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EXISTENCE OF SOLUTIONS FOR THREE-POINT BOUNDARY VALUE PROBLEM AT RESONANCE

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ABSTRACT. In this paper, we study the existence of solutions for three-point boundary value problem at resonance by using the continuation theorem of Mawhin. Some known results are improved.

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1. Introduction

Il'in and Moiseev [6] firstly studied the following multi-point boundary value problem

$$x'' = f(t,x,x') + e(t), \quad t \in (0,1), \tag{1}$$

$$x'(0) = 0\ ,\ x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i). \tag{2}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, a_i \in R$ with all of the a_i's having the same sign. Moreover, the authors [6] proved that the existence of sulutions for boundary value problem (1)(2) can be studied via the existence of solutions for the equation (1) subject to the three-point boundary value condition.

$$x'(0) = 0\ ,\ x(1) = \alpha x(\eta), \quad \eta \in (0,1) \tag{3}$$

Motivated by [6], many people investigated the three-point boundary value problem and got a lot of results, please see [1-2,4,8-9]. In 1997, by using the continuation theorem of Mawhin, W.Feng and J.R.L.Webb [8] studied the existence of solutions for the boundary value problem(1)(3)at resonance ($\alpha = 1$) under the following assumptions:

(A_1) there exists a constant $M \geq 0$, such that

$$x[f(t,x,0+e(t))] > 0, \quad for\ |x| > M,\ t \in [0,1];$$

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(A_2) $pg(t,x,p) \leq 0$, for all $(t,x,p) \in [0,1] \times [-M,M] \times R$;

(A_3) $|h(t,x,p)| \leq a(t)|x| + b(t)|p| + u(t)|x|^r + v(t)|p|^k + c(t)$, for $(t,x,p) \in [0,1] \times [-M,M] \times R$, where $0 \leq r,\ k < 1$ and $a,b,u,v,c \in L^1[0,1]$.
where f having the decomposition, that is, $f(t,x,p) = g(t,x,p) + h(t,x,p)$

In this paper, by using Gronwall inequality, we obtain the existence of solutions for three-point boundary value problem (1)(3)at resonance and improve the known resluts.

2. Background notation and lemmas

For convenience of the readers, we give some notation and the continuation theorem of Mawhin.

In the following, we shall use the classical Banach spaces $C^1[0,1], L^1[0,1]$. For $x(t) \in C^1[0,1]$, we use the norms $\|x\|_\infty = \max_{t\in[0,1]}|x(t)|$ and $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$. We denote the norm in $L^1[0,1]$ by $\|\cdot\|_1$ and use the Sobelev space $W^{2,1}(0,1)$ defined by

$$W^{2,1}(0,1) = \{x : [0,1] \to R|\ x, x' \text{ absolutely continuous on } [0,1], x'' \in L^1[0,1]\}$$

with the norm $\|x\|_{W^{2,1}} = \sum_{j=0}^{2} \int_0^1 |x^{(j)}(t)|dt$.

Let X and Z be real Banach spaces and $L : D(L) \subset X \to Z$ be a linear operator which is Fredholm of index zero (that is, Im (L)(the image of L) is closed in Z, and Ker(L)(the kernel of L) and $Z/Im(L)$ (the co-kernel of L) are finite dimensional with equal dimension). Let $P : X \to \text{Ker}(L)$ and $Q : Z \to Z_1$, where $X = \text{Ker}(L) \oplus X_1$ and $Z = \text{Im}(L) \oplus Z_1$, be continous projections. Let L_1 denote L restricted to $D(L) \cap X_1$, an invertible operator into Im(L), and write $K = L_1^{-1}$. Let Ω be a bounded, open subset of X such that $D(L) \cap \Omega \neq \emptyset$ and let $N : \bar{\Omega} \to Z$ be an L-compact mapping, that is, the maps $QN : \bar{\Omega} \to Z$ and $K(I-Q)N : \bar{\Omega} \to X$ are compact.

Lemma 1. [3] *Let L be Fredholm of index zero and let N be L-compact on $\bar{\Omega}$. Assume that the following assumptions are satisfied.*

(B_1) $Lx + \lambda Nx \neq 0$, for any $(x,\lambda) \in [(D(L)/Ker(L)) \cap \partial\Omega] \times [0,1]$,

(B_2) $Nx \notin Im(L)$, for each $x \in Ker(L) \cap \partial\Omega$,

(B_3) $deg(QN|_{Ker(L)}, \Omega \cap Ker(L), 0) \neq 0$, where $Q : Z \to Z$ is a continuous projection as above.
Then the equation $Lx + Nx = 0$ has at least one solution in $D(L) \cap \bar{\Omega}$.

Lemma 2. *(Gronwall inequality) Suppose that $g(t)$, $\varphi(t)$ be continuous on $[a,b]$ and the following inequality*

$$\varphi(t) \leq \lambda + \int_a^t (r + g(\tau)\varphi(\tau))d\tau$$

holds, where $g(t) \geq 0$, $\lambda \geq 0$, and $r \geq 0$. Then

$$\varphi(t) \leq [\lambda + r(b-a)]e^{\int_a^t g(\tau)d\tau} \qquad a \leq t \leq b.$$

3. Existence results

Theorem 1. *Assume that $f : [0,1] \times R^2 \to R$ is continuous and f has the decomposition $f(t,x,p) = g(t,x,p) + h(t,x,p)$ which satisfying*

(H_1) there exists a constant $M \geq 0$, such that

$$x[f(t,x,0+e(t))] > 0, \quad for\ |x| > M,\ t \in [0,1],$$

(H_2) $pg(t,x,p) \leq 0$, for all $(t,x,p) \in [0,1] \times [-M,M] \times R$,

(H_3) $|h(t,x,p)| \leq a(t)|x| + b(t)|p| + u(t)|x|^r + v(t)|p|^k + c(t)$, $\forall (t,x,p) \in [0,1] \times [-M,M] \times R$, where $a,b,u,v,c \in L^1[0,1]$, $0 \leq k \leq 1$.

Then the BVP(1)(3) with $\alpha = 1$ has at least one solution in $C^1[0,1]$.

Proof. Set $X = C^1[0,1]$ and $Z = L^1[0,1]$, then X and Z are Banach spaces. Define L to be the linear operator from $D(L) \subset X$ to Z with

$$D(L) = \{x(t) \in W^{2,1}[0,1] | x'(0) = 0,\ x(1) = x(\eta)\}$$

that is, $Lx = x''$, $x \in D(L)$. We define $N : X \to Z$ by setting

$$N(x)(t) = -f(t,x(t),x'(t)) - e(t),\ t \in (0,1).$$

The continuous projections $P : X \to \mathrm{Ker}(L)$, $Q : Z \to \mathrm{Im}(Q)$ can be defined by

$$(Px)(t) = x(0),$$

$$Qy = \frac{2}{1-\eta^2} \int_\eta^1 \int_0^\tau y(s)dsd\tau.$$

It can be proved that L is Fredholm of index zero [see 9]. We shall prove the conditions of Lemma 1 are satisfied.

Consider the following nomotopic equation of the equation of (1)

$$x'' = \lambda[f(t,x(t),x'(t)) + e(t)],\ \lambda \in [0,1] \tag{4}$$

Then the equation (4) is equivalent to the operator equation:

$$Lx + \lambda Nx = 0.$$

Step 1. Let $U_1 = \{x(t) \in D(L) : Lx + \lambda Nx = 0,\ \lambda \in [0,1]\}$. We can prove that U_1 is bounded.

For any $x(t) \in U_1$, let $t_0 \in [0,1]$ be such that $|x(t_0)| = \max_{t\in[0,1]} |x(t)|$. Assume that $|x(t_0)| > M$, then we have the following two cases:

Case 1. $t_0 \neq 0$.

If $x(t_0) > M$, then $x'(t_0) = 0$, $x''(t_0) \leq 0$, so we have

$$0 \geq x(t_0)x''(t_0) = \lambda x(t_0)[f(t_0,x(t_0),0) + e(t_0)] > 0$$

a contradiction.

If $x(t_0) < -M$, then $x'(t_0) = 0$, $x''(t_0) \geq 0$, so we have

$$0 \geq x(t_0)x''(t_0) = \lambda x(t_0)[f(t_0,x(t_0),0) + e(t_0)] > 0$$

a contradiction again.

Case 2. $t_0 = 0$.

If $x(0) > M$, then by the assumption (H_1), we have

$$x''(0) = \lambda[f(0, x(0), 0) + e(0)] > 0.$$

This implies that $x'(t)$ is increasing for t small enough. Since $x'(0) = 0$, we can get that $x'(t) > 0$ for t small enough. Thus $x(t)$ is increasing, contradicting $x(0) = \max_{t\in[0,1]} |x(t)|$.

If $x(0) < -M$, then a similar argument shows that $x(t)$ is decreasing and a contradiction is obtained. Thus we have shown

$$\|x\|_\infty \leq M, \quad \text{for each } x \in U_1. \tag{5}$$

Next, for $x \in U_1$, we shall prove that there exists a constant $M_1 \geq 0$ such that $\|x'\|_\infty \leq M$.

Multiplying (4) by $x'(t)$, we show

$$\begin{aligned} x'x'' &= \lambda x'[f(t, x, x') + e(t)] \\ &= \lambda x' g(t, x, x') + \lambda x' h(t, x, x') + \lambda x' e(t). \end{aligned} \tag{6}$$

If for any $t \in [0, 1]$, then $|x'(t)| \leq 1$. This implies that $\|x'\|_\infty \leq 1$. Otherwise, there exists $t_1 \in [0, 1]$ such that $|x'(t_1)| = \max_{t\in[0,1]} |x'(t)| > 1$. Without loss of generality, we suppose $x'(t_1) > 1$.

Since $x'(0) = 0$ and $x'(t)$ is continuous, there exists an interval $[t_2, t_1] \subset [0, 1]$ such that $x'(t_2) = 1$ and $x'(t) \geq 1$, for any $t \in [t_2, t_1]$. By (H_2) and (H_3), intergrating on both sides of (6) from t_2 to t, we obtain

$$\begin{aligned} \frac{1}{2}(x'(t))^2 &= \frac{1}{2}(x'(t_2))^2 + \lambda \int_{t_2}^{t} x' g(s, x(s), x'(s))ds \\ &\quad + \lambda \int_{t_2}^{t} x' h(s, x(s), x'(s))ds + \lambda \int_{t_2}^{t} x' e(s)ds \\ &\leq \frac{1}{2} + \int_{t_2}^{t} |x'||h|ds + \int_{t_2}^{t} |x'||e|ds \\ &\leq \frac{1}{2} + \int_{t_2}^{t} [a(s)|x'||x| + b(s)|x'|^2 + u(s)|x'||x|^r \\ &\quad + v(s)|x'||x'|^k + c(s)|x'| + |e(s)||x'|]ds \\ &\leq \frac{1}{2} + \int_{t_2}^{t} |x'|^2[a(s)M + b(s) + u(s)M^r + v(s) + c(s) + |e(s)|]ds \\ &= \frac{1}{2} + \int_{t_2}^{t} |x'|^2 G(s)ds, \end{aligned} \tag{7}$$

where $G(s) = a(s)M + b(s) + u(s)M^r + v(s) + c(s) + |e(s)|$. It implies that $G(s) \in L^1[0, 1]$ and $G(s) > 0$, for $s \in [0, 1]$ hold.

Let $u(t) = x'(t)^2$. Then

$$u(t) \leq 1 + \int_{t_2}^{t} 2G(s)u(s)ds \leq 1 + \int_{0}^{t} 2G(s)u(s)ds.$$

By Gronwall inequality, we show

$$u(t) \leq e^{\int_0^t 2G(s)ds} \leq e^{\int_0^1 2G(s)ds} \doteq M_2^2,$$

where $M_2 \geq 0$(independent of λ). So $|x'(t)| \leq M_2$, for any $t \in [t_2, t_1]$. Combining with $x'(t_1) = \max_{t\in[0,1]} |x'(t)|$, we have $|x'(t)| \leq M_2$, for any $t \in [0,1]$. Let $M_1 = \max\{1, M_2\}$, then for $x \in U_1$,

$$\|x'\|_\infty \leq M_1. \tag{8}$$

By (5) and (8), we have shown $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\} \leq \max\{M, M_1\}$, namely, U_1 is bounded.

Step 2. Let $U_2 = \{x \in \mathrm{Ker}(L) : Nx \in \mathrm{Im}(L)\}$.

Assunme that $x \in U_2$ and $x \equiv C_0$, for $t \in [0,1]$. Then $C_0 > M$ implies that

$$\int_\eta^1 \int_0^\tau (f(t, C_0, 0) + e(t))dtd\tau > 0,$$

and $C_0 < -M$ implies that

$$\int_\eta^1 \int_0^\tau (f(t, C_0, 0) + e(t))dtd\tau < 0.$$

In both cases, $N(C_0) = f(t, C_0, 0) + e(t) \notin \mathrm{Im}(L)$. Therefore $\|x\|_\infty = |C_0| \leq M$, namely $\|x\| \leq M$.

Step 3. Let $U_3 = \{x \in \mathrm{Ker}(L) : H(x,\mu) = \mu QNx + (1-\mu)x = 0,\ \mu \in [0,1]\}$.

For any $x \in U_3$, then $x \equiv C_0$, we have

$$\frac{2\mu}{1-\eta^2} \int_\eta^1 \int_0^\tau [f(s, C_0, 0) + e(s)]dsd\tau = -(1-\mu)C_0.$$

If $\mu = 0$, then$C_0 = 0$. Otherwise, suppose $C_0 > M$, then $f(s, C_0, 0) + e(s) > 0$, for any $s \in [0,1]$, contradicting $-(1-\mu)C_0 \leq 0$. If $C_0 < -M$, then $f(s, C_0, 0) + e(s) < 0$, for any $s \in [0,1]$, contradicting $-(1-\mu)C_0 \geq 0$. Therefore, $U_3 \subset \{x \in \mathrm{Ker}(L) : \|x\| \leq M\}$, that is, U_3 is bounded.

Now, writing

$$X = \mathrm{Ker}(L) \oplus \mathrm{Ker}(P), Z = \mathrm{Im}(L) \oplus \mathrm{Im}(Q) \quad \text{and} \quad L_1 = L|_{D(L)\cap\mathrm{Ker}(P)},$$

it can be proved that the operator $K = L_1^{-1} : \mathrm{Im}(L) \to D(L) \cap \mathrm{Ker}(P)$ is the linear operator defined by $(Ky)(t) = \int_0^t (t-s)y(s)ds$, for $y \in \mathrm{Im}(L)$, (see [9]).

By the Arzéla-Ascoli Theorem, we can get that K is compact [3], so N is L-compact. Let $\Omega \subset X$ be a bounded open set such that $U_{i=1}^3 U_i \subset \Omega$. The above proof shows that the Lemma 1 is satisfied. Thus, $Lx + Nx = 0$ has at least one solution in $D(L) \cap \bar{\Omega}$, that is, the boundary value problem (1)(3) has a solution with $\alpha = 1$. □

If adding the assumption

(H_4) $f(t,x_1,p_1) > f(t,x_2,p_2)$, for all $x_1,x_2,p_1,p_2 \in R$, $x_1 > x_2$, $p_1 \leq p_2$,

we have the following result.

Theorem 2. *The assumption $(H_1)-(H_3)$ and (H_4) hold, then BVP(1)(3) with $\alpha = 1$ has a unique solution.*

Proof. we have proved that there exists at least one solution for BVP(1)(3) under the assumption $(H_1)-(H_3)$ in Theroem 1. Next, by (H_4), we shall get the uniqueness of the solution for BVP(1)(3).

Assume to the contrary that BVP(1)(3) has two different solutions $x_1(t)$ and $x_2(t)$. Let $y(t) = x_1(t) - x_2(t)$, by the boundary condition (3) with $\alpha = 1$, without loss of generality, there exists a point t_0 such that $y(t_0) = \max_{t\in[0,1]} y(t) > 0$ (if $y(t_0) \leq 0$, a similar argument). If $t_0 \in [0,1]$, then $y(t_0) > 0$, $y'(t_0) = 0$.

It implies that there exists an interval $[t_0,t_1]$ such that $y(s) > 0$, $y'(s) \leq 0$ for all $s \in [t_0,t_1]$. By (H_4), for all $s \in [t_0,t_1]$,

$$\int_{t_0}^{s} (x_1''(t) - x_2''(t))dt = \int_{t_0}^{s} (f(t,x_1(t),x_1'(t)) - f(t,x_2(t),x_2'(t)))dt > 0,$$

that is, $y'(s) = x_1'(s) - x_2'(s) > 0$, contrdicting $y'(s) \leq 0$. We deduce that BVP(1)(3)has a unique solution. □

Let $\eta \to 1$ in Theorem 1, we have the following result.

Theorem 3. *Let f, g, h be mappings as in Theorem 1. Then for $e(t) \in L^1[0,1]$, the Neumann boundary value problem*

$$x''(t) = f(t,x(t),x'(t)) + e(t) \tag{9}$$

$$x'(0) = x'(1) = 0 \tag{10}$$

has at least one solution in $C^1[0,1]$.

Proof. By Theorem 1, for any given $\eta \in (0,1)$, BVP(1)(3) with $\alpha = 1$ has at least one solution $x_\eta(t)$ such that $x_\eta'(0) = 0$, $x_\eta(\eta) = x_\eta(1)$. From above proof, we can obtain

$$\|x_\eta\|_\infty \leq M, \text{ for } \eta \in (0,1). \tag{11}$$

Next we claim that there exists a constant M_3(independent of η) such that

$$\|x_\eta'\|_\infty \leq M_3, \text{ for } \eta \in (0,1). \tag{12}$$

If $\|x_\eta'\|_\infty \leq 1$, then $M_3 = 1$. Otherwise, there exists $\eta \in (0,1)$ such that $\|x_\eta'\|_\infty \geq 1$. Suppose $|x_\eta'(t_0)| = \|x_\eta'\|_\infty > 1$. By the continuity of $x_\eta'(t)$ and $x_\eta'(0) = 0$, we can find an interval $[\mu,v] \subset [0,1]$, $t_0 \in [\mu,v]$ such that $|x_\eta'(\mu)| = 1$ and $|x_\eta'(t)| \geq 1$, for $t \in [\mu,v]$. Without loss of generality, we suppose that

$x'_\eta(t) \geq 1$, for all $t \in [\mu, v]$. Multiplying (4) by $x'_\eta(t)$ and combining with $(H_2)(H_3)$, we have

$$\begin{aligned} x'_\eta(t)x''_\eta(t) &= \lambda x'_\eta(t)f(t, x_\eta(t), x'_\eta(t)) + \lambda x'_\eta(t)e(t) \\ &= \lambda x'_\eta(t)[h(t, x_\eta(t), x'_\eta(t)) + g(t, x_\eta(t), x'_\eta(t))] + \lambda x'_\eta(t)e(t) \\ &\leq x'_\eta(t)[a(t)|x_\eta(t)| + b(t)|x'_\eta(t)| + u(t)|x'_\eta(t)|^r \\ &\quad + v(t)|x'_\eta(t)|^k + c(t)] + x'_\eta(t)e(t) \end{aligned} \tag{13}$$

Integrating (13) from μ to t, a similar argument in Theorem 1, we have

$$\frac{1}{2}x'_\eta(t)^2 \leq \frac{1}{2} + \int_\mu^t |x'_\eta(s)|^2 G(s)ds \tag{14}$$

where $G(s) = Ma(s) + b(s) + u(s)M^r + v(s) + c(s) + |e(s)|$.

By Gronwall inequality, we get

$$x'_\eta(t)^2 \leq e^{\int_0^1 2G(s)ds} \doteq M_3^2,$$

where $M_3 > 0$ (independent of λ). That is, $|x'_\eta(t)| \leq M_3$, for any $t \in [\mu, v]$. In connection with $|x'_\eta(t_0)| = \|x'_\eta\|_\infty$ and $t_0 \in [\mu, v]$, it implies that

$$\|x'_\eta\|_\infty \leq M_3, \quad \text{for } \eta \in (0,1).$$

By (11)(12) and applying Arzela-Ascoli theorem, this shows that $\{x_\eta(t)\}$, for $\eta \in (0,1)$ is relative compact in $C^1[0,1]$. Without loss of generality, we suppose that

$$\lim_{\eta\to 1} x_\eta(t) = x(t), \text{ and } \lim_{\eta \to 1} x'_\eta(t) = x'(t), \text{ for } t \in [0,1]. \tag{15}$$

For any given $\eta \in (0,1)$, we have

$$x''_\eta(t) = f(t, x_\eta(t), x'_\eta(t)) + e(t), \quad x'_\eta(0) = 0, \quad x_\eta(\eta) = x_\eta(1).$$

So

$$x'_\eta(t) = \int_0^t f(s, x_\eta(s), x'_\eta(s))ds + \int_0^t e(s)ds, \text{ for } t \in (0,1) \tag{16}$$

$$x'_\eta(0) = 0, \quad x_\eta(\eta) = x_\eta(1) \tag{17}$$

Let $\eta \to 1$ in (16)(17), we obtain

$$x'(t) = \int_0^t f(s, x(s), x'(s))ds + \int_0^t e(s)ds, \text{ for } t \in (0,1),$$

$$x'(0) = 0, \quad x'(1) = 0.$$

Thus the Neumann boundary value problem (9)(10) has at least one solution in $C^1[0,1]$. □

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