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A NUMERICAL METHOD FOR SINGULARLY PERTURBED SYSTEM OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS OF CONVECTION DIFFUSION TYPE WITH A DISCONTINUOUS SOURCE TERM

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ABSTRACT. In this paper, a numerical method that uses standard finite difference scheme defined on Shishkin mesh for a weakly coupled system of two singularly perturbed convection-diffusion second order ordinary differential equations with a discontinuous source term is presented. An error estimate is derived to show that the method is uniformly convergent with respect to the singular perturbation parameter. Numerical results are presented to illustrate the theoretical results.

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1. Introduction

Singular perturbation problems appear in many branches of applied mathematics, ike fluid dynamics,quantum mechanics, turbulent interaction of waves and currents, electro analytic chemistry etc. The solutions of such equations have boundary and interior layers. The convergence of the numerical approximations generated by standard numerical methods applied to such problems depends adversely on the singular perturbation parameter. Robust parameter-uniform numerical methods have been developed over the last 20 years. Most of this work has concentrated on problems involving single differential equations. Only a few authors have developed numerical methods for singularly perturbed system of ordinary differential equations. As said above the classical numerical methods fail to produce good approximations for singularly perturbed system of equations also. Various methods are available in the literature in order to

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obtain numerical solution to singularly perturbed system of second order differential equations when the source term are smooth on Ω [7]-[11]. Some authors have developed numerical methods for single equations with non-smooth data [4]. A.Tamilselvan et al [5] have developed a numerical method for a system of two second order ordinary differential equations of reaction-diffusion type with a discontinuous source term.

The objective of the present paper is to develop an ε-uniform numerical method for a system of singularly perturbed convection-diffusion equations with a discontinuous source term. This discontinuity gives rise to a weak interior layer in the exact solution of the problem, in addition to the boundary layer at the outflow boundary point.

Note: Through out this paper, C denotes a generic constant is independent of the singular perturbation parameter ε and the dimension of the discrete problem N. Let $y : D \longrightarrow \mathbb{R}$, where $D \subset \mathbb{R}$. The appropriate norm for studying the convergence of numerical solution to the exact solution of a singular perturbation problem is the maximum norm

$$\| y \| = \max_{x \in \bar{D}} |y(x)|.$$

In case of vectors $\bar{y} = (y_1, y_2)^T$, we define

$$|\bar{y}(x)| = (|y_1(x)|, |y_2(x)|)^T \quad \text{and} \quad \| \bar{y} \| = \max\{\| y_1 \|, \| y_2 \|\}.$$

2. Continuous problem

Find $y_1, y_2 \in C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ such that

$$P_1\bar{y}(x) \equiv -\varepsilon y_1''(x) - a_1(x)y_1'(x) + b_{11}(x)y_1(x) + b_{12}(x)y_2(x) = f_1(x),\ x \in \Omega^- \cup \Omega^+,$$
$$P_2\bar{y}(x) \equiv -\varepsilon y_2''(x) - a_2(x)y_2'(x) + b_{21}(x)y_1(x) + b_{22}(x)y_2(x) = f_2(x),\ x \in \Omega^- \cup \Omega^+, \tag{1}$$

$$y_1(0) = p, \quad y_1(1) = q, \quad y_2(0) = r, \quad y_2(1) = s, \tag{2}$$

where $\varepsilon > 0$ is a small parameter,

$$\begin{aligned} a_1(x) \geq \alpha_1 > 0, \quad a_2(x) \geq \alpha_2 > 0 \\ b_{12}(x) \leq 0, \quad b_{21}(x) \leq 0, \\ b_{11}(x) + b_{12}(x) \geq 0, \quad \text{and} \quad b_{21}(x) + b_{22}(x) \geq 0, \forall x \in \bar{\Omega}, \end{aligned} \tag{3}$$

where $\Omega = (0, 1)$, $\Omega^- = (0, d)$, $\Omega^+ = (d, 1)$, $d \in \Omega$. For $i, j = 1, 2, a_i(x), b_{ij}(x)$ are assumed to be smooth on $\bar{\Omega}$; $f_i(x)$ are assumed to be smooth on $\Omega^- \cup \Omega^+$; $f_i(x)$ and their derivatives are assumed to have right and left limits at $x = d$. The above weakly coupled system of singularly perturbed BVP can be written in the vector form as

$$\begin{aligned} \mathbf{P}\bar{y} = \quad & -\varepsilon \bar{y}''(x) - \mathbf{A}(x)\bar{y}'(x) + \mathbf{B}(x)\bar{y}(x) = \bar{f}(x), \\ & \bar{y}(0) = (p, r)^T, \quad \bar{y}(1) = (q, s)^T \end{aligned}$$

where $\mathbf{A}(x) = \begin{pmatrix} a_1(x) & 0 \\ 0 & a_2(x) \end{pmatrix}$, $\mathbf{B}(x) = \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix}$ and $\bar{f}(x) = (f_1(x), f_2(x))^T$. We denote the jump discontinuity at d in any function ω with $[\omega](d) = \omega(d+) - \omega(d-)$. The above differential operators satisfy the following maximum principle.

Theorem 1. *(Maximum principle) Suppose* $y_1, y_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$. *Further suppose that* $\bar{y} = (y_1, y_2)^T$ *satisfies* $\bar{y}(0) \geq \bar{0}$, $\bar{y}(1) \geq \bar{0}$, $P_1\bar{y}(x) \geq \bar{0}$, $P_2\bar{y}(x) \geq \bar{0}$ *and* $[\bar{y}'](d) \leq \bar{0}$. *Then if there exists a function* $\bar{s} = (s_1, s_2)^T, s_1, s_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$, *such that* $\bar{s}(0) > \bar{0}$, $\bar{s}(1) > \bar{0}$, $P_1\bar{s}(x) > \bar{0}$, $P_2\bar{s}(x) > \bar{0}$ *and* $[\bar{s}'](d) < \bar{0}$, *then* $\bar{y}(x) \geq \bar{0}$, $\forall x \in \bar{\Omega}$.

Proof. Define
$$\zeta = \max\{\max_{x\in\bar{\Omega}}(\frac{-y_1}{s_1})(x), \max_{x\in\bar{\Omega}}(\frac{-y_2}{s_2})(x)\}.$$
Assume that the theorem is not true. Then $\zeta > 0$ and there exists a point $x_0 \in \Omega$, such that either $(\frac{-y_1}{s_1})(x_0) = \zeta$ or $(\frac{-y_2}{s_2})(x_0) = \zeta$ or both. Also $(\bar{y} + \zeta\bar{s})(x) \geq \bar{0}$, $\forall x \in \bar{\Omega}$.

Case(i): $(\frac{-y_1}{s_1})(x_0) = \zeta$. That is, $(y_1 + \zeta s_1)(x_0) = 0$. Therefore $(y_1 + \zeta s_1)$ attains its minimum at $x = x_0$. Then,
$$\begin{aligned} 0 < P_1(\bar{y} + \zeta\bar{s})(x_0) &= -\varepsilon(y_1 + \zeta s_1)''(x_0) - a_1(x_0)(y_1 + \zeta s_1)'(x_0) \\ &\quad + b_{11}(x_0)(y_1 + \zeta s_1)(x_0) + b_{12}(x_0)(y_2 + \zeta s_2)(x_0) \leq 0, \end{aligned}$$
which is a contradiction.

Case(ii): Similar to **Case(i)**.

Case(iii): $(\frac{-y_1}{s_1})(x_0) = \zeta, x_0 = d$. That is, $(y_1 + \zeta s_1)(x_0) = 0$. Therefore $(y_1 + \zeta s_1)$ attains its minimum at $x = x_0$. Then,
$$0 \leq [(y_1 + \zeta s_1)'](x_0) = [y_1'](d) + \zeta[s_1'](d) < 0,$$
which is a contradiction.

Case(iv):Similar to **Case(iii)**. Hence $\bar{y}(x) \geq \bar{0}$, $\forall x \in \bar{\Omega}$. □

Corollary 2. *Consider the differential equation (1) subject to the conditions (2) - (3).Let* $\bar{s} = (s_1, s_2)^T$ *where*
$$s_1 = s_2 = \begin{cases} \frac{1}{4} - \frac{x}{8} + \frac{d}{8}, & x \in \Omega^- \cup \{d\}, \\ \frac{1}{4} - \frac{x}{4} + \frac{d}{4}, & x \in \Omega^+ \cup \{1\}. \end{cases}$$
Then the above maximum principle is true for the BVP (1).

Lemma 1. *If* $y_1, y_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ *then*
$$|y_i(x)| \leq C \max\{|y_1(0)|, |y_2(0)|, |y_1(1)|, |y_2(1)|, \| P_1\bar{y} \|_{\Omega^-\cup\Omega^+}, \| P_2\bar{y} \|_{\Omega^-\cup\Omega^+}\},$$
$x \in \bar{\Omega}$, $i = 1, 2$.

Proof. Let $A = C \max\{|y_1(0)|, |y_2(0)|, |y_1(1)|, |y_2(1)|, \| P_1\bar{y} \|_{\Omega^-\cup\Omega^+}, \| P_2\bar{y} \|_{\Omega^-\cup\Omega^+}\}, x \in \bar{\Omega}$, $i = 1, 2$.

Define barrier functions $\bar{w}^{\pm}(x) = (w_1^{\pm}(x), w_2^{\pm}(x))^T$ as

$$w_1^{\pm}(x) = As_1(x) \pm y_1(x), \quad \text{and} \quad w_2^{\pm}(x) = As_2(x) \pm y_2(x),$$

and observe that $\bar{w}^{\pm}(0) \geq \bar{0}$, $\bar{w}^{\pm}(1) \geq \bar{0}$, $P_1\bar{w}^{\pm}(x) \geq \bar{0}$, $P_2\bar{w}^{\pm}(x) \geq \bar{0}$. and $[\bar{w}^{\pm}]'(d) \leq \bar{0}$. Then $\bar{w}^{\pm}(x) \geq \bar{0}, \forall x \in \bar{\Omega}$ by Theorem 1, which completes the proof. □

The sharper bounds on the derivatives of the solution are obtained by decomposing the solution $\bar{y}$ into regular and singular components as, $\bar{y} = \bar{v} + \bar{w}$, where $\bar{v} = (v_1, v_2)^T$ and $\bar{w} = (w_1, w_2)^T$. The regular component $\bar{v}$ can be written in the form $\bar{v} = \bar{v}_0 + \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \bar{v}_1 + \begin{pmatrix} \varepsilon^2 & 0 \\ 0 & \varepsilon^2 \end{pmatrix} \bar{v}_2$, where $\bar{v}_0 = (v_{01}, v_{02})^T$, $\bar{v}_1 = (v_{11}, v_{12})^T$, $\bar{v}_2 = (v_{21}, v_{22})^T$ are defined respectively to be the solutions of the problems

$$-\mathbf{A}\bar{v}_0' + \mathbf{B}\bar{v}_0 = \bar{f}, \quad \bar{v}_0(1) = \bar{y}(1),\, x \in \Omega^- \cup \Omega^+,$$
$$-\mathbf{A}\bar{v}_1' + \mathbf{B}\bar{v}_1 = \bar{v}_0'', \quad \bar{v}_1(1) = \bar{0},\, x \in \Omega^- \cup \Omega^+$$

and

$$\mathbf{P}\bar{v}_2 = \bar{v}_1'', \quad x \neq d,\, \bar{v}_2(0) = \bar{v}_2(d) = \bar{v}_2(1) = \bar{0}.$$

Thus the regular component $\bar{v}$ is the solution of

$$\mathbf{P}\bar{v} = \bar{f}, x \in \Omega^- \cup \Omega^+,\, \bar{v}(0) = \bar{v}_0(0) + \varepsilon\bar{v}_1(0), \bar{v}(d) = \bar{v}_0(d) + \varepsilon\bar{v}_1(d), \bar{v}(1) = \bar{y}(1).$$

Further we decompose $\bar{w}$ as $\bar{w} = \bar{w}_1 + \bar{w}_2$ where $\bar{w}_1 = (w_{11}, w_{12})^T$, $\bar{w}_2 = (w_{21}, w_{22})^T$. Thus $w_1 = w_{11} + w_{21}, and w_2 = w_{12} + w_{22}$. Note that $\bar{w}_1$ is the solution of

$$\mathbf{P}\bar{w}_1 = \bar{0}, \quad x \in \Omega \quad \bar{w}_1(0) = \bar{u}(0) - \bar{v}(0), \quad \bar{w}_1(1) = \bar{0} \tag{4}$$

and $\bar{w}_2$ is the solution of

$$\mathbf{P}\bar{w}_2 = \bar{0}, \quad x \in \Omega^- \cup \Omega^+, \tag{5}$$
$$\bar{w}_2(0) = \bar{0}, \quad \bar{w}_2(1) = \bar{0} \tag{6}$$
$$[\bar{w}_2'](d) = -[\bar{v}'](d). \tag{7}$$

The following lemma provides the bound on the derivatives of the regular and singular components of the solution $\bar{y}$.

Lemma 2. *The solution $\bar{y}$ can be decomposed into the sum $\bar{y} = \bar{v} + \bar{w}$ where $\bar{v}$ and $\bar{w}$ are regular and singular components. Further, regular components and their derivatives satisfy the bounds*

$$\| v_j^{(k)} \| \leq C(1 + \varepsilon^{2-k}), \quad k = 0, 1, 2, 3, \quad j = 1, 2,\, x \in \Omega.$$

Proof. Using appropriate barrier functions, applying Theorem 1 and adopting the method of proof used in [4] and [5], the present lemma can be proved. □

Lemma 3. *For each integer k, satisfying $0 \leq k \leq 3$, the solution $\bar{w}_1$ of (4) satisfies the bounds*

$$|w_{1j}^{(k)}(x)| \leq C\varepsilon^{-k} e^{-\alpha x/\varepsilon}, \quad \forall x \in \bar{\Omega}, \quad j = 1, 2.$$

Proof. Using appropriate barrier functions, applying Theorem 1 and adopting the method of proof used in [4] and [5] , the present lemma can be proved. □

Lemma 4. *For each integer k, satisfying $0 \leq k \leq 3$, the solution $\bar{w}_2$ of (5) satisfies the bounds for $j = 1, 2$*

$$|w_{2j}(x)| \leq C\varepsilon$$

$$|w_{2j}^{(k)}(x)| \leq \begin{cases} C\varepsilon^{1-k} e^{-\alpha x/\varepsilon}, & x \in \Omega^-, \\ C\varepsilon^{1-k} e^{-\alpha(x-d)/\varepsilon}, & x \in \Omega^+. \end{cases}$$

Proof. Using appropriate barrier functions, applying Theorem 1 and adopting the method of proof used in [4] and [5], the present lemma can be proved. □

3. Discrete problem

A fitted mesh method for the BVP (1)-(2) is now introduced. On Ω a piecewise uniform mesh of N mesh interval is constructed as follows.

The interval $\bar{\Omega}^-$ is subdivided into two subintervals $[0, \sigma_1]$ and $[\sigma_1, d]$ for some σ_1 satisfy $0 < \sigma_1 \leq \frac{d}{2}$. On each subinterval a uniform mesh with $N/4$ mesh-intervals is placed. The subinterval $[d, d+\sigma_2]$ and $[d+\sigma_2, 1]$ of $\bar{\Omega}^+$are treated analogously for some σ_2 that satisfy $0 < \sigma_2 \leq \frac{1-d}{2}$. The interior points of the mesh are denoted by

$$\Omega_\varepsilon^N = \{x_i : 1 \leq i \leq \frac{N}{2} - 1\} \cup \{x_i : \frac{N}{2} + 1 \leq i \leq N - 1\}.$$

Clearly $x_{N/2} = d$ and $\bar{\Omega}_\varepsilon^N = \{x_i\}_0^N$. It is fitted to the BVP (1)-(2) by choosing σ_1 and σ_2 to be the following functions of N and ε

$$\sigma_1 = \min\{\frac{d}{2}, \frac{\varepsilon}{\alpha} \ln N\} \quad \text{and} \quad \sigma_2 = \min\{\frac{1-d}{2}, \frac{\varepsilon}{\alpha} \ln N\},$$

where $\alpha = \min\{\alpha_1, \alpha_2\}$. We now introduce the following notations for the four mesh widths

$$h_1 = \frac{4\sigma_1}{N}, \quad h_2 = \frac{4(d-\sigma_1)}{N}, \quad h_3 = \frac{4\sigma_2}{N} \quad \text{and} \quad h_4 = \frac{4(1-d-\sigma_2)}{N}.$$

Then the fitted mesh method for the BVP (1)-(2) is

$$P_1^N \bar{Y}(x_i) \equiv -\varepsilon\delta^2 Y_1(x_i) - a_1(x_i) D^+ Y_1(x_i) + b_{11}(x_i) Y_1(x_i) + b_{12}(x_i) Y_2(x_i) = f_1(x_i),$$
$$P_2^N \bar{Y}(x_i) \equiv -\varepsilon\delta^2 Y_2(x_i) - a_2(x_i) D^+ Y_2(x_i) + b_{21}(x_i) Y_1(x_i) + b_{22}(x_i) Y_2(x_i) = f_2(x_i),$$
$$Y_1(x_0) = y_1(0), \quad Y_2(x_0) = y_2(0), \quad Y_1(x_N) = y_1(1), \quad Y_2(x_N) = y_2(1),$$

$$D^- \bar{Y}(x_{N/2}) = D^+ \bar{Y}(x_{N/2}). \tag{8}$$

The finite difference operator δ^2 is the central difference operator defined as

$$\delta^2 Y_j(x_i) = \frac{(D^+ - D^-)Y_j(x_i)}{(x_{i+1} - x_{i-1})/2}, \quad j = 1, 2, \quad \text{where}$$
$$D^+ Y_j(x_i) = \frac{Y_j(x_{i+1}) - Y_j(x_i)}{x_{i+1} - x_i} \quad \text{and} \quad D^- Y_j(x_i) = \frac{Y_j(x_i) - Y_j(x_{i-1})}{x_i - x_{i-1}}.$$

The difference operator $\mathbf{P}^N$ can be defined as

$$\mathbf{P}^N \bar{Y}(x_i) \equiv \begin{pmatrix} P_1^N \bar{Y}(x_i) \\ P_2^N \bar{Y}(x_i) \end{pmatrix} \equiv \begin{pmatrix} -\varepsilon\delta^2 & 0 \\ 0 & -\varepsilon\delta^2 \end{pmatrix} \bar{Y}(x_i) - \begin{pmatrix} a_1(x_i)D^+ & 0 \\ 0 & a_2(x_i)D^+ \end{pmatrix} \bar{Y}(x_i)$$
$$+ \begin{pmatrix} b_{11}(x_i) & b_{12}(x_i) \\ b_{21}(x_i) & b_{22}(x_i) \end{pmatrix} \bar{Y}(x_i) = \bar{f}(x_i).$$

Analogous to the continuous results stated in Theorem 1 and Lemma 1 one can prove the following results.

Theorem 3. (*Discrete maximum principle*) *Suppose that a mesh function $\bar{\omega}_i$ satisfies $\bar{\omega}_0 \geq \bar{0}, \bar{\omega}_N \geq \bar{0}, {P_1}^N \bar{\omega}_i \geq \bar{0}, {P_2}^N \bar{\omega}_i \geq \bar{0}$ for all $x_i \in \Omega_\varepsilon^N$ and $D^+ \bar{\omega}_{\frac{N}{2}} - D^- \bar{\omega}_{\frac{N}{2}} \leq \bar{0}$. Then if there exists a mesh function $\bar{s}_i$ such that $\bar{s}_0 > \bar{0}$, $\bar{s_N} > \bar{0}$, ${P_1}^N \bar{s}_i > \bar{0}$, ${P_2}^N \bar{s}_i > \bar{0}$ for all $x_i \in \Omega_\varepsilon^N$ and $D^+ \bar{s}_{\frac{N}{2}} - D^- \bar{s}_{\frac{N}{2}} \leq \bar{0}$, then $\bar{\omega}_i \geq \bar{0}$ for all $x_i \in \bar{\Omega}_\varepsilon^N$.*

Theorem 4. *If $\bar{Y}(x_i)$ is the solution of the problem (8), then*

$$|\bar{Y}(d)| \leq C.$$

The discrete solution $\bar{Y}(x_i)$ can be decomposed into the sum $\bar{Y}(x_i) = \bar{V}(x_i) + \bar{W}(x_i)$. Define the function $\bar{V}$ to be the solution of

$$\mathbf{P}^N \bar{V}(x_i) = \bar{f}(x_i), \quad \forall x_i \in \Omega_\varepsilon^N \setminus \{d\}$$
$$\bar{V}(0) = \bar{v}(0), \quad \bar{V}(d) = \bar{v}(d), \quad \bar{V}(1) = \bar{v}(1).$$

We define $\bar{W}$ to be the solution of

$$\mathbf{P}^N \bar{W}(x_i) = \bar{0}, \quad \forall x_i \in \Omega_\varepsilon^N \setminus \{d\} \tag{9}$$
$$\bar{W}(0) = \bar{w}(0), \quad \bar{W}(1) = \bar{w}(1) \tag{10}$$
$$[D\bar{W}(d)] = -[D\bar{V}(d)], \tag{11}$$

where, throughout this section, we denote the jump in the derivative of a mesh function Z at the point $x_i = d$ by

$$[D\bar{Z}(d)] = D^+ \bar{Z}(d) - D^- \bar{Z}(d).$$

Analogously to the continuous case we can further decompose $\bar{W}$ as $\bar{W} = \bar{W}_1 + \bar{W}_2$ The error in the numerical solution can be written in the form $(\bar{Y} - \bar{y})(x_i) = (\bar{V} - \bar{v})(x_i) + (\bar{W} - \bar{w})(x_i)$ where $\bar{W}_1$ is defined as the solution of

$$\mathbf{P}^N \bar{W}_1 = 0 \qquad \forall x_i \in \Omega_\varepsilon^N \cup \{d\} \tag{12}$$
$$\bar{W}_1(0) = \bar{w}(0), \qquad \bar{W}_1(1) = 0 \tag{13}$$

and $\bar{W}_2$ is defined as the solution of

$$\mathbf{P}^N \bar{W}_2 = 0 \qquad \forall x_i \in \Omega_\varepsilon^N \cup \{d\} \tag{14}$$

$$\bar{W}_2(0) = \bar{w}(0), \qquad \bar{W}_2(1) = \bar{0}, \tag{15}$$

$$[D\bar{W}_2(d)] = -[D\bar{V}(d)] - [D\bar{W}_1(d)]. \tag{16}$$

Lemma 5. *At each mesh point $x_i \in \bar{\Omega}_\varepsilon^N$, the error of the regular component satisfies the estimate*

$$|(\bar{V} - \bar{v})(x_i)| \le \begin{pmatrix} CN^{-1}(d - x_i) \\ CN^{-1}(d - x_i) \end{pmatrix} \quad on \quad \Omega^-$$

$$|(\bar{V} - \bar{v})(x_i)| \le \begin{pmatrix} CN^{-1}(1 - x_i) \\ CN^{-1}(1 - x_i) \end{pmatrix} \quad on \quad \Omega^+.$$

Proof. Considering the differential and difference equations, and following the usual procedure, we get

$$\begin{aligned} | \mathbf{P}^N(\bar{V} - \bar{v})(x_i) | &= | (\mathbf{P} - \mathbf{P}^N)\bar{v}(x_i) | \\ &\le \begin{pmatrix} \frac{\varepsilon}{3}(x_{i+1} - x_{i-1}) \parallel v_1^{(3)} \parallel + \frac{a_1(x_i)}{2}(x_i - x_{i-1}) \parallel v_1^{(2)} \parallel \\ \frac{\varepsilon}{3}(x_{i+1} - x_{i-1}) \parallel v_2^{(3)} \parallel + \frac{a_2(x_i)}{2}(x_i - x_{i-1}) \parallel v_2^{(2)} \parallel \end{pmatrix} \\ &\le \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix}. \end{aligned} \tag{17}$$

Now, defining the mesh functions $\bar{\Psi}^{\pm}(x_i)$ as

$$\bar{\Psi}^{\pm}(x_i) = \begin{pmatrix} CN^{-1}(d - x_i) \\ CN^{-1}(d - x_i) \end{pmatrix} \pm (\bar{V} - \bar{v})(x_i),$$

and observing that $\bar{\Psi}^{\pm}(x_0) = \bar{0}$, $\bar{\Psi}^{\pm}(x_N) > \bar{0}$, $\mathbf{P}^N\bar{\Psi}^{\pm}(x_i) \ge \bar{0}$ and $D^+\bar{\Psi}^{\pm}_{\frac{N}{2}} - D^-\bar{\Psi}^{\pm}_{\frac{N}{2}} \le \bar{0}$. We, by Theorem 3, get $\bar{\Psi}^{\pm}(x_i) \ge \bar{0}$, $x_i \in \Omega^-$, which leads to the desired result.

Similar proof can be given on Ω^+. □

Lemma 6. *At each mesh point $x_i \in \Omega_\varepsilon^N$, the error of the singular component satisfies the estimate*

$$|(\bar{W}_1 - \bar{w}_1)(x_i)| \le \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}.$$

Proof. Using the procedure adopted in [4], proof follows. □

Note that the jump at $x = d$ in the derivative of the weak interior layer function $\bar{w}_2$. In the following lemma we establish the bound for $\bar{W}_2$.

Lemma 7. *Suppose $\bar{W}_2$ be the solution of (14), then it satisfies the following $\varepsilon-$uniform bound*

$$|[D\bar{W}_2(d)]| \le \begin{pmatrix} C(1 + \varepsilon^{-1}N^{-1}) \\ C(1 + \varepsilon^{-1}N^{-1}) \end{pmatrix}.$$

Proof. At the point $x = d$, first we consider

$$D^-\bar{V}(d) = D^-(\bar{V} - \bar{v})(d) + D^-\bar{v}(d).$$

Note that

$$\| \bar{v}' \|_{\Omega^-} \le \begin{pmatrix} C \\ C \end{pmatrix}.$$

Hence, $|D^-\bar{v}(d)| \le \begin{pmatrix} C \\ C \end{pmatrix}$ and

$$|D^-(\bar{V} - \bar{v})(d)| = |\frac{(\bar{V} - \bar{v})(d) - (\bar{V} - \bar{v})(d - H_1)}{H_1}| \le \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix}.$$

Therefore, $|D^-\bar{V}(d)| \le \begin{pmatrix} C(1 + N^{-1}) \\ C(1 + N^{-1}) \end{pmatrix}$.

Now we consider

$$D^+\bar{V}(d) = D^+(\bar{V} - \bar{v})(d) + D^+\bar{v}(d)$$

and $\| \bar{v}' \|_{\Omega^+} \le \begin{pmatrix} C \\ C \end{pmatrix}$. As in [1, lemma 3.14], $|\varepsilon D^+(\bar{V} - \bar{v})(d)| \le \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix}$. Therefore,

$$|D^+\bar{V}(d)| \le \begin{pmatrix} \frac{CN^{-1}}{\varepsilon} \\ \frac{CN^{-1}}{\varepsilon} \end{pmatrix} + \begin{pmatrix} C \\ C \end{pmatrix} = \begin{pmatrix} C(1 + \varepsilon^{-1}N^{-1}) \\ C(1 + \varepsilon^{-1}N^{-1}) \end{pmatrix}.$$

On Ω^-,

$$|\bar{W}_1(x_i)| \le \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix}$$

implies that $|D^-\bar{W}_1(d)| \le \begin{pmatrix} C \\ C \end{pmatrix}$. On Ω^+, $D^+W_1(d) = D^+(\bar{W}_1 - \bar{w}_1)(d) + D^+\bar{w}_1(d)$. Note that $\| \bar{w}_1' \| \le \begin{pmatrix} C \\ C \end{pmatrix}$. Hence $|D^+\bar{W}_1(d)| \le |D^+(\bar{W}_1 - \bar{w}_1)(d)| + \begin{pmatrix} C \\ C \end{pmatrix}$. Using the known results and the fact that $\| \bar{w}_1^{(k)} \|_{\Omega^+} \le \begin{pmatrix} C\varepsilon^{-k}e^{-\alpha d/\varepsilon} \\ C\varepsilon^{-k}e^{-\alpha d/\varepsilon} \end{pmatrix}$, we can show that $|D^+(\bar{W}_1 - \bar{w}_1)(x_i)| \le \begin{pmatrix} C \\ C \end{pmatrix}$, which implies that $|D^+\bar{W}_1(d)| \le \begin{pmatrix} C \\ C \end{pmatrix}$. Therefore,

$$|[D\bar{W}_2(d)]| \le \begin{pmatrix} C(1 + \varepsilon^{-1}N^{-1}) \\ C(1 + \varepsilon^{-1}N^{-1}) \end{pmatrix}.$$

□

Lemma 8. *The following $\varepsilon-$uniform bound*

$$|\bar{W}_2(x_i)| \le C\varepsilon|[D\bar{W}_2(d)]|$$

is valid, where $\bar{W}_2$ is the solution of (14).

Proof. Consider the following barrier functions $\phi_j^{\pm}, \quad j = 1, 2$ where

$$\phi_j^{\pm}(x_i) = \frac{C\varepsilon|[DW_{j2}(d)]|}{\alpha} \begin{cases} 1, & x_i \leq d \\ \psi_j(x_i), & x_i \geq d \end{cases} \pm W_{2j}$$

where $\bar{\psi} = (\psi_1, \psi_2)^T$ is the solution of

$$\begin{aligned} -\varepsilon\delta^2\bar{\psi}(x_i) - \alpha D^+\bar{\psi}(x_i) = \bar{0}, \quad x_i \in \Omega^N \cap \Omega^+, \\ \bar{\psi}(d) = \bar{1}, \quad \bar{\psi}(1) = \bar{0}, \\ D^+\bar{\psi}(x_i) < \bar{0}, \quad x_i \geq d. \end{aligned}$$

Using the procedure adopted in [4], the remaining proof can be given. □

Lemma 9. *At each mesh point $x_i \in \Omega_\varepsilon^N$, the error of the regular component satisfies the estimate*

$$|(\bar{W}_2 - \bar{w}_2)(x_i)| \leq \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}.$$

Proof. From (7), we have $[v'(d)] + [w_2'(d)] = 0$ and so

$$\begin{aligned} [D(\bar{W}_2 - \bar{w}_2)(d) &= [D\bar{W}_2(d)] - [D\bar{w}_2(d)] \\ &= [\bar{v}'(d)] - [D\bar{V}(d)] + [\bar{w}_d'(d)] - [D\bar{W}_2(d)] - [D\bar{W}_1(d)]. \end{aligned}$$

Note that

$$[\bar{v}'(d)] - [D\bar{V}(d)] = \bar{v}'(d+) - D^+\bar{v}(d) + D^-\bar{v}(d) - \bar{v}'(d-) + [D(\bar{V} - \bar{v})(d)].$$

From the proof of previous lemma,

$$|[D(\bar{V} - \bar{v})(d)]| \leq \begin{pmatrix} C\varepsilon^{-1}N^{-1} \\ C\varepsilon^{-1}N^{-1} \end{pmatrix}$$

and

$$|[\bar{v}'(d)] - [D\bar{v}(d)]| \leq \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix}.$$

Hence

$$|[\bar{v}'(d)] - [D\bar{V}(d)]| \leq \begin{pmatrix} C\varepsilon N^{-1} \\ C\varepsilon N^{-1} \end{pmatrix}.$$

Similarly,

$$\begin{aligned} |[\bar{w}_2'(d)] - [D\bar{w}_2(d)]| &\leq |D^+\bar{W}_2(d) - \bar{w}_2'(d+)| + |D^-\bar{W}_2(d) - \bar{w}_2'(d-)| \\ &\leq \begin{pmatrix} Ch_2|w_2^{(2)}(d)| + CH_1|w_2^{(2)}(d-)| \\ Ch_2|w_2^{(2)}(d)| + CH_1|w_2^{(2)}(d-)| \end{pmatrix} \\ &\leq \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}, \end{aligned}$$

since $e^{-\alpha(d-H_1)/\varepsilon} = e^{-\alpha(d-\frac{4d}{N})/\varepsilon} \leq e^{-\alpha d/2\varepsilon}$. Finally,

$$|[D\bar{w}_1(d)]| \le \begin{pmatrix} C(h_2+H_1)|\bar{w}_1^{(2)}(d-H_1)| \\ C(h_2+H_1)|\bar{w}_1^{(2)}(d-H_1)| \end{pmatrix}$$
$$\le \begin{pmatrix} C(h_2+H_1)\varepsilon^{-2}e^{-\alpha(d-H_1)/\varepsilon} \\ C(h_2+H_1)\varepsilon^{-2}e^{-\alpha(d-H_1)/\varepsilon} \end{pmatrix}$$
$$\le \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}.$$

Also, we know that
$$|\varepsilon[D(\bar{W}_1-\bar{w}_1)(d)]| \le \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}.$$

Thus
$$|[D(\bar{W}_2-\bar{w}_2)(d)]| \le \begin{pmatrix} \frac{CN^{-1}\ln N}{\varepsilon} \\ \frac{CN^{-1}\ln N}{\varepsilon} \end{pmatrix}.$$

Let us now consider the truncation error at the interval mesh points. Using standard truncation error bounds and the bounds on the derivatives of $\bar{w}_2$, for $x_i \in (0,\sigma_1)$, we get
$$|P^N(\bar{W}_2-\bar{w}_2)(x_i)| \le \begin{pmatrix} Ch_1\varepsilon^{-1}\sigma_1 \\ Ch_1\varepsilon^{-1}\sigma_1 \end{pmatrix}$$
$$\le \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}.$$

For $x_i \in [\sigma_1, d)$,
$$|P^N(\bar{W}_2-\bar{w}_2)(x_i)| \le \begin{pmatrix} C \parallel \varepsilon\bar{w}_2'' \parallel_{(x_{i-1},x_{i+1})} + C \parallel \bar{w}_2' \parallel_{[x_i,x_{i+1}]} \\ C \parallel \varepsilon\bar{w}_2'' \parallel_{(x_{i-1},x_{i+1})} + C \parallel \bar{w}_2' \parallel_{[x_i,x_{i+1}]} \end{pmatrix}$$
$$\le \begin{pmatrix} Ce^{-\alpha\sigma_1/\varepsilon} \\ Ce^{-\alpha\sigma_1/\varepsilon} \end{pmatrix}$$
$$\le \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix}.$$

For $x_i \in (d, d+\sigma_2)$,
$$|P^N(\bar{W}_2-\bar{w}_2)| \le \begin{pmatrix} Ch_2\varepsilon^{-1} \\ Ch_2\varepsilon^{-1} \end{pmatrix} \le \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}$$

and for $x_i \in [d+\sigma_2, 1)$,
$$|P^N(\bar{W}_2-\bar{w}_2)| \le \begin{pmatrix} C \parallel \varepsilon\bar{w}_2'' \parallel_{(x_{i-1},x_{i+1})} + C \parallel \bar{w}_2' \parallel_{[x_i,x_{i+1}]} \\ C \parallel \varepsilon\bar{w}_2'' \parallel_{(x_{i-1},x_{i+1})} + C \parallel \bar{w}_2' \parallel_{[x_i,x_{i+1}]} \end{pmatrix}$$
$$\le \begin{pmatrix} Ce^{-\alpha(\sigma_2-h_2)/\varepsilon} \\ Ce^{-\alpha(\sigma_2-h_2)/\varepsilon} \end{pmatrix}$$
$$\le \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix}.$$

Combining all these give

$$|P^N(\bar{W}_2 - \bar{w}_2)| \leq \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}$$

and

$$|[D(\bar{W}_2 - \bar{w}_2)(d)]| \leq \begin{pmatrix} C\dfrac{N^{-1}\ln N}{\varepsilon} \\ C\dfrac{N^{-1}\ln N}{\varepsilon} \end{pmatrix}.$$

Consider the following barrier function

$$\phi_j(x_i) = CN^{-1}\ln N \begin{cases} 1, \; x_i \leq d \\ \psi_j, \; x_i \geq d \end{cases} + CN^{-1}\ln N(1 - x_i), \quad \text{for } j = 1, 2$$

where ψ_j is the solution of the problem

$$-\varepsilon\delta^2\psi_j - \alpha D^+\psi_j = 0, \quad \psi_j(d) = 1, \; \psi_j(1) = 0, \quad \text{for } j = 1, 2.$$

The proof is completed in the usual way using theorem 3. □

Theorem 5. *Let* $\bar{y}(x) = (y_1(x), y_2(x))^T$, $x \in \bar{\Omega}$ *be the solution of (1). Further let* $\bar{Y}(x_i) = (Y_1(x_i), Y_2(x_i))^T$, $x_i \in \bar{\Omega}_\varepsilon^N$ *be the numerical solution of problem (8). Then we have*

$$\sup_{0<\varepsilon\leq 1} \| Y_1 - y_1 \|_{\Omega_\varepsilon^N} \leq CN^{-1}\ln N$$

and

$$\sup_{0<\varepsilon\leq 1} \| Y_2 - y_2 \|_{\Omega_\varepsilon^N} \leq CN^{-1}\ln N.$$

Proof. Combining the Lemmas 5, 6 and 9 we get the required result. □

4. Numerical results

In this section, two examples are given to illustrate the numerical method discussed in this paper.

Consider the following singularly perturbed boundary value problems.

Example 1.

$$-\varepsilon y_1''(x) - 0.8y_1'(x) + 3y_1(x) - y_2(x) = \begin{cases} 2, & x < 0.5 \\ -1.0, & x \geq 0.5, \end{cases}$$

$$-\varepsilon y_2''(x) - y_2'(x) - y_1(x) + 3y_2(x) = \begin{cases} 1.8, & x < 0.5 \\ -0.8, & x \geq 0.5, \end{cases}$$

$$y_1(0) = 0, \quad y_1(1) = 2, \quad y_2(0) = 0, \quad y_2(1) = 2.$$

Example 2.

$$-\varepsilon y_1''(x) - (1+x)y_1'(x) + (2+x)y_1(x) - (1+x)y_2(x) = \begin{cases} 1+x, & x < 0.5 \\ -e^x, & x \geq 0.5, \end{cases}$$

$$-\varepsilon y_2''(x) + (2-x)y_2'(x) - (1+x)y_1(x) + (2+x)y_2(x) = \begin{cases} x, & x < 0.5 \\ 2x-1, & x \geq 0.5, \end{cases}$$

$y_1(0) = 0, \quad y_1(1) = 1, \quad y_2(0) = 0, \quad y_2(1) = 1.$

The ε−uniform rate of convergence is determined using the double mesh error

$$D_{\varepsilon,j}^N = \max_{\varepsilon} |Y_{\varepsilon,j}^N - \bar{Y}_{\varepsilon,j}^{2N}|$$

which is the difference between the values of the j^{th} component of the solution on a mesh of N points and the interpolated value of the solution, at the same point, on a mesh of $2N$ points. Here the range of the singular perturbation parameter is taken as $\varepsilon = \{2^{-1}, ..., 2^{-30}\}$.

For each values of N,

$$D_j^N = \max_{\varepsilon} D_{\varepsilon,j}^N, \quad j = 1,2; \quad p_j^N = \log_2(\frac{D_j^N}{D_j^{2N}}), \quad j = 1,2.$$

are computed.

TABLE 1. Values of $D_{\varepsilon,1}^N$, D_1^N and p_1^N for the solution component Y_1

	Number of mesh points N					
	32	64	128	256	512	1024
D_1^N	3.0993e-002	1.7890e-002	1.1151e-002	6.4060e-003	3.6164e-003	2.0081e-003
p_1^N	**0.7928**	**0.6820**	**0.7997**	**0.8249**	**0.8487**	-

TABLE 2. Values of $D_{\varepsilon,2}^N$, D_2^N and p_2^N for the solution component Y_2

	Number of mesh points N					
	32	64	128	256	512	1024
D_2^N	2.7238e-002	1.5904e-002	9.6979e-003	5.9419e-003	3.4324e-003	1.9536e-003
p_2^N	**0.7762**	**0.7136**	**0.7067**	**0.7917**	**0.8131**	-

TABLE 3. Values of $D_{\varepsilon,2}^N$, D_2^N and p_1^N for the solution component Y_1 of example2

	Number of mesh points N					
	32	64	128	256	512	1024
D_2^N	3.5256e-002	2.1983e-002	1.3450e-002	7.6762e-003	4.3244e-003	2.4092e-003
p_1^N	**0.6815**	**0.7088**	**0.8091**	**0.8279**	**0.8440**	-

TABLE 4. Values of $D_{\varepsilon,2}^N$, D_2^N and p_2^N for the solution component Y_2 of example 2

	Number of mesh points N					
	32	64	128	256	512	1024
D_2^N	2.5072e-002	1.6011e-002	1.1992e-002	7.5088e-003	4.6753e-003	2.7115e-003
p_2^N	**0.6470**	**0.4170**	**0.6754**	**0.6835**	**0.7859**	-

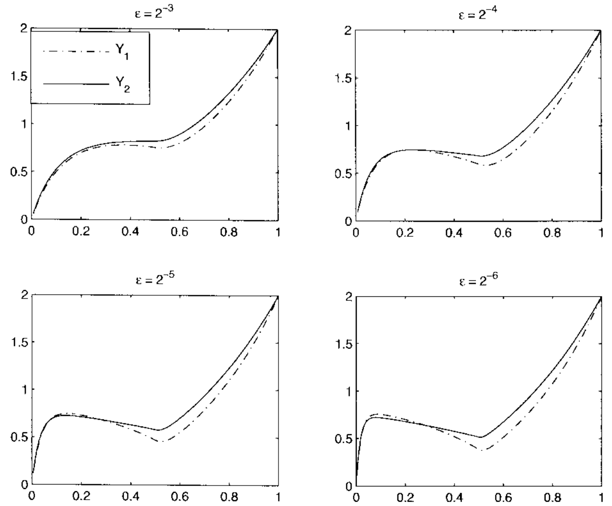

FIGURE 1. For various values of ε and $N = 128$, the solution graph of Y_1 and Y_2 of example1

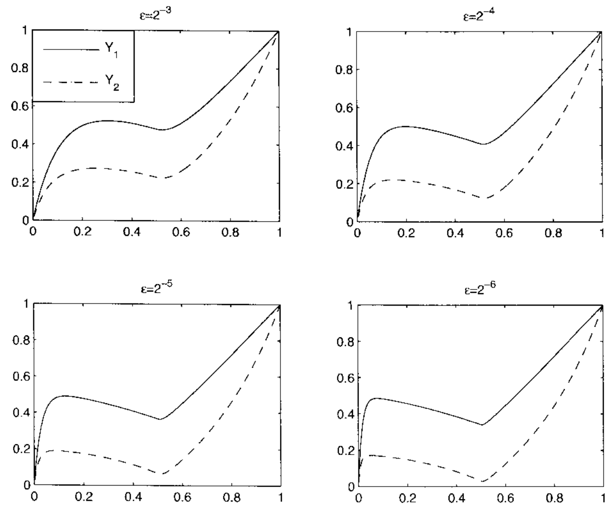

FIGURE 2. For various values of ε and $N = 128$, the solution graph of Y_1 and Y_2 of example2

5. Conclusion

A two point boundary value problem for a system of two second order ordinary differential equations of convection diffusion type with a discontinuous source term is considered. We constructed a numerical method for solving this problem,which generates $\varepsilon-$uniform convergent numerical approximations to the solution. An error estimate is derived to show that the method is uniformly convergent with respect to the singular perturbation parameter. Two numerical examples are presented which are in agreement with the theoretical results.

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