J. Appl. Math. & Informatics Vol. 27(2009), No. 5 - 6, pp. 1247 - 1256
Website: http://www.kcam.biz

FINITE DIFFERENCE SCHEMES FOR A GENERALIZED NONLINEAR CALCIUM DIFFUSION EQUATION

S.M. Choo

Abstract. Finite difference schemes are considered for a nonlinear Ca^{2+} diffusion equations with stationary and mobile buffers. The scheme inherits mass conservation as for the classical solution. Stability and L^∞ error estimates of approximate solutions for the corresponding schemes are obtained. using the extended Lax-Richtmyer equivalence theorem.

AMS Mathematics Subject Classification : 65M06, 65M12, 65M15
Key words and phrases : Nonlinear Ca^{2+} diffusion equation, buffer, convection, nonlinear difference schemes, mass conservation, error estimates, Lax-Richtmyer equivalence theorem.

1. Introduction

Consider the generalized nonlinear Ca^{2+} diffusion equation in cells

$$\frac{\partial[Ca^{2+}]}{\partial t} = \frac{\partial}{\partial x}\left\{F([Ca^{2+}],[B_m],[CaB_m])\frac{\partial[Ca^{2+}]}{\partial x}\right\} - k_s^+[Ca^{2+}][B_s] \qquad (1)$$

Received December 26, 2008. Accepted June 26, 2009.
This work was supported by the 2008 Research Fund of University of Ulsan.

$$+ k_s^-[CaB_s] - k_m^+[Ca^{2+}][B_m] + k_m^-[CaB_m] - \alpha_{Ca}[Ca^{2+}] - \beta_{Ca}\frac{\partial[Ca^{2+}]}{\partial x},$$

$$\frac{\partial[B_m]}{\partial t} = \frac{\partial}{\partial x}\left\{G\Big([Ca^{2+}],[B_m],[CaB_m]\Big)\frac{\partial[B_m]}{\partial x}\right\} - k_m^+[Ca^{2+}][B_m]$$

$$+ k_m^-[CaB_m] - \alpha_{B_m}[B_m] - \beta_{B_m}\frac{\partial[B_m]}{\partial x},$$

$$\frac{\partial[CaB_m]}{\partial t} = \frac{\partial}{\partial x}\left\{H\Big([Ca^{2+}],[B_m],[CaB_m]\Big)\frac{\partial[CaB_m]}{\partial x}\right\} + 2k_m^+[Ca^{2+}][B_m]$$

$$- 2k_m^-[CaB_m] - \alpha_{CaB_m}[CaB_m] - \beta_{CaB_m}\frac{\partial[CaB_m]}{\partial x},$$

$$\frac{\partial[CaB_s]}{\partial t} = k_s^+[Ca^{2+}][B_s] - k_s^-[CaB_s],$$

$$[B_s] = [B_s]_{tot} - [CaB_s], \quad x \in \Omega = (0,L), 0 < t \le T$$

with the initial conditions

$$[Ca^{2+}](x,0) = [Ca^{2+}]_0(x), \quad [B_m](x,0) = [B_m]_0(x), \qquad (2)$$
$$[CaB_m](x,0) = [CaB_m]_0(x), \quad [CaB_s](x,0) = [CaB_s]_0(x)$$

and the boundary conditions for $u \in \{[Ca^{2+}],[B_m],[CaB_m]\}$,

$$\frac{\partial u}{\partial x}(x,t) = 0, \quad x \in \{0,L\}, t \in (0,T] \qquad (3)$$

where $\alpha_i, \beta_i (i = Ca, B_m, CaB_m, CaB_s)$, [Ca^{2+}], [B$_s$], [B$_m$], [CaB$_s$],[CaB$_m$] are damping term, convection term, concentrations of free Calcium ion, stationary and mobile buffers, and Ca^{2+} bounded to stationary and mobile buffer sites([1],[11]), respectively. The diffusion coefficients F, G, H are dependent on concentrations of free Calcium ion and buffers. The total concentration of the stationary buffer $[B_s]_{tot}$ is constant, and k^+, k_- are association, and dissociation constants, respectively and all constants are positive.

Studies on Calcium dynamics belong to the area of electrophysiology, in which almost all systems are described by ordinary differential equations([2],[6]–[8]]) but recently some systems are modeled by partial differential equations having temporal and spatial terms([5],[10],[12]). In the case of $\alpha_i = \beta_i = 0$, Wagner and Keizer[13] have described the Ca^{2+} buffering as the partial differential equations (1)–(4) with constants F, G and H. There is no numerical analysis of equations with nonlinear Ca^{2+} diffusion coefficients. Following the finite difference approaches in [3]–[4], we can analysis numerical schemes for the generalized nonlinear Ca^{2+} buffering model.

In this paper, we consider estimates of approximate solutions for finite difference methods. In Section 2, we introduce the finite difference schemes for

(1)–(4), which has the property of mass conservation as for the classical solution with zero damping and convection terms. Some lemmas are necessary to obtain error estimates. In Section 3, we briefly recall the Lax-Richtmyer equivalence theorem[9] and obtain stability and error estimates for the equation.

2. Finite difference schemes

Let $h = L/\mathcal{M}$ be the uniform step size in the spatial direction for a positive integer $\mathcal{M}$ and $\Omega_h = \left\{x_i = ih | i = -1, 0, \cdots, \mathcal{M}, \mathcal{M}+1\right\}$. Let $k = T/N$ denote the uniform step size in the temporal direction for a positive integer N. Denote $V_i^n = V(x_i, t_n)$ for $t_n = nk, n = 0, 1, \cdots, N$. For a function V^n defined on Ω_h, define the difference operators as for $0 \le i \le \mathcal{M}$,

$$\nabla_+ V_i^n = \left(V_{i+1}^n - V_i^n\right)/h, \quad \nabla_- V_i^n = \nabla_+ V_{i-1}^n, \quad \nabla^2 V_i^n = \nabla_+(\nabla_- V_i^n).$$

Further, define operators $V^{n+\frac{1}{2}}$ and $\partial_t V^n$ as

$$V_i^{n+\frac{1}{2}} = \left(V_i^{n+1} + V_i^n\right)/2 \quad \text{and} \quad \partial_t V_i^n = \left(V_i^{n+1} - V_i^n\right)/k.$$

Then the approximate solutions $[U_1]_i^{n+1}, [U_2]_i^{n+1}, [U_3]_i^{n+1}, [U_4]_i^{n+1} (0 \le i \le \mathcal{M}, 0 \le n \le N-1)$ for (1)–(3) are defined as solutions of

$$\begin{aligned}
\partial_t [U_1]_i^n &= F_{ui}^{n+\frac{1}{2}} \nabla^2 [U_1]_i^{n+\frac{1}{2}} + \mathcal{F}_{ui}^{n+\frac{1}{2}} - k_s^+ [U_1]_i^{n+\frac{1}{2}} [S]_i^{n+\frac{1}{2}} + k_s^- [U_4]_i^{n+\frac{1}{2}} \qquad (4)\\
&\quad - k_m^+ [U_1]_i^{n+\frac{1}{2}} [U_2]_i^{n+\frac{1}{2}} + k_m^- [U_3]_i^{n+\frac{1}{2}} - \alpha_1 [U_1]_i^{n+\frac{1}{2}} - \beta_1 \bar{\nabla} [U_1]_i^{n+\frac{1}{2}},\\
\partial_t [U_2]_i^n &= G_{ui}^{n+\frac{1}{2}} \nabla^2 [U_2]_i^{n+\frac{1}{2}} + \mathcal{G}_{ui}^{n+\frac{1}{2}} - k_m^+ [U_1]_i^{n+\frac{1}{2}} [U_2]_i^{n+\frac{1}{2}} + k_m^- [U_3]_i^{n+\frac{1}{2}}\\
&\quad - \alpha_2 [U_2]_i^{n+\frac{1}{2}} - \beta_2 \bar{\nabla} [U_2]_i^{n+\frac{1}{2}},\\
\partial_t [U_3]_i^n &= H_{ui}^{n+\frac{1}{2}} \nabla^2 [U_3]_i^{n+\frac{1}{2}} + \mathcal{H}_{ui}^{n+\frac{1}{2}} + 2k_m^+ [U_1]_i^{n+\frac{1}{2}} [U_2]_i^{n+\frac{1}{2}}\\
&\quad - 2k_m^- [U_3]_i^{n+\frac{1}{2}} - \alpha_3 [U_3]_i^{n+\frac{1}{2}} - \beta_3 \bar{\nabla} [U_3]_i^{n+\frac{1}{2}},\\
\partial_t [U_4]_i^n &= k_s^+ [U_1]_i^{n+\frac{1}{2}} [S]_i^{n+\frac{1}{2}} - k_s^- [U_4]_i^{n+\frac{1}{2}},\\
[S]_i^n &= [B_s]_{tot} - [U_4]_i^n
\end{aligned}$$

with the initial conditions

$$\begin{aligned}
[U_1]_i^0 &= [Ca^{2+}]_0(x_i), \quad [U_2]_i^0 = [B_m]_0(x_i),\\
[U_3]_i^0 &= [CaB_m]_0(x_i), \quad [U_4]_i^0 = [CaB_s]_0(x_i)
\end{aligned} \qquad (5)$$

and the Neumann boundary conditions

$$\frac{\nabla_+ + \nabla_-}{2} [U_j]_i^n = 0, \quad j = 1, 2, 3, \quad i = 0, \mathcal{M}, \; 1 \le n \le N \qquad (6)$$

where $\bar{\nabla} = (\nabla_- + \nabla_+)/2$, $\alpha_1 = \alpha_{Ca}$, $\beta_1 = \beta_{Ca}$, $\alpha_2 = \alpha_{B_m}$, $\beta_2 = \beta_{B_m}$, $\alpha_3 = \alpha_{CaB_m}$, $\beta_3 = \beta_{CaB_m}$, $F_{ui}^{n+\frac{1}{2}} = F([U_1]_i^{n+\frac{1}{2}}, [U_2]_i^{n+\frac{1}{2}}, [U_3]_i^{n+\frac{1}{2}})$, and

$$\mathcal{F}_{ui}^{n+\frac{1}{2}} = \frac{1}{2}\left(\nabla_- F_i^{n+\frac{1}{2}}\right)\nabla_-[U_1]_i^{n+\frac{1}{2}} + \frac{1}{2}\left(\nabla_+ F_i^{n+\frac{1}{2}}\right)\nabla_+[U_1]_i^{n+\frac{1}{2}}.$$

$G_{ui}^{n+\frac{1}{2}}, H_{ui}^{n+\frac{1}{2}}, \mathcal{G}_{ui}^{n+\frac{1}{2}}$ and $\mathcal{H}_{ui}^{n+\frac{1}{2}}$ are defined as for $F_{ui}^{n+\frac{1}{2}}$ and $\mathcal{F}_{ui}^{n+\frac{1}{2}}$.

Note that the discretized Neumann boundary conditions (6) are equal to $[U_j]_{-1}^n = [U_j]_1^n$ and $[U_j]_{\mathcal{M}+1}^n = [U_j]_{\mathcal{M}-1}^n$ for $j = 1, 2, 3$.

In order to consider the error estimates, we now introduce the discrete L^2-inner product and the corresponding discrete L^2-norm on Ω_h

$$(V, W)_h = h\sum_{i=0}^{\mathcal{M}}{}'' V_i W_i = h\left\{(V_0 W_0 + V_{\mathcal{M}} W_{\mathcal{M}})/2 + \sum_{i=1}^{\mathcal{M}-1} V_i W_i\right\},$$

$$\|V\|_h = (V, V)_h^{1/2}.$$

For the maximum norm, we define $\|V\|_\infty = \max_{0\le i\le\mathcal{M}} |V_i|$.

Hereafter, whenever there is no confusion, $(\cdot,\cdot)$ and $\|\cdot\|$ will denote $(\cdot,\cdot)_h$ and $\|\cdot\|_h$, respectively.

It follows from summation by parts and the definition of difference operators that Lemma 1 holds.

Lemma 1. *For functions V and W defined on Ω_h and satisfying the boundary conditions (6), the following identity and inequality hold.*

(1) $(\nabla^2 V, W) = -h\sum_{i=1}^{\mathcal{M}}(\nabla_- V_i)(\nabla_- W_i)$.

(2) $\max\left\{\|\nabla_+ V\|^2, \|\nabla_- V\|^2\right\} \le -2(\nabla^2 V, V)$.

Using Lemma 2.5 in [4] and Lemma 1, we obtain the following lemma.

Lemma 2. *For V defined on Ω_h, the following inequality holds.*

$$\|V\|_\infty^2 \le 3\|V\|^2 + 8\|V\|\|\bar{\nabla} V\|.$$

For the classical solution of (1)–(3) with zero damping and convection terms, we obtain

$$\frac{d}{dt}\int_0^L [Ca^{2+}] + [B_m] + [CaB_m] + [CaB_s]dx = \int_0^L \sum_{j=1}^{3} \frac{\partial}{\partial x}\left(D_j[E]_j\right)dx = 0$$

where $D_1 = F, [E]_1 = [Ca^{2+}], D_2 = G, [E]_2 = [B_m], D_3 = H$, and $[E]_3 = [CaB_m]$. Using the definitions of $\nabla_-, \nabla_+, \nabla^2$ and boundary conditions (6), we can also show that the scheme (4)–(6) has the property of mass conservation.

Theorem 1. *Let U_j be the solution of (4)–(6) with $\alpha_i = \beta_i = 0 (1 \le j \le 4, 1 \le i \le 3)$. Then the conservation of mass holds. That is, for $1 \le n \le N$,*

$$h \sum_{i=0}^{\mathcal{M}}{}'' \sum_{j=1}^{4} [U_j]_i^n = h \sum_{i=0}^{\mathcal{M}}{}'' \sum_{j=1}^{4} [U_j]_i^0.$$

3. Convergence of approximate solution

We recall the extension of Lax-Richtmyer equivalence theorem in Lopez-Marcos and Sanz-Serna[9] which makes us avoid the difficulty of direct proof for convergence arising specially in nonlinear problems. Let u be a solution of a problem $\Phi(u) = 0$ and u_h be a discrete evaluation of u on Ω_h. Let U_h be an approximate solution of u, which is obtained by solving the discrete equation

$$\Phi_h(U_h) = 0, \tag{7}$$

where $\Phi_h : \mathbf{X}_h \to \mathbf{Y}_h$ is a continuous mapping and $\mathbf{X}_h, \mathbf{Y}_h$ are normed spaces having the same dimension. The scheme (7) is said to be convergent if (7) has a solution U_h such that $\lim_{h \to 0} \|U_h - u_h\|_{\mathbf{X}_h} = 0$. The discretization (7) is said to be consistent if $\lim_{h \to 0} \|\Phi_h(u_h)\|_{\mathbf{Y}_h} = 0$. The scheme (7) is said to be stable in the threshold R_h if there exists a constant C such that for an open ball $B(u_h, R_h) \subset \mathbf{X}_h$,

$$\|V_h - W_h\|_{\mathbf{X}_h} \le C \|\Phi_h(V_h) - \Phi_h(W_h)\|_{\mathbf{Y}_h}, \quad \forall\, V_h, W_h \in B(u_h, R_h).$$

The following theorem is the extended Lax-Richtmyer equivalence theorem which gives existence and convergence of approximate solutions. For the proof, see [9].

Theorem 2. *Assume that the discrete equation (7) is consistent and stable in the threshold R_h. Let $B(u_h, R_h)$ be the ball with center u_h and radius R_h. If Φ_h is continuous in $B(u_h, R_h)$ and $\|\Phi_h(u_h)\|_{\mathbf{Y}_h} = o(R_h)$ as $h \to 0$, then (7) has a unique solution U_h in $B(u_h, R_h)$ and there exists a constant C such that*

$$\|U_h - u_h\|_{\mathbf{X}_h} \le \Theta \|\Phi_h(u_h)\|_{\mathbf{Y}_h}.$$

According to Theorem 2, we have only to show that (7) is consistent and stable in the threshold in order to show the unique existence and convergence of approximate solutions.

Let Z_h^n be the set of all functions defined on Ω_h and $\hat{Z}_h^n$ be the subset of Z_h^n satisfying the discretized Neumann boundary conditions (6) at time level

n $(0 \le n \le N)$. We take $\mathbf{X}_h = \mathbf{Y}_h = \left(\prod_{n=0}^{N} \hat{Z}_h^n\right)^3 \times \left(\prod_{n=0}^{N} Z_h^n\right)$ and define a mapping $\Phi_h : \mathbf{X}_h \to \mathbf{Y}_h$ by $\Phi_h(\mathbf{U}) = \widetilde{\mathbf{U}}$, where for $n = 0, \cdots, N-1$

$$\begin{aligned}
\widetilde{[U_1]}_i^{n+1} &= \partial_t[U_1]_i^n - F_{ui}^{n+\frac{1}{2}}\nabla^2[U_1]_i^{n+\frac{1}{2}} - \mathcal{F}_{ui}^{n+\frac{1}{2}} \\
&\quad + k_s^+[U_1]_i^{n+\frac{1}{2}}\left([B_s]_{tot} - [U_4]_i^{n+\frac{1}{2}}\right) - k_s^-[U_4]_i^{n+\frac{1}{2}} \\
&\quad + k_m^+[U_1]_i^{n+\frac{1}{2}}[U_2]_i^{n+\frac{1}{2}} - k_m^-[U_3]_i^{n+\frac{1}{2}} + \alpha_1[U_1]_i^{n+\frac{1}{2}} + \beta_1\bar{\nabla}[U_1]_i^{n+\frac{1}{2}}, \\
\widetilde{[U_2]}_i^{n+1} &= \partial_t[U_2]_i^n - G_{ui}^{n+\frac{1}{2}}\nabla^2[U_2]_i^{n+\frac{1}{2}} - \mathcal{G}_{ui}^{n+\frac{1}{2}} \\
&\quad + k_m^+[U_1]_i^{n+\frac{1}{2}}[U_2]_i^{n+\frac{1}{2}} - k_m^-[U_3]_i^{n+\frac{1}{2}} + \alpha_2[U_2]_i^{n+\frac{1}{2}} + \beta_2\bar{\nabla}[U_2]_i^{n+\frac{1}{2}}, \\
\widetilde{[U_3]}_i^{n+1} &= \partial_t[U_3]_i^n - H_{ui}^{n+\frac{1}{2}}\nabla^2[U_3]_i^{n+\frac{1}{2}} - \mathcal{H}_{ui}^{n+\frac{1}{2}} \\
&\quad - 2k_m^+[U_1]_i^{n+\frac{1}{2}}[U_2]_i^{n+\frac{1}{2}} + 2k_m^-[U_3]_i^{n+\frac{1}{2}} + \alpha_3[U_3]_i^{n+\frac{1}{2}} + \beta_3\bar{\nabla}[U_3]_i^{n+\frac{1}{2}}, \\
\widetilde{[U_4]}_i^{n+1} &= \partial_t[U_4]_i^n - k_s^+[U_1]_i^{n+\frac{1}{2}}\left([B_s]_{tot} - [U_4]_i^{n+\frac{1}{2}}\right) + k_s^-[U_4]_i^{n+\frac{1}{2}}
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
&\widetilde{[U_1]}_i^0 = [U_1]_i^0 - [Ca^{2+}]_0(x_i), \quad \widetilde{[U_2]}_i^0 = [U_2]_i^0 - [B_m]_0(x_i), \\
&\widetilde{[U_3]}_i^0 = [U_3]_i^0 - [CaB_m]_0(x_i), \quad \widetilde{[U_4]}_i^0 = [U_4]_i^0 - [CaB_s]_0(x_i).
\end{aligned} \tag{9}$$

We take norms $\|\cdot\|_{\mathbf{X}_h}$ and $\|\cdot\|_{\mathbf{Y}_h}$ on $\mathbf{X}_h$ and $\mathbf{Y}_h$, respectively, such that

$$\|\mathbf{U}\|_{\mathbf{X}_h}^2 = \max_{0\le n\le N}\sum_{j=1}^{4}\|U_j^n\|^2 + k\sum_{n=0}^{N-1}\left\{-\sum_{j=1}^{3}(\nabla^2 U_j^{n+\frac{1}{2}}, U_j^{n+\frac{1}{2}}) + \sum_{j=1}^{4}\|U_j^{n+\frac{1}{2}}\|^2\right\}$$

and

$$\|\widetilde{\mathbf{U}}\|_{\mathbf{Y}_h}^2 = \sum_{j=1}^{4}\|\widetilde{U}_j^0\|^2 + k\sum_{n=1}^{N}\sum_{j=1}^{4}\|\widetilde{U}_j^n\|^2.$$

The consistency of the scheme (4)–(6) is obtained using Taylor's Theorem and the Mean Value Theorem.

Theorem 3. *Let $u = \big([Ca^{2+}], [B_m], [CaB_m], [CaB_s]\big)$ be the solution of (1)–(3) with bounded derivatives $\dfrac{\partial^3 u_i}{\partial t^3}(1 \le i \le 4)$, $\dfrac{\partial^4 u_i}{\partial x^4}(1 \le i \le 3)$ and $\dfrac{\partial^2 u_4}{\partial x^2}$. Assume F, G and H are $C^3(\mathbb{R}^4)$-functions. Then there exists a constant C such that*

$$\|\Phi_h(u_h)\|_{\mathbf{Y}_h} \le C\Big(k^2 + h^2\Big).$$

We now consider the stability of the approximate solution in the threshold R_h.

Theorem 4. *Let $\Phi_h(\mathbf{U}) = \widetilde{\mathbf{U}}$, $\Phi_h(\mathbf{V}) = \widetilde{\mathbf{V}}$ and $R_h = O\left(k^{\frac{1}{2}+\ell_1}h^{1+\ell_2}\right)$ with positive constants ℓ_1 and ℓ_2. Assume that the conditions in Theorem 3 hold and the diffusion coefficients are bounded below by a positive constant D_0. Then there exists a constant C such that for any $\mathbf{U}$ and $\mathbf{V}$ in $B(u_h, R_h)$,*

$$\|\mathbf{U}-\mathbf{V}\|_{\mathbf{X}_h} \le C\|\Phi_h(\mathbf{U}) - \Phi_h(\mathbf{V})\|_{\mathbf{Y}_h}.$$

Proof. Let $e_j^n = [U_j]^n - [V_j]^n$ and $\widetilde{K}_j^n = \widetilde{[U_j]}^n - \widetilde{[V_j]}^n$ with $1 \le j \le 4$. Replacing $[U_j]^n$ and $\widetilde{[U_j]}^n$ in (8) by $[V_j]^n$ and $\widetilde{[V_j]}^n$, respectively, and subtracting these results from (8), we obtain

$$\begin{aligned}
&\partial_t e_1^n - F_u^{n+\frac{1}{2}}\nabla^2 e_1^{n+\frac{1}{2}} + (k_s^+[B_s]_{tot} + \alpha_1)e_1^{n+\frac{1}{2}} \qquad (10)\\
&\quad = -\left(F_u^{n+\frac{1}{2}} - F_v^{n+\frac{1}{2}}\right)\nabla^2[V_1]^{n+\frac{1}{2}} + \mathcal{F}_u^{n+\frac{1}{2}} - \mathcal{F}_v^{n+\frac{1}{2}}\\
&\quad + k_s^+\left(e_1^{n+\frac{1}{2}}[U_4]^{n+\frac{1}{2}} + [U_1]^{n+\frac{1}{2}}e_4^{n+\frac{1}{2}}\right) + k_s^- e_4^{n+\frac{1}{2}} + k_m^- e_3^{n+\frac{1}{2}}\\
&\quad - k_m^+\left(e_1^{n+\frac{1}{2}}[U_2]^{n+\frac{1}{2}} + [U_1]^{n+\frac{1}{2}}e_2^{n+\frac{1}{2}}\right) + \beta_1\bar{\nabla}e_1^{n+\frac{1}{2}} + \widetilde{K}_1^{n+1}\\
&\partial_t e_2^n - G_u^{n+\frac{1}{2}}\nabla^2 e_2^{n+\frac{1}{2}} + \alpha_2 e_2^{n+\frac{1}{2}}\\
&\quad = -\left(G_u^{n+\frac{1}{2}} - G_v^{n+\frac{1}{2}}\right)\nabla^2[V_2]^{n+\frac{1}{2}} + \mathcal{G}_u^{n+\frac{1}{2}} - \mathcal{G}_v^{n+\frac{1}{2}}\\
&\quad - k_m^+\left(e_1^{n+\frac{1}{2}}[U_2]^{n+\frac{1}{2}} + [U_1]^{n+\frac{1}{2}}e_2^{n+\frac{1}{2}}\right) + k_m^- e_3^{n+\frac{1}{2}}\\
&\quad + \beta_2\bar{\nabla}e_2^{n+\frac{1}{2}} + \widetilde{K}_2^{n+1},\\
&\partial_t e_3^n - H_u^{n+\frac{1}{2}}\nabla^2 e_3^{n+\frac{1}{2}} + (2k_m^- + \alpha_3)e_3^{n+\frac{1}{2}}\\
&\quad = -\left(H_u^{n+\frac{1}{2}} - H_v^{n+\frac{1}{2}}\right)\nabla^2[V_3]^{n+\frac{1}{2}} + \mathcal{H}_u^{n+\frac{1}{2}} - \mathcal{H}_v^{n+\frac{1}{2}}\\
&\quad + 2k_m^+\left(e_1^{n+\frac{1}{2}}[U_2]^{n+\frac{1}{2}} + [U_1]^{n+\frac{1}{2}}e_2^{n+\frac{1}{2}}\right) + \beta_3\bar{\nabla}e_3^{n+\frac{1}{2}} + \widetilde{K}_3^{n+1},\\
&\partial_t e_4^n + k_s^- e_4^{n+\frac{1}{2}} = k_s^+[B_s]_{tot}e_1^{n+\frac{1}{2}} - k_s^+\left(e_1^{n+\frac{1}{2}}[U_4]^{n+\frac{1}{2}} + [U_1]^{n+\frac{1}{2}}e_4^{n+\frac{1}{2}}\right)\\
&\quad + \widetilde{K}_4^{n+1}.
\end{aligned}$$

It follows from the definition of $\|\cdot\|_{\mathbf{X}_h}$ that for $\mathbf{V}$ in the ball $B(u_h, R_h)$,

$$\begin{aligned}
\|\nabla^2[V_1]^{n+1/2}\| &\le \left\|\nabla^2\left([V_1]^{n+1/2} - u_{1h}^{n+1/2}\right)\right\| + \left\|\nabla^2 u_{1h}^{n+1/2}\right\| \qquad (11)\\
&\le C\frac{k^{1/2+\ell_1}h^{1+\ell_2}}{k^{1/2}h} + C \le C(k^{\ell_1}h^{\ell_2} + 1),\\
\|[V_1]^n\|^2 + \|\nabla_-[V_1]^n\|^2 &+ \|\nabla_+[V_1]^n\|^2 \qquad (12)\\
&\le C\left(k^{1+2\ell_1}h^{2+2\ell_2}\right) + \|u_{1h}^n\|^2 + \|\nabla_- u_{1h}^n\|^2 + \|\nabla_+ u_{1h}^n\|^2\\
&\le C\left(k^{1+2\ell_1}h^{2+2\ell_2} + 1\right).
\end{aligned}$$

Note that from Lemma 2 and (11)–(12),

$$\left\|\left(F_u^{n+\frac{1}{2}}-F_v^{n+\frac{1}{2}}\right)\nabla^2[V_1]^{n+1/2}\right\| \le C\sum_{j=1}^{3}\|e_j^{n+1/2}\|_\infty\|\nabla^2[V_1]^{n+1/2}\| \tag{13}$$

$$\le C(k^{\ell_1}h^{\ell_2}+1)\sum_{j=1}^{3}\left(\|e_j^{n+1/2}\|\|\bar\nabla e_j^{n+1/2}\|+\|e_j^{n+1/2}\|^2\right)$$

$$\le C\left\{1+\sum_{j=1}^{2}(k^{\ell_1}h^{\ell_2}+1)^j\right\}\sum_{j=1}^{3}\left\{\|e_j^{n+1/2}\|^2-\frac{D_0}{4}(\nabla^2 e_j^{n+1/2},e_j^{n+1/2})\right\}.$$

Similarly to (13), we obtain

$$\left\|\mathcal{F}_u^{n+\frac{1}{2}}-\mathcal{F}_v^{n+\frac{1}{2}}\right\| \le C\sum_{j=1}^{3}\left\{\|e_j^{n+1/2}\|^2-\frac{D_0}{4}(\nabla^2 e_j^{n+1/2},e_j^{n+1/2})\right\}. \tag{14}$$

Taking inner products between (10) and $e_j^{n+\frac{1}{2}}$ and summing these results (13)–(14), we obtain for a constant C

$$\sum_{j=1}^{4}\partial_t\|e_j^n\|^2-\sum_{j=1}^{3}\frac{D_0}{2}\left(\nabla^2 e_j^{n+\frac{1}{2}},e_j^{n+\frac{1}{2}}\right)+\sum_{j=1}^{4}\tau_j\|e_j^{n+\frac{1}{2}}\|^2 \tag{15}$$

$$\le C\left(\|e_3^{n+\frac{1}{2}}\|+\sum_{j=1}^{3}\|\bar\nabla e_j^{n+\frac{1}{2}}\|+\sum_{j\in\{1,2,4\}}\|e_j^{n+\frac{1}{2}}\|_\infty\right)\sum_{j=1}^{4}\|e_j^{n+\frac{1}{2}}\|$$

$$+\sum_{j=1}^{4}\|\widetilde{K}_j^{n+1}\|^2$$

where $\tau_1=k_s^+[B_s]_{tot}+\alpha_1,\tau_2=\alpha_2,\tau_3=k_m^-+\alpha_3$ and $\tau_4=k_s^-$.
Applying Lemma 1–2 and the discrete Gronwall's inequality to (15), we obtain for $0\le m\le N-1$,

$$\sum_{j=1}^{4}\|e_j^{m+1}\|^2+k\sum_{n=0}^{m}\left\{-\sum_{j=1}^{3}(\nabla^2 e_j^{n+\frac{1}{2}},e_j^{n+\frac{1}{2}})+\sum_{j=1}^{4}\|e_j^{n+\frac{1}{2}}\|^2\right\}$$

$$\le C\sum_{j=1}^{4}\left(\|e_j^0\|^2+k\sum_{n=1}^{m+1}\|\widetilde{K}_j^n\|^2\right).$$

Since

$$e_j^0=[U_j]^0-[V_j]^0=[\widetilde{U}_j]^0-[\widetilde{V}_j]^0=\widetilde{K}_j^0,$$

the desired result is obtained. □

It follows from Theorem 1 that for $k = O(h^{\alpha})$ with $\dfrac{1+\ell_2}{\frac{3}{2}-\ell_1} < \alpha < \dfrac{1-\ell_2}{\frac{1}{2}+\ell_1}$,

$$\frac{\|\Phi_h(u_h)\|_{\mathbf{Y}_h}}{R_h} = O\left(\frac{k^2+h^2}{k^{\frac{1}{2}+\ell_1}h^{1+\ell_2}}\right) \to 0 \quad \text{as} \quad h \to 0. \tag{16}$$

Hence, applying Theorems 3–4 and (16) to Theorem 2, we obtain the following error estimate for (4)–(6).

Theorem 5. *Suppose that hypotheses of Theorem 4 hold. Let* $\mathbf{U} = ([U_1],[U_2],[U_3],[U_4])$ *be the solution of (4)–(6). Then for* $k = O(h^{\alpha})$ *with* $\dfrac{1+\ell_2}{\frac{3}{2}-\ell_1} < \alpha < \dfrac{1-\ell_2}{\frac{1}{2}+\ell_1}$, *there exists a constant* C *such that*

$$\|\mathbf{U} - u_h\|_{\mathbf{X}_h} \leq C(k^2+h^2).$$

References

1. N. L. Allbritton, T. Meyer and L. Stryer, *Range of messenger action of calcium ion and inositol 1,4,5-trisphosphate*, Science. 258(1992), 1812-1815.
2. V. E. Bondarenko, G. P. Szigeti, G. C. Bett, S. J. Kim and R. L. Rasmusson, *Computer model of action potential of mouse ventricular myocytes*, Am J Physiol Heart Circ Physiol. 287(2004), H1378-403.
3. S. M. Choo and S. K. Chung, *A conservative nonlinear difference scheme for the viscous Cahn-Hilliard equation*, J. Appl. Math. Comput. 16(2004), 53-68.
4. S. M. Choo, S. K. Chung and Y. J. Lee, *Error estimates of nonstandard finite difference schemes for generalized Cahn-Hilliard and Kuramoto-Sivashinsky equations*, J. Korean Math. Soc. 41(2005), 1121-1136.
5. M. Falcke, *Buffers and oscillations in intracellular Ca2+ dynamics*, Biophys J. 84(2003), 28-41.
6. J. L. Greenstein and R. L. Winslow, *An integrative model of the cardiac ventricular myocyte incorporating local control of Ca2+ release*, Biophys J. 83(2002), 2918-2945.
7. T. J. Hund and Y. Rudy, *Rate dependence and regulation of action potential and calcium transient in a canine cardiac ventricular cell model*, Circulation. 110(2004), 3168-3174.
8. M. S. Jafri, J. J. Rice and R. L. Winslow, *Cardiac Ca2+ dynamics: the roles of ryanodine receptor adaptation and sarcoplasmic reticulum load*, Biophys J. 74(1998), 1149-1168.
9. J. C. Lopez-Marcos and J. M. Sanz-Serna, *Stability and convergence in numerical analysis III: Linear investigation of nonlinear stability*, IMA J. Numer. Anal. 8(1988), 71-84.
10. A. Michailova, F. DelPrincipe, M. Egger and E. Niggli, *Spatiotemporal features of Ca2+ buffering and diffusion in atrial cardiac myocytes with inhibited sarcoplasmic reticulum*, Biophys J. 83(2002), 3134-3151.
11. R. E. Milner, K. S. Famulski and M. Michalak, *Calcium binding proteins in the sarcoplasmic/endoplasmic reticulum of muscle and nonmuscle cells*, Mol Cell Biochem. 112(1992), 1-13.
12. D. E. Strier, A. C. Ventura and S. P. Dawson, *Saltatory and continuous calcium waves and the rapid buffering approximation*, Biophys J. 85(2003), 3575-3586.

13. J. Wagner and J. Keizer, *Effects of rapid buffers on Ca2+ diffusion and Ca2+ oscillations*, Biophys J. 67(1994), 447-456.

S. M. Choo received his degrees of B.S. and M.S. from Seoul National University. He earned his Ph.D. at Seoul National University under the direction of S.K. Chung. He has been at University of Ulsan since September, 2001. His research interest are numerical analysis and systems biology.

School of Electrical Engineering, University of Ulsan, Ulsan 680–749, Korea
e-mail: `smchoo@ulsan.ac.kr`