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SOFT BCC-ALGEBRAS

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ABSTRACT. Molodtsov [D. Molodtsov, Soft set theory – First results, Comput. Math. Appl. 37 (1999) 19–31] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. In this paper we apply the notion of soft sets by Molodtsov to the theory of BCC-algebras. The notion of (trivial, whole) soft BCC-algebras and soft BCC-subalgebras are introduced, and several examples are provided. Relations between a fuzzy subalgebra and a soft BCC-algebra are given, and the characterization of soft BCC-algebras is established.

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1. Introduction

The real world is inherently uncertain, imprecise and vague. Rough set theory, proposed by Pawlak in 1982, focusesed on the uncertainty caused by indiscernible elements with different values in decision attributes. It approximates the underlying set with two crisp sets. Therefore, the cardinality of elements in these two sets has a direct influence on the uncertainty of their corresponding rough set as a whole. Also, rough set theory can be seen as a new mathematical approach to vagueness. Worldwide, there has been a rapid growth in interest in rough set theory and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on rough sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years. It seems that the rough set approach is fundamentally important in artificial intelligence and cognitive sciences, especially in research areas such as machine learning, intelligent systems, inductive reasoning, pattern recognition, knowledge discovery,

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decision analysis, and expert systems. Recently, Pawlak and Skowron [15] presented the basic concepts of rough set theory and point out some rough set-based research directions and applications. They [16] presented some extensions of the rough set approach, and outlined a challenge for the rough set based research. Also, they [17] discussed methods based on the combination of rough sets and Boolean reasoning with applications in pattern recognition, machine learning, data mining and conflict analysis. This paper is dedicated to the memory of Zdzisław Pawlak[1], a great scientist and a great human being.

Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [19]. In response to this situation Zadeh [20] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [21]. To solve complicated problems in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as the probability theory, the theory of (intuitionistic) fuzzy sets, the theory of vague sets, the theory of interval mathematics, and the theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [14]. Maji et al. [13] and Molodtsov [14] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [14] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [13] described the application of soft set theory to a decision making problem. Maji et al. [12] also studied several operations on the theory of soft sets. Chen et al. [2] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. The study of structures of fuzzy sets in algebraic structures are carried out by several authors (see [1, 3, 4, 7, 8, 9, 10, 11, 18]). Aktaş and Çağman [1] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups. Jun [6] applied the notion of soft sets by Molodtsov to the theory of BCK/BCI-algerbas, and introduced the notion of soft BCK/BCI-algebras and soft subalgebras, and then derived their basic properties. In this paper,

[1]Prof. Pawlak passed away on 7 April 2006.

we deal with the algebraic structure of BCC-algebras by applying soft set theory. We discuss the algebraic trends of soft sets in BCC-algebras. We introduce the notion of (trivial, whole) soft BCC-algebras and soft BCC-subalgebras, and give several examples. We give relations between a fuzzy subalgebra and a soft BCC-algebra. We establish the characterization of soft BCC-algebras. We also discuss the intersection, union, "AND" operation, and "OR" operation of soft BCC-algebras and soft BCC-subalgebras.

2. Preliminaries

A BCK-algebra is an important calss of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *BCC-algebra* we mean a system $(X; \to, 0) \in K(\tau)$ in which the following axioms hold:

(C1) $(\forall x, y, z \in X)\ ((x \to y) \to (z \to y)) \to (x \to z) = 0)$.
(C2) $(\forall x \in X)\ (0 \to x = 0\ \&\ x \to 0 = x)$.
(C3) $(\forall x, y \in X)\ (x \to y = 0\ \&\ y \to x = 0 \Rightarrow x = y)$.

For any BCC-algebra X, the relation $\leq$ defined by

$$(\forall x, y \in X)\,(x \leq y \iff x \to y = 0)$$

is a partial order on X. Note that any BCK-algebra is a BCC-algebra, but the converse is not true. In a BCC-algebras X, the following hold (see [5]).

(a1) $(\forall x \in X)\ (x \leq x)$.
(a2) $(\forall x, y \in X)\ (x \to y \leq x)$.
(a3) $(\forall x, y, z \in X)\ (x \leq y \Longrightarrow x \to z \leq y \to z\ \&\ z \to y \leq z \to x)$.

A nonempty subset S of a BCC-algebra X is said to be a *subalgebra* of X if $x \to y \in S$ whenever $x, y \in S$. A mapping $f : X \to Y$ of BCC-algebras is called a *homomorphism* if $f(x \to y) = f(x) \to f(y)$ for all $x, y \in X$. For a homomorphism $f : X \to Y$ of BCC-algebras, the *kernel* of f, denoted by $\ker(f)$, is defined to be the set

$$\ker(f) = \Big\{x \in X \mid f(x) = 0\Big\}.$$

Let X be a BCC-algebra. A fuzzy set $\mu : X \to [0, 1]$ is called a *fuzzy subalgebra* of X if $\mu(x \to y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

Molodtsov [14] defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. Let $\mathscr{P}(U)$ denotes the power set of U and $A \subset E$.

Definition 1. [14] A pair (δ, A) is called a *soft set* over U, where δ is a mapping given by

$$\delta : A \to \mathscr{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $\delta(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (δ, A). Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [14].

Definition 2. [12] Let (δ, A) and (γ, B) be two soft sets over a common universe U. The *intersection* of (δ, A) and (γ, B) is defined to be the soft set (ρ, C) satisfying the following conditions:

(i) $C = A \cap B$,
(ii) $(\forall e \in C)$ $(\rho(e) = \delta(e)$ or $\gamma(e)$, (as both are same sets)).

In this case, we write $(\delta, A)\tilde{\cap}(\gamma, B) = (\rho, C)$.

Definition 3. [12] Let (δ, A) and (γ, B) be two soft sets over a common universe U. The *union* of (δ, A) and (γ, B) is defined to be the soft set (ρ, C) satisfying the following conditions:

(i) $C = A \cup B$,
(ii) for all $e \in C$,

$$\rho(e) = \begin{cases} \delta(e) & \text{if } e \in A \setminus B, \\ \gamma(e) & \text{if } e \in B \setminus A, \\ \delta(e) \cup \gamma(e) & \text{if } e \in A \cap B. \end{cases}$$

In this case, we write $(\delta, A)\tilde{\cup}(\gamma, B) = (\rho, C)$.

Definition 4. [12] If (δ, A) and (γ, B) are two soft sets over a common universe U, then "(δ, A) *AND* (γ, B)" denoted by $(\delta, A)\tilde{\wedge}(\gamma, B)$ is defined by $(\delta, A)\tilde{\wedge}(\gamma, B)$ $= (\rho, A \times B)$, where $\rho(x, y) = \delta(x) \cap \gamma(y)$ for all $(x, y) \in A \times B$.

Definition 5. [12] If (δ, A) and (γ, B) are two soft sets over a common universe U, then "(δ, A) *OR* (γ, B)" denoted by $(\delta, A)\tilde{\vee}(\gamma, B)$ is defined by $(\delta, A)\tilde{\vee}(\gamma, B) =$ $(\rho, A \times B)$, where $\rho(x, y) = \delta(x) \cup \gamma(y)$ for all $(x, y) \in A \times B$.

Definition 6. [12] For two soft sets (δ, A) and (γ, B) over a common universe U, we say that (δ, A) is a *soft subset* of (γ, B), denoted by $(\delta, A)\tilde{\subset}(\gamma, B)$, if it satisfies:

(i) $A \subset B$,
(ii) For every $\varepsilon \in A$, $\delta(\varepsilon)$ and $\gamma(\varepsilon)$ are identical approximations.

3. Soft BCC-algebras

In what follows let X and A be a BCC-algebra and a nonempty set, respectively, and R will refer to an arbitrary binary relation between an element of A and an element of X, that is, R is a subset of $A \times X$ without otherwise specified. A set-valued function $\delta : A \to \mathscr{P}(X)$ can be defined as $\delta(x) = \{y \in X \mid (x, y) \in R\}$ for all $x \in A$. The pair (δ, A) is then a soft set over X.

Definition 7. Let (δ, A) be a soft set over X. Then (δ, A) is called a *soft BCC-algebra* over X if $\delta(x)$ is a subalgebra of X for all $x \in A$.

Let us illustrate this definition using the following examples.

Example 1. Let $X = \{0, a, b, c, d\}$ be a BCC-algebra with the following Cayley table:

$\to$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

(1) Let (δ, A) be a soft set over X, where $A = X$ and $\delta : A \to \mathscr{P}(X)$ is a set-valued function defined by $\delta(x) = \Big\{y \in X \mid y \to (y \to x) \in \{0, a\}\Big\}$ for all $x \in A$. Then $\delta(0) = \delta(a) = X$, $\delta(b) = \{0, a, c, d\}$, $\delta(c) = \{0, a, b, d\}$, and $\delta(d) = \{0, a, b, c\}$ are subalgebras of X. Therefore (δ, A) is a soft BCC-algebra over X.

(2) Let (ρ, B) be a soft set over X, where $B = \{a, c, d\}$ and $\rho : B \to \mathscr{P}(X)$ is a set-valued function defined by $\rho(x) = \Big\{y \in X \mid (y \to x) \to x \in \{0, b\}\Big\}$ for all $x \in B$. Then $\rho(a) = \{0, a, b\}$, $\rho(c) = \{0, b, c\}$ and $\rho(d) = \{0, b, d\}$ are subalgebras of X. Therefore (ρ, B) is a soft BCC-algebra over X.

Example 2. Let $X = \{0, 1, 2, 3\}$ be a BCC-algebra with the following Cayley table:

$\to$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	1
3	3	3	1	0

Let (γ, A) be a soft set over X, where $A = \{1, 2\}$ and $\gamma : A \to \mathscr{P}(X)$ is a set-valued function defined by $\gamma(x) = \Big\{y \in X \mid (y \to x) \to x \in \{0, 1\}\Big\}$ for all $x \in A$. Then $\gamma(1) = \{0, 1\}$ and $\gamma(2) = X$ which are subalgebras of X. Hence (γ, A) is a soft BCC-algebra over X.

Theorem 1. *Let (δ, A) be a soft BCC-algebra over X. If B is a subset of A, then $(\delta|_B, B)$ is a soft BCC-algebra over X.*

Proof. Straightforward. □

The following example shows that there exists a soft set (ρ, A) over X such that

(i) (ρ, A) is not a soft BCC-algebra over X.
(ii) there exists a subset B of A such that $(\rho|_B, B)$ is a soft BCC-algebra over X.

Example 3. Let $X = \{0, 1, 2, 3, 4\}$ be a BCC-algebra with the following Cayley table:

$\rightarrow$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	0
3	3	2	1	0	1
4	4	4	2	2	0

Let (ρ, A) be a soft set over X, where $A = X$ and $\rho : A \rightarrow \mathscr{P}(X)$ is a set-valued function defined by $\rho(x) = \Big\{y \in X \mid y \rightarrow x \in \{0, 3, 4\}\Big\}$ for all $x \in A$. Then (ρ, A) is not a soft BCC-algebra over X because $\rho(0) = \{0, 3, 4\}$ is not a subalgebra of X. If we take $B \subsetneq A \setminus \{0\}$, then $(\rho|_B, B)$ is a soft BCC-algebra over X.

Theorem 2. *Let (δ, A) and (γ, B) be two soft BCC-algebras over X. If $A \cap B \neq \emptyset$, then the intersection $(\delta, A)\tilde{\cap}(\gamma, B)$ is a soft BCC-algebra over X.*

Proof. Using Definition 2, we can write $(\delta, A)\tilde{\cap}(\gamma, B) = (\rho, C)$, where $C = A \cap B$ and $\rho(x) = \delta(x)$ or $\gamma(x)$ for all $x \in C$. Note that $\rho : C \rightarrow \mathscr{P}(X)$ is a mapping, and therefore (ρ, C) is a soft set over X. Since (δ, A) and (γ, B) are soft BCC-algebras over X, it follows that $\rho(x) = \delta(x)$ is a subalgebra of X, or $\rho(x) = \gamma(x)$ is a subalgebra of X for all $x \in C$. Hence $(\rho, C) = (\delta, A)\tilde{\cap}(\gamma, B)$ is a soft BCC-algebra over X. □

Corollary 1. *Let (δ, A) and (γ, A) be two soft BCC-algebras over X. Then their intersection $(\delta, A)\tilde{\cap}(\gamma, A)$ is a soft BCC-algebra over X.*

Proof. Straightforward. □

Theorem 3. *Let (δ, A) and (γ, B) be two soft BCC-algebras over X. If A and B are disjoint, then the union $(\delta, A)\tilde{\cup}(\gamma, B)$ is a soft BCC-algebra over X.*

Proof. Using Definition 3, we can write $(\delta, A)\tilde{\cup}(\gamma, B) = (\rho, C)$, where $C = A \cup B$ and for every $x \in C$,

$$\rho(x) = \begin{cases} \delta(x) & \text{if } x \in A \setminus B, \\ \gamma(x) & \text{if } x \in B \setminus A, \\ \delta(x) \cup \gamma(x) & \text{if } x \in A \cap B. \end{cases}$$

Since $A \cap B = \emptyset$, either $x \in A \setminus B$ or $x \in B \setminus A$ for all $x \in C$. If $x \in A \setminus B$, then $\rho(x) = \delta(x)$ is a subalgebra of X since (δ, A) is a soft BCC-algebra over X. If $x \in B \setminus A$, then $\rho(x) = \gamma(x)$ is a subalgebra of X since (γ, B) is a soft BCC-algebra over X. Hence $(\rho, C) = (\delta, A)\widetilde{\cup}(\gamma, B)$ is a soft BCC-algebra over X. □

In Theorem 3, if A and B are not disjoint, then the result is not valid as seen in the following example.

Example 4. (1) Consider the BCC-algebra X in Example 3. Let (δ, A) and (γ, B) be two soft sets over X, where $A = \{0, 1, 2, 3\}$, $B = \{0, 3, 4\}$, and $\delta : A \to \mathscr{P}(X)$ and $\gamma : B \to \mathscr{P}(X)$ are set-valued functions defined by

$$\delta(a) = \Big\{y \in X \mid y \to a \in \{0, 3\}\Big\} \text{ and } \gamma(b) = \Big\{y \in X \mid y \to b \in \{0, 4\}\Big\}$$

for all $a \in A$ and $b \in B$, respectively. Then (δ, A) and (γ, B) are soft BCC-algebras over X. Note that A and B are not disjoint and the union $(\delta, A)\widetilde{\cup}(\gamma, B)$ is not a soft BCC-algebra over X since $\delta(0) \cup \gamma(0) = \{0, 3, 4\}$ is not a subalgebra of X.

(2) Consider the BCC-algebra X in Example 2. For $A := \{1\}$, let $\delta : A \to \mathscr{P}(X)$ be a set-valued function defined by

$$\delta(x) = \Big\{y \in X \mid x \to y \in \{1\}\Big\}$$

for all $x \in A$. Then (δ, A) is a soft BCC-algebra over X since $\delta(1) = \{0, 2\}$ is a subalgebra of X. Now, let $B := \{0, 1\}$ which is not disjoint with A, and let $\gamma : B \to \mathscr{P}(X)$ be a set-valued function defined by

$$\gamma(x) = \Big\{y \in X \mid y \to (x \to y) \in \{0, 3\}\Big\}$$

for all $x \in B$. Then $\gamma(0) = \gamma(1) = \{0, 3\}$ is a subalgebra of X, and hence (γ, B) is a soft BCC-algebra over X. But if $(\rho, C) := (\delta, A)\widetilde{\cup}(\gamma, B)$, then $\rho(1) = \delta(1) \cup \gamma(1) = \{0, 2, 3\}$ is not a subalgebra of X since $2 \to 3 = 1 \notin \rho(1)$. This means that $(\rho, C) = (\delta, A)\widetilde{\cup}(\gamma, B)$ is not a soft BCC-algebra over X.

Theorem 4. *If (δ, A) and (γ, B) are soft BCC-algebras over X, then $(\delta, A)\widetilde{\wedge}(\gamma, B)$ is a soft BCC-algebra over X.*

Proof. By means of Definition 4, we know that $(\delta, A)\widetilde{\wedge}(\gamma, B) = (\rho, A \times B)$, where $\rho(x, y) = \delta(x) \cap \gamma(y)$ for all $(x, y) \in A \times B$. Since $\delta(x)$ and $\gamma(y)$ are subalgebras of X, the intersection $\delta(x) \cap \gamma(y)$ is also a subalgebra of X. Hence $\rho(x, y)$ is a subalgebra of X for all $(x, y) \in A \times B$, and therefore $(\delta, A)\widetilde{\wedge}(\gamma, B) = (\rho, A \times B)$ is a soft BCC-algebra over X. □

The following example shows that there are two soft BCC-algebras (δ, A) and (γ, B) over X such that $(\delta, A)\widetilde{\vee}(\gamma, B)$ is not a soft BCC-algebra over X.

Example 5. (1) Consider the BCC-algebra X in Example 3. Let (δ, A) and (γ, B) be two soft sets over X, where $A = \{0,3\}$, $B = \{2\}$, and $\delta : A \to \mathscr{P}(X)$ and $\gamma : B \to \mathscr{P}(X)$ are set-valued functions defined by

$$\delta(a) = \Big\{x \in X \mid x \to a \in \{0,3\}\Big\} \text{ and } \gamma(b) = \Big\{x \in X \mid x \to b = 2\Big\} \cup \{0\}$$

for all $a \in A$ and $b \in B$, respectively. Then $\delta(0) = \{0,3\}$, $\delta(3) = \{0,1,2,3\}$, and $\gamma(2) = \{0,4\}$ are subalgebras of X. Hence (δ, A) and (γ, B) are soft BCC-algebras over X. But, $(\delta, A)\tilde{\vee}(\gamma, B)$ is not a soft BCC-algebra over X since $\delta(0) \cup \gamma(2) = \{0,3,4\}$ is not a subalgebra of X.

(2) Let (δ, A) and (γ, B) be two soft BCC-algebras over X which is described in Example 4(2). Then $(\rho, A \times B) := (\delta, A)\tilde{\vee}(\gamma, B)$ is not a soft BCC-algebra over X since $\rho(1,1) = \delta(1) \cup \gamma(1) = \{0,2,3\}$ is not a subalgebra of X.

Definition 8. A soft BCC-algebra (δ, A) over X is said to be *trivial* (resp., *whole*) if $\delta(x) = \{0\}$ (resp., $\delta(x) = X$) for all $x \in A$.

Example 6. Let $X = \{0, a, b, c, d\}$ be a BCC-algebra which is described in Example 1. Consider $A = X$ and a set-valued function $\delta : A \to \mathscr{P}(X)$ defined by

$$\delta(x) = \Big\{y \in X \mid x \to y \in \{0, x\}\Big\}$$

for all $x \in A$. Then $\delta(0) = \delta(a) = \delta(b) = \delta(c) = \delta(d) = X$. Hence (δ, A) is a whole soft BCC-algebra over X. Now, let $\gamma : A \to \mathscr{P}(X)$ be a set-valued function defined by

$$\gamma(x) = \Big\{y \in X \mid y \to (x \to y) \in \{0\}\Big\}$$

for all $x \in A$. Then $\gamma(0) = \gamma(a) = \gamma(b) = \gamma(c) = \gamma(d) = \{0\}$, and so (γ, A) is a trivial soft BCC-algebra over X.

Let $f : X \to Y$ be a mapping of BCC-algebras. For a soft set (δ, A) over X, $(f(\delta), A)$ is a soft set over Y where $f(\delta) : A \to \mathscr{P}(Y)$ is defined by $f(\delta)(x) = f(\delta(x))$ for all $x \in A$.

Lemma 1. *Let $f : X \to Y$ be a homomorphism of BCC-algebras. If (δ, A) is a soft BCC-algebra over X, then $(f(\delta), A)$ is a soft BCC-algebra over Y.*

Proof. For every $x \in A$, we have $f(\delta)(x) = f(\delta(x))$ is a subalgebra of Y since $\delta(x)$ is a subalgebra of X and its homomorphic image is also a subalgebra of Y. Hence $(f(\delta), A)$ is a soft BCC-algebra over Y. □

Example 7. Let $X = \{0, a, b, c, d\}$ and $Y = \{0,1,2,3\}$ be BCC-algebras which are described in Example 1 and Example 2, respectively. Define a map $f : X \to Y$ by

$$f(0) = f(c) = f(d) = 0, f(a) = 2 \text{ and } f(b) = 1.$$

We can verify that f is a homomorphism of BCC-algebras. If (δ, A) is a soft BCC-algebra over X which is given in Example 1, then

$$f(\delta)(0) = f(\delta)(a) = f(\delta)(c) = f(\delta)(d) = \{0, 1, 2\} \text{ and } f(\delta)(b) = \{0, 2\}$$

are subalgebras of Y. This means that $(f(\delta), A)$ is a soft BCC-algebra over Y. Also, if we take a soft BCC-algebra (ρ, B) over X which is defined in Example 1, then $(f(\rho), B)$ is a soft BCC-algebra over Y since $f(\rho)(a) = \{0, 1, 2\}$ and $f(\rho)(c) = f(\rho)(d) = \{0, 1\}$ are subalgebras of Y.

Theorem 5. *Let $f : X \to Y$ be a homomorphism of BCC-algebras and let (δ, A) be a soft BCC-algebra over X.*

(i) *If $\delta(x) \subseteq \ker(f)$ for all $x \in A$, then $(f(\delta), A)$ is a trivial soft BCC-algebra over Y.*
(ii) *If f is onto and (δ, A) is whole, then $(f(\delta), A)$ is a whole soft BCC-algebra over Y.*

Proof. (i) Assume that $\delta(x) \subseteq \ker(f)$ for all $x \in A$. Then $f(\delta)(x) = f(\delta(x)) = \{0_Y\}$ for all $x \in A$. Hence $(f(\delta), A)$ is a trivial soft BCC-algebra over Y by Lemma 1 and Definition 8.

(ii) Suppose that f is onto and (δ, A) is whole. Then $\delta(x) = X$ for all $x \in A$, and so $f(\delta)(x) = f(\delta(x)) = f(X) = Y$ for all $x \in A$. It follows from Lemma 1 and Definition 8 that $(f(\delta), A)$ is a whole soft BCC-algebra over Y. □

Definition 9. Let (δ, A) and (γ, B) be two soft BCC-algebras over X. Then (δ, A) is called a *soft BCC-subalgebra* of (γ, B), denoted by $(\delta, A)\tilde{<}(\gamma, B)$, if it satisfies:

(i) $A \subset B$,
(ii) $\delta(x)$ is a subalgebra of $\gamma(x)$ for all $x \in A$.

Example 8. Let (δ, A) be a soft BCC-algebra over X which is given in Example 1(1). Let $B = \{a, c, d\}$ be a subset of A and let $\gamma : B \to \mathscr{P}(X)$ be a set-valued function defined by

$$\gamma(x) = \Big\{y \in X \mid y \to (y \to x) \in \{0, a\}\Big\}$$

for all $x \in B$. Then

$$\gamma(a) = X, \gamma(c) = \{0, a, b, d\} \text{ and } \gamma(d) = \{0, a, b, c\}$$

are subalgebras of $\delta(a)$, $\delta(c)$ and $\delta(d)$, respectively. Hence (γ, B) is a soft BCC-subalgebra of (δ, A).

Theorem 6. *Let (δ, A) and (γ, A) be two soft BCC-algebras over X.*

(i) *If $\delta(x) \subset \gamma(x)$ for all $x \in A$, then $(\delta, A)\tilde{<}(\gamma, A)$.*

(ii) *If* $B = \{0\}$ *and* (ρ, B), (ρ, X) *are soft BCC-algebras over* X, *then* $(\rho, B)\tilde{<}(\rho, X)$.

Proof. Straightforward. □

Theorem 7. *Let* (δ, A) *be a soft BCC-algebra over* X *and let* (γ_1, B_1) *and* (γ_2, B_2) *be soft BCC-subalgebras of* (δ, A). *Then*

(i) $(\gamma_1, B_1)\tilde{\cap}(\gamma_2, B_2)\tilde{<}(\delta, A)$.

(ii) $B_1 \cap B_2 = \emptyset \Rightarrow (\gamma_1, B_1)\tilde{\cup}(\gamma_2, B_2)\tilde{<}(\delta, A)$.

Proof. (i) Using Definition 2, we can write $(\gamma_1, B_1)\tilde{\cap}(\gamma_2, B_2) = (\gamma, B)$, where $B = B_1 \cap B_2$ and $\gamma(x) = \gamma_1(x)$ or $\gamma_2(x)$ for all $x \in B$. Obviously, $B \subset A$. Let $x \in B$. Then $x \in B_1$ and $x \in B_2$. If $x \in B_1$, then $\gamma(x) = \gamma_1(x)$ is a subalgebra of $\delta(x)$ since $(\gamma_1, B_1)\tilde{<}(\delta, A)$. If $x \in B_2$, then $\gamma(x) = \gamma_2(x)$ is a subalgebra of $\delta(x)$ since $(\gamma_2, B_2)\tilde{<}(\delta, A)$. Hence

$$(\gamma_1, B_1)\tilde{\cap}(\gamma_2, B_2) = (\gamma, B)\tilde{<}(\delta, A).$$

(ii) Assume that $B_1 \cap B_2 = \emptyset$. We can write $(\gamma_1, B_1)\tilde{\cup}(\gamma_2, B_2) = (\gamma, B)$ where $B = B_1 \cup B_2$ and

$$\gamma(x) = \begin{cases} \gamma_1(x) & \text{if } x \in B_1 \setminus B_2, \\ \gamma_2(x) & \text{if } x \in B_2 \setminus B_1, \\ \gamma_1(x) \cup \gamma_2(x) & \text{if } x \in B_1 \cap B_2 \end{cases}$$

for all $x \in B$. Since $(\gamma_i, B_i))\tilde{<}(\delta, A)$ for $i = 1, 2$, $B = B_1 \cup B_2 \subset A$ and $\gamma_i(x)$ is a subalgebra of $\delta(x)$ for all $x \in B_i$, $i = 1, 2$. Since $B_1 \cap B_2 = \emptyset$, $\gamma(x)$ is a subalgebra of $\delta(x)$ for all $x \in B$. Therefore $(\gamma_1, B_1)\tilde{\cup}(\gamma_2, B_2) = (\gamma, B)\tilde{<}(\delta, A)$. □

In Theorem 7(ii), if B_1 and B_2 are not disjoint, then the result is not valid as seen in the following example.

Example 9. Let $X = \{0, 1, 2, 3, 4\}$ be a BCC-algebra which is given in Example 3. For $A = X$, let $\delta : A \to \mathscr{P}(X)$ be a set-valued function defined by

$$\delta(x) = \Big\{y \in X \mid (x \to y) \to x \in \{0\}\Big\}$$

for all $x \in A$. Then $\delta(x) = X$ for all $x \in A$, and so (δ, A) is a whole soft BCC-algebra over X. Now let (γ_1, B_1) be a soft set over X, where $B_1 = \{3\} \subset A$ and $\gamma_1 : B_1 \to \mathscr{P}(X)$ is a set-valued function defined by

$$\gamma_1(x) = \Big\{y \in X \mid y \to (y \to x) \in \{0, 2\}\Big\}$$

for all $x \in B_1$. Then $\gamma_1(3) = \{0, 2, 4\}$ is a subalgebra of $\delta(3)(= X)$, i.e., $(\gamma_1, B_1)\tilde{<}(\delta, A)$. If we take $B_2 := \{0, 3\} \subset A$ that is not disjoint with B_1, and define a set-valued function $\gamma_2 : B_2 \to \mathscr{P}(X)$ by

$$\gamma_2(x) = \Big\{y \in X \mid x \to y \in \{0, 3\}\Big\}$$

for all $x \in B_2$, then $(\gamma_2, B_2)\tilde{<}(\delta, A)$ since $\gamma_2(0) = X$ and $\gamma_2(3) = \{0,3\}$ are subalgebras of $X(= \delta(0) = \delta(3))$. But $(\gamma_1, B_1)\tilde{\cup}(\gamma_2, B_2)$ is not a soft BCC-subalgebra of (δ, A) because $\gamma_1(3) \cup \gamma_2(3) = \{0,2,3,4\}$ is not a subalgebra of $\delta(3)(= X)$.

Theorem 8. *Let $f : X \to Y$ be a homomorphism of BCC-algebras and let (δ, A) and (γ, B) be soft BCC-algebras over X. Then*

$$(\delta, A)\tilde{<}(\gamma, B) \Rightarrow (f(\delta), A)\tilde{<}(f(\gamma), B).$$

Proof. Assume that $(\delta, A)\tilde{<}(\gamma, B)$. Let $x \in A$. Then $A \subset B$ and $\delta(x)$ is a subalgebra of $\gamma(x)$. Since f is a homomorphism, $f(\delta)(x) = f(\delta(x))$ is a subalgebra of $f(\gamma(x)) = f(\gamma)(x)$, and therefore $(f(\delta), A)\tilde{<}(f(\gamma), B)$. □

Theorem 9. *For every fuzzy subalgebra μ of X, there exists a soft BCC-algebra (δ, A) over X.*

Proof. If μ is a fuzzy subalgebra of X, then $U(\mu; t) := \Big\{x \in X \mid \mu(x) \geq t\Big\}$ is a subalgebra of X for all $t \in \text{Im}(\mu)$. Take $A = \text{Im}(\mu)$ and let $\delta : A \to \mathscr{P}(X)$ be a set-valued function defined by $\delta(t) = U(\mu; t)$ for all $t \in A$. Then (δ, A) is a soft BCC-algebra over X. □

Conversely, the following theorem is straightforward.

Theorem 10. *For any fuzzy set μ in X, if a soft BCC-algebra (δ, A) over X is given by $A = \text{Im}(\mu)$ and $\delta(t) = U(\mu; t)$ for all $t \in A$, then μ is a fuzzy subalgebra of X.*

Let μ be a fuzzy set in X and let (δ, A) be a soft set over X in which $A \subseteq [0, 1]$ and $\delta : A \to \mathscr{P}(X)$ is a set valued function defined by

$$(\forall t \in A)\Big(\delta(t) = \{x \in X \mid \mu(x) + t > 1\}\Big). \quad (1)$$

Then there exists $t \in A$ such that $\delta(t)$ is not a subalgebra of X as seen in the following example.

Example 10. Let $X = \{0, a, b, c, d\}$ be a BCC-algebra with the following Cayley table:

$\to$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	b
c	c	b	a	0	b
d	d	a	d	a	0

Let $\mu : X \to [0, 1]$ be a fuzzy set defined by $\mu(0) = 0.6$, $\mu(a) = 0.3$, $\mu(b) = 0.1$, $\mu(c) = 0.7$ and $\mu(d) = 0.8$. If we take $A = \{0.1, 0.3, 0.6, 0.7, 0.8\}$, then

$$\delta(0.6) = \Big\{x \in X \mid \mu(x) + 0.6 > 1\} = \{0, c, d\Big\}$$

which is not a subalgebra of X because $c \to d = b \notin \delta(0.6)$.

Theorem 11. *Let μ be a fuzzy set in X and let (δ, A) be a soft set over X in which $A = [0,1]$ and $\delta : A \to \mathscr{P}(X)$ is given by (1). Then the following assertions are equivalent.*

(i) *μ is a fuzzy subalgebra of X.*
(ii) *(δ, A) is a soft BCC-algebra over X.*

Proof. Assume that μ is a fuzzy subalgebra of X. Let $t \in A$ and let $x, y \in \delta(t)$. Then $\mu(x)+t > 1$ and $\mu(y)+t > 1$. Since μ is a fuzzy subalgebra of X, it follows that $\mu(x \to y) \geq \min\{\mu(x), \mu(y)\}$ so that

$$\begin{aligned}\mu(x \to y) + t &\geq \min\left\{\mu(x), \mu(y)\right\} + t \\ &= \min\left\{\mu(x) + t, \mu(y) + t\right\} > 1.\end{aligned}$$

Hence $x \to y \in \delta(t)$, and so $\delta(t)$ is a subalgebra of X for all $t \in A$. Therefore (δ, A) is a soft BCC-algebra over X.

Conversely, suppose that (δ, A) is a soft BCC-algebra over X. Let $x_0, y_0 \in X$ be such that $\mu(x_0 \to y_0) < \min\{\mu(x_0), \mu(y_0)\}$. Take $t_0 \in A$ such that

$$\mu(x_0 \to y_0) + t_0 < 1 < \min\left\{\mu(x_0), \mu(y_0)\right\} + t_0.$$

Then $\mu(x_0) + t_0 > 1$ and $\mu(y_0) + t_0 > 1$, which imply that $x_0, y_0 \in \delta(t_0)$. But $x_0 \to y_0 \notin \delta(t_0)$, a contradiction. Hence $\mu(x \to y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. Therefore μ is a fuzzy subalgebra of X. □

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