bar{p}^T \nabla \sum_{i\in I_\alpha} \bar{y}_i(g_i(\bar{x}) + \bar{x}^t \bar{w}_i) \geq 0$ or*

(b) $\nabla^2\left(\sum_{i\in I_\alpha} \bar{y}_i g_i(\bar{x})\right)$ *is negative definite and $\bar{p}^T \nabla \sum_{i\in I_\alpha} \bar{y}_i(g_i(\bar{x}) + \bar{x}^t \bar{w}_i) \leq 0$*

(A_2): *the set of vectors*

$$\left\{\left[\nabla^2\left(f(\bar{x}) - \sum_{i\in I_0} \bar{y}_i g_i(\bar{x})\right)\right]_j, \left[\nabla^2\left(\sum_{i\in I_\alpha} \bar{y}_i g_i(\bar{x})\right)_j\right\}, \; j = 1, 2, \ldots, n,$$

$\alpha = 1, 2, \ldots, r$, are linearly independent,

where $\left[\nabla^2\left(f(\bar{x}) - \sum_{i\in I_0} \bar{y}_i g_i(\bar{x})\right)\right]_j$ is jth row of the matrix $\left[\nabla^2\left(f(\bar{x}) - \sum_{i\in I_\alpha} \bar{y}_i g_i(\bar{x})\right)\right]$ and $\left[\nabla^2\left(\sum_{i\in I_\alpha} \bar{y}_i g_i(\bar{x})\right)\right]_j$ is jth row of the matrix $\left[\nabla^2\left(\sum_{i\in I_\alpha} \bar{y}_i g_i(\bar{x})\right)\right]$.

(A_3): the vectors $\left\{\sum_{i\in I_\alpha} \bar{y}_i(\nabla g_i(\bar{x}) + \bar{w}_i)\right\}$, $\alpha = 1, 2, \ldots, r$, are linearly independent.

If for all feasible $(x, z, y, u, w_1, w_2, \ldots, w_m, p)$, $f(\cdot) + (\cdot)^T + \sum_{i\in I_0} y_i(g_i(\cdot) + (\cdot)^T w_i)$ is second order pseudoinvex and $\sum_{i\in I_\alpha} y_i(g_i(\cdot) + (\cdot)^T w_i)$, $\alpha = 1, 2, \ldots, r$, is second order quasi-invex with respect to same η, then $\bar{x}$ is an optimal solution of the problem (NP).

Proof. Since $(\bar{x}, \bar{z}, \bar{y}, \bar{w}, \bar{p})$, where $\bar{w} = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_m)$ is an optimal solution of (MixSD), by generalized Fritz John necessary optimality conditions, there exists, $\tau_0 \in R$, $\theta \in R^n$, $\tau_\alpha \in R$, $\alpha = 1, 2, \ldots, r$, $\beta \in R$, and $\mu \in R^m$, such that

$$\tau_0\left\{ -(\nabla f(\bar{x}) + \bar{z}) - \sum_{i\in I_o} \bar{y}_i(\nabla g_i(\bar{x}) + \bar{w}_i) + \frac{1}{2}\bar{p}\nabla\left[\nabla^2\left(f(\bar{x}) + \sum_{i\in I_o} \bar{y}_i g_i(\bar{x})\right)\bar{p}\right]\right\}$$
$$+\theta\{\nabla^2(f(\bar{x}) + \bar{y}^T g(\bar{x})) + \nabla(\nabla^2(f(\bar{x}) + \bar{y}^T g(\bar{x}))\bar{p})\}$$
$$+\sum_{\alpha=1}^{r} \tau_\alpha\{\sum_{i\in I_\alpha} \bar{y}_i(\nabla g_i(\bar{x}) + \bar{w}) - \frac{1}{2}\bar{p}^T\nabla\left[\left(\nabla^2 \sum_{i\in I_\alpha} \bar{y}_i g_i(\bar{x})\right)\bar{p}\right]\} = 0$$

$$\tau_0\left\{g_i(\bar{x}) + \bar{x}^T\bar{w}_i - \frac{1}{2}\bar{p}^T\nabla^2 g_i(\bar{x})\bar{p}\right\} + \theta^T\{\nabla g_i(\bar{x}) + \bar{w}_i + \nabla^2 g_i(\bar{x})\bar{p}\} + \mu_i = 0, i \in I_0$$

$$\tau_\alpha\left\{g_i(\bar{x}) + \bar{x}^T\bar{w}_i - \frac{1}{2}\bar{p}^T\nabla^2 g_i(\bar{x})\bar{p}\right\} + \theta^T\{(\nabla g_i(\bar{x}) + \bar{w}_i + \nabla^2 g_i(\bar{x})\bar{p})\} + \mu_i = 0,$$

$i \in I_0, \quad \alpha = 1, 2, \ldots, r$

$$(\tau_0 \bar{p} + \theta)^T \left\{ \nabla^2 \left(f(\bar{x}) - \sum_{i \in I_0} \bar{y}_i g(\bar{x}) \right) \right\} + \sum_{\alpha=1}^{r} (\tau_\alpha \bar{p} + \theta)^T \left\{ \nabla^2 \sum_{i \in I_\alpha} \bar{y}_i g_i(\bar{x}) \right\} = 0$$

$$\tau_\alpha \left\{ \sum_{i \in I_\alpha} \bar{y}_i (g_i(\bar{x}) + \bar{x}^T \bar{w}_i) - \frac{1}{2} \bar{p} \nabla^2 \sum_{i \in I_\alpha} \bar{y}_i g_i(\bar{x}) \bar{p} \right\} = 0, \quad i \in I_0, \ \alpha = 1, 2, \ldots, r$$

$$\tau_0 \bar{p} + \theta \ \in N_c(\bar{z}), \qquad (\tau_0 \bar{x} + \theta) y_i \in N_{D_i}(\bar{w}), \quad i \in I_0$$

$$(\tau_0 \bar{x} + \theta) y_i \in N_{D_i}(\bar{w}), \quad i \in I_\alpha, \alpha = 1, 2, \ldots, r, \quad \mu^T y = 0'$$

$$(\tau_0, \tau_1, \ldots, \tau_r, \mu) \geq 0, \quad (\tau_0, \tau_1, \ldots, \tau_r, \theta, \mu) \neq 0.$$

The relation (12),in view of assumption (A_2) yields, $\tau_\alpha \bar{p} + \theta = 0, \quad \alpha = 1, 2, \ldots, r$, Multiplying (11) by $\bar{y}_i, i \in I_\alpha$, $\alpha = 1, 2, \ldots, r$, and summing with respect to $i \in I_\alpha$, $\alpha = 1, 2, \ldots, r$, we get

$$\tau_\alpha \left\{ \sum_{i \in I_\alpha} \bar{y}_i (g_i(\bar{x}) + \bar{x}^T \bar{w}_i) - \frac{1}{2} \bar{p} \nabla^2 \sum_{i \in I_\alpha} \bar{y}_i g_i(\bar{x}) \bar{p} \right\}$$
$$+ \theta^T \left\{ \sum_{i \in I_\alpha} \bar{y}_i (\nabla g_i(\bar{x}) + \bar{w}_i + \nabla^2 \sum_{i \in I_\alpha} \bar{y}_i g_i(\bar{x}) \bar{p}) \right\} = 0, \quad \alpha = 1, 2, \ldots, r$$

Using (13) we get,

$$\theta^T \left\{ \sum_{i \in I_\alpha} \bar{y}_i \left(\nabla g_i(\bar{x}) + \bar{w}_i + \nabla^2 \sum_{i \in I_\alpha} \bar{y}_i g_i(\bar{x}) \bar{p} \right) \right\} = 0, \quad \alpha = 1, 2, \ldots, r$$

By using the equality constraint of the dual in (9), we get

$$(\tau_\alpha \bar{p} + \theta)^T \left\{ \nabla^2 \left(f(\bar{x}) - \sum_{i \in I_0} \bar{y}_i g_i(\bar{x}) \right) + \nabla \left[\nabla^2 \left(f(\bar{x}) + \sum_{i \in I_0} \bar{y}_i g_i(\bar{x}) \right) \right] \bar{p} \right\}$$
$$+ \sum_{\alpha=1}^{r} (\tau_\alpha \bar{p} + \theta)^T \left\{ \nabla^2 \left(\sum_{i \in I_\alpha} \bar{y}_i g_i(\bar{x}) \right) + \nabla (\nabla^2 \sum_{i \in I_\alpha} \bar{y}_i g_i(\bar{x}) \Big) \bar{p} \right\}$$
$$+ \tau_0 \left\{ \nabla \sum_{i \in M - I_0} \bar{y}_i (g_i(\bar{x}) + \bar{x}^T \bar{w}_i) + \nabla^2 \sum_{i \in M - I_0} \bar{y}_i g_i(\bar{x}) \bar{p} \right\}$$
$$- \frac{1}{2} \tau_0 \bar{p}^T \left\{ \nabla \left[\nabla^2 \left(f(\bar{x}) + \sum_{i \in I_0} \bar{y}_i g_i(\bar{x}) \right) \right] \bar{p} \right\}$$
$$+ \sum_{\alpha=1}^{r} \tau_\alpha \left\{ \nabla \sum_{i \in I_\alpha} \bar{y}_i (g_i(\bar{x}) + \bar{x}^T \bar{w}_i) + \nabla^2 \left(\sum_{i \in I_\alpha} \bar{y}_i g_i(\bar{x}) \right) \bar{p} \right\}$$
$$+ \sum_{\alpha=1}^{r} \frac{1}{2} \tau_\alpha \bar{p}^T \left\{ \nabla \left[\nabla^2 \left(\sum_{i \in I_\alpha} \bar{y}_i g_i(\bar{x}) \right) \right] \bar{p} \right\} = 0$$

From (20), it implies,

$$\sum_{\alpha=1}^{r}(\tau_\alpha-\tau_0)\Big\{\sum_{i\in I_\alpha}\bar{y}_i(\nabla g_i(\bar{x})+\bar{w}_i\Big)+\nabla^2\Big(\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\Big)\bar{p}\Big\}$$
$$+\frac{1}{2}\theta^T\Big\{\nabla\Big[\nabla^2\Big(f(\bar{x})+\sum_{i\in I_0}\bar{y}_i g_i(\bar{x})\Big)\Big]\bar{p}+\nabla\Big[\nabla^2\Big(\sum_{i\in M-I_0}\bar{y}_i g_i(\bar{x})\Big)\Big]\bar{p}\Big\}=0$$

This implies

$$\sum_{\alpha=1}^{r}(\tau_\alpha-\tau_0)\Big\{\sum_{i\in I_\alpha}\bar{y}_i(\nabla g_i(\bar{x})+\bar{w}_i)+\nabla^2\Big(\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\Big)\bar{p}\Big\}$$
$$+\frac{1}{2}\theta^T\{\nabla(\nabla^2(f(\bar{x})+\bar{y}^T g(\bar{x}))\bar{p})\}=0$$

Assume that $\tau_\alpha=0$, for all $\alpha\in\{0,1,2,\ldots,r\}$. Then $\theta=0$ from (20), $\mu=0$, i.e., $(\tau_0,\tau_1,\ldots,\tau_r,\theta,)=0$. This contradicts the Fritz John condition (20). Thus there exists an $\alpha\in\{0,1,2,\ldots,r\}$ such that $\tau_\alpha>0$.

The relation (20) can be rewritten as $\tau_0\bar{p}+\theta=0$, $\tau_\alpha\bar{p}+\theta=0$, $\alpha=1,2,\ldots,r$, which implies $(\tau_0-\tau_\alpha)\bar{p}=0$ We claim $\bar{p}=0$. Suppose that $\bar{p}\neq 0$. Then (23) yields $\tau_0=\tau_\alpha$, $\alpha=1,2,\ldots,r$. So from (20) we have $\theta=-\tau_0\bar{p}$. Using this in (21), we obtain

$$-\tau_0\bar{p}\Big\{\sum_{i\in I_\alpha}\bar{y}_i(\nabla g(\bar{x})+\bar{w}_i)+\nabla^2\Big(\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\Big)\bar{p}\Big\}=0$$

$$\Rightarrow\qquad \bar{p}\Big\{\sum_{i\in I_\alpha}\bar{y}_i(\nabla g(\bar{x})+\bar{w}_i)+\nabla^2\Big(\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\Big)\bar{p}\Big\}=0.$$

From the assumption (A_1), i.e., for $\alpha=1,2,\ldots,r$,

$$\bar{p}\sum_{i\in I_\alpha}\bar{y}_i(g(\bar{x})+\bar{w}_i)\geq 0,\quad \bar{p}\nabla^2\Big(\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\Big)\bar{p}\geq 0,$$

it follows

$$\bar{p}\sum_{i\in I_\alpha}\bar{y}_i(g(\bar{x})+\bar{w}_i)+\bar{p}^T\nabla^2\Big(\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\Big)\bar{p}\neq 0.$$

This is contradicted by (23). Hence $\bar{p}=0$. Using $\bar{p}=0$ in (22) we have

$$\sum_{\alpha=1}^{r}(\tau_\alpha-\tau_0)\Big\{\sum_{i\in I_\alpha}\bar{y}_i(\nabla g_i(\bar{x})+\bar{w}_i)\Big\}=0.$$

By (A_3), this implies $\tau_0=\tau_\alpha>0$, $\alpha=1,2,\ldots,r$.

Since $\theta=0$, (10) and (11) implies

$$\tau_0(g_i(\bar{x})+\bar{x}_i^T\bar{w}_i)+\mu_i=0,\quad g_i(\bar{x})+\bar{x}_i^T\bar{w}=-\frac{\mu_i}{\tau_0}\leq 0, i\in I_0$$

$$\tau_\alpha(g_i(\bar{x})+\bar{x}_i^T\bar{w})+\mu_i=0, i\in I_\alpha\quad \alpha=1,2,\ldots,r$$

Comparing these, we have $g_i(\bar{x}) + \bar{x}_i^T \bar{w} = -\frac{\mu_i}{\tau_\alpha} \leq 0, i \in I_0, i \in I_\alpha, \alpha = 1, 2, \ldots, r$. From (15) and (16) we have $\bar{x}^T \bar{w}_i = S(.\bar{x}|D_i), i \in I_0, i \in I_\alpha, \alpha = 0, 1, 2, \ldots, r$. The relation (25) along with this implies $g_i(\bar{x}) + S(.\bar{x}|D_i) \leq 0 \quad i = 0, 1, 2, \ldots, m$. This shows that $\bar{x}$is feasible for (NP)

Multiplying (25) by $\bar{y}_i$, $i \in I_0$, and $\bar{y}_i$, $i \in I_\alpha$ $\alpha = 1, 2, \ldots, r$, and adding and using $\mu^T y = 0$,

$$\sum_{i\in I_0} \bar{y}_i(g(\bar{x}) + \bar{w}_i \bar{x}) = 0, \quad \sum_{i\in I_\alpha} \bar{y}_i(g(\bar{x}) + \bar{w}_i \bar{x}) = 0,$$

$$(f(\bar{x}) + \bar{x}^T \bar{z}) - \sum_{i\in I_0} \bar{y}_i(g_i(\bar{x}) + \bar{w}_i^T \bar{x}) - \frac{1}{2}\bar{p}^T \left[\nabla^2 \Big(f(\bar{x}) + \sum_{i\in I_o} \bar{y}_i g_i(\bar{x})\Big)\bar{p}\right]$$
$$= f(\bar{x}) + \bar{x}^T \bar{z} \quad \text{using p=0} \quad \text{and (26)}$$
$$= f(\bar{x}) + S(\bar{x}|C.), \quad \text{by (14)}$$

If, for all feasible $(\bar{x}, \bar{z}, \bar{u}, \bar{w}_1, \ldots, \bar{w}_m, \bar{p})$, $f(\cdot) + (\cdot)^T + \sum_{i\in I_0} y_i(g_i(\cdot) + (\cdot)^T w_i)$ is second order pseudoinvex and $\sum_{i\in I_\alpha} y_i(g_i(\cdot) + (\cdot)^T w_i)$, $\alpha = 1, 2, \ldots, r$, is second order quasi-invex for $z \in C$ and $w_i \in D_i$ with respect to same η, by Theorem 1,then $\bar{x}$ is an optimal solution of the problem (NP). □

5. Special cases

If $p = 0$, the mixed type dual (MixSD) to the following to the following first order mixed type dual formulated in [10].

(MixSD) : Maximize $f(u) + u^T z + \sum_{i\in I_0} y_i(g_i(u) + u^T w_i)$

Subject to

$$(\nabla f(u) + u^T z) + \sum_{i=1}^{m} y_i(\nabla g_i(u) + u^T w_i) = 0$$

$$\sum_{i\in I_\alpha} y_i(g_i(u) + u^T w_i) \geq 0, \quad \alpha = 1, 2, \ldots, r.$$

$$y \geq 0, \quad z \in C, \quad w_i \in D_i, \quad i = 1, 2, \ldots, m.$$

where $I_\alpha \subseteq M = \{1, 2, \ldots, m\}$, $\alpha = 0, 1, 2, \ldots, r$ with $\bigcup_{i=0}^{r} I_\alpha = M$ and $I_\alpha \bigcap I_\beta = \phi$ if $\alpha \neq \beta$.

As discussed in [6], we may write for positive semi definite matrix B, $S(.x|C) = (x^T Bx)^{\frac{1}{2}}$ by taking $C = \{By | y^T By \leq 1.\}$. If the support function appearing in the constraints suppressed but the support function in the objective function of (NP) is retained and replaced by $(x^T Bx)^{\frac{1}{2}}$, then we have the following pair of problems treated by Zhang and Mond [28] and re-examined Zhang and Yang for correcting the converse duality theorem proved in [29].

(P) : Minimize$f(x) + (x^T Bx)^{\frac{1}{2}}$
subject to
$g(x) \leq 0,$

(SD) Maximize $f(u) - \sum_{i \in I_0} y_i g_i(u) + u^T z$
$-\frac{1}{2}\nabla^2 p^T \left[f(u) + \sum_{i \in I_0} y_i g_i(u) \right] p$
subject to
$\nabla f(u) - y^T g(u) + z + \nabla^2 (f(u) + y^T g(u))p = 0$
$\sum_{i \in I_\alpha} y_i (g_i(u) + u^T w_i) - \frac{1}{2} p^T \nabla^2 \Big(\sum_{i = I_\alpha} y_i g_i(u) \Big) p \geq 0, \quad \alpha = 1, 2, \ldots, r,$
$w^T z \leq 1, \quad y \geq 0.$

where $I_\alpha \subseteq M = \{1, 2, \ldots, m\}$, $\alpha = 0, 1, 2, \ldots, r$ with $\bigcup_{i=0}^{r} I_\alpha = M$ and $I_\alpha \bigcap I_\beta = \phi$ if $\alpha \neq \beta$.

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