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SOLVABILITY OF IMPULSIVE EVOLUTION DIFFERENTIAL INCLUSIONS WITH NONLOCAL CONDITIONS IN BANACH SPACE

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ABSTRACT. In this paper, we prove existence results for first order impulsive evolution differential inclusions with nonlocal condition by using a fixed point theorem for condensing multi-valued maps. An example is also given to illustrate the obtained results.

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1. Introduction

The starting point of this paper is the work in paper [5], where the authors have investigated the existence of solutions for the first order controllability system.

$$\begin{aligned} \frac{d}{dt}[y(t) - g(t, y(t))] &\in A(t)y(t) + F(t, y(t)) + (Bu)(t), \quad t \in J = [0, b] \\ y(0) &= y_0 \end{aligned}$$

by using a fixed point theorem for multivalued maps to Dhage combined with an evolution system. We are going to study the nonlocal impulsive Cauchy problem for this evolution system without controllability using a fixed point theorem for condensing multi-valued maps. We shall study the existence of solutions for the following evolution system in a Banach space X (with norm $\|\cdot\|$)

$$\frac{d}{dt}[y(t) - G(t, y(t))] \in A(t)y(t) + F(t, y(t)), t \in J = [0, a], \ t \neq t_k, \quad (1)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, 2, \ldots, m, \quad (2)$$

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$$y(0) + g(y) = y_0, \tag{3}$$

where $F : J \times X \to P(X)$ is a multi-valued map, $A(t)$ generates an evolution system, $G : J \times X \to X$ is a given function and $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, where $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$.

The nonlocal Cauchy problem was considered by Byszewski [2] and the importance of nonlocal conditions in different fields has been discussed in [2] and [6] and the references. In the past several years results on existence, uniqueness and stability of differential and functional differential abstract evolution Cauchy problem with nonlocal conditions have been studied by Byszewski and Lakshmikantham [4], by Byszewski [3], by Ezzinibi et al. [8], by Fu [10], by Fu and Ezzinbi [11] et. al. For results about impulsive differential systems, we refer readers to [13].

The rest of this paper is organized as follows: In section 2 we recall briefly some basic definitions and preliminary facts about multi-valued maps and evolution systems which will be used throughout this paper. The existence theorem for the problem (1)-(3) and its proof is arranged in section 3. Finally, in section 4 an example is presented to illustrate the applications of the obtained result.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space. $C(J, X)$ is the Banach space of continuous functions from J to X with the norm $\|x\|_J = \sup\{\|x(t)\| : t \in J\}$. $B(x)$ denotes the Banach space of bounded linear operators from X to X, with the norm $\|N\|_{B(X)} = \sup\{\|N(x)\| : \|x\| = 1\}$. A measurable function $x : J \to X$ is Bochner integrable if and only if $\|x\|$ is Lebegue integrable (For properties of the Bochner integral see Yosida [16]). $L^1(J, X)$ denotes the Banach space of Bochner integrable functions $x : J \to X$with norm $\|x\|_{L^1} = \int_0^a \|x(t)\|dt$.

We use the notations $P(X) = \{Y \in 2^X : Y \neq \emptyset\}, P_{cl}(X) = \{Y \in P(X) : Y \text{closed}\}$,
$P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}, P_c(X) = \{Y \in P(X) : Y \text{ convex}\}$, and $P_{cp}(X) = \{Y \in P(X) : Y \text{ compact}\}$.

A multi-valued map $F : X \to P(X)$ is said to be convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$. F is said to be bounded on bounded sets if $F(B) = \bigcup_{x \in B} F(x)$ is bounded in X for all $B \in P_b(X)$)(i.e., $\sup_{x \in B}\{\sup\{\|y\| : y \in F(x)\}\} < \infty$).

F is called upper semi-continuous (u.s.c) on X if for each $y_0 \in X$ the set $F(y_0)$ is a nonempty, closed subset of X, and if for each open subset $(\mho)$ of X containing $F(y_0)$, there exists an open neighborhood $\mathcal{N}$ of y_0 such that $F(\mathcal{N}) \subseteq (\mho)$.

F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in P_b(X)$. If the multi-valued map F is completely continuous with nonempty compact values, then F is u.s.c. if and only if F has a closed graph (i.e., $x_n \to x, y_n \to y, y_n \in F(x_n)$ imply $y_* \in F(x_*)$). We say that F has a fixed point if there is $x \in X$ such that $x \in F(x)$.

A multi-valued map $F : J \to P_{cl}(X)$ is said to be measurable if for each $x \in X$ the function $Y : J \to R$ defined by $Y(t) = d(x, F(t)) = \inf\{\|x - z\| : z \in F(t)\}$ is measurable.

An upper-semi continuous multi-valued map $F : X \to P(X)$ is said to be condensing [7] if for any subset $B \subset X$ with $\alpha(B) \neq 0$ we have $\alpha(F(B)) < \alpha(B)$, where α denotes the Kuratowski measure of non-compactness [1].

For more details on multi-valued maps we refer to the books by Deimling [7], by Hu [12].

Let $\{A(t) : t \in J\}$ be a family of linear operators and satisfy:

(A1) The domain $D(A(t)) = D$ of $A(t)$ is dense in X and independent of $t, A(t)$ is closed linear operator.

(A2) For each $t \in J$, the resolvent $R(\lambda, A(t))$ exists for all λ with $Re\lambda \leq 0$ and there exists $k > 0$ such that $\|R(\lambda, A(t))\| \leq \frac{k}{|\lambda|+1}$.

(A3) There exist constants $H > 0$ and $0 < \alpha \leq 1$ such that for $t, s, \tau \in J$

$$\|(A(t) - A(s))A^{-1}(\tau)\| \leq H|t - s|^{\alpha}.$$

(A4) For each $t \in J$ and some $\lambda \in \rho(A(t))$, the resolvent $R(\lambda, A(t))$ is compact operator.

To set the framework for our main existence results, we need to introduce the following definitions.

Definition 1 [15] A two parameter family of bounded linear operators $U(t, s), 0 \leq s \leq t \leq a$, on X is called an evolution system if the following two conditions are satisfied: (i)$U(s, s) = I, U(t, r)U(r, s) = U(t, s)$ for $0 \leq s \leq r \leq t \leq a$.
(ii)$(t, s) \to U(t, s)$ is strongly continuous for $0 \leq s \leq t \leq a$.

Definition 2 [5] A function $y \in C(J, X)$ is called a mild solution of the problem (1.1)-(1.3) if the following holds: $y(0) + g(y) = y_0$ for each $0 \leq t \leq a$; $\Delta y|_{t=t_k} = I_k(y(t_k^-)), k = 1, \ldots, m$, the restriction of $y(\cdot)$ to the interval $[0, a) - \{t_1, t_2, \ldots, t_m\}$ is continuous and for each $s \in [0, t)$ the function $A(s)U(t, s)g(s, y(s))$ is integrable and the impulsive integral inclusion

$$\begin{aligned} y(t) &= U(t,0)[y_0 - g(y) - G(0, y(0))] + G(t, y(t)) \\ &\quad + \int_0^t U(t,s)A(s)G(s, y(s))ds + \int_0^t U(t,s)F(s, y(s))ds \\ &\quad + \sum_{0<t_k<t} U(t, t_k)I_k(y(t_k^-)), \end{aligned}$$

is satisfied.

Our main results are based on the following lemmas.

Lemma 1. ([14]) *Let X be a Banach space. Let $F : J \times X \to P_{b,cl,c}(X)$ satisfy that*

(i) For each $y \in X, (t, y) \to F(t, y)$ is measurable with respect to t.

(ii) For each $t \in J, (t, y) \to F(t, y)$ is u.s.c. with respect to y.

(iii) For each fixed $y \in C(J,X)$, *the set*

$$S_{F,y} = \{f \in L^1(J,X) : f(t) \in F(t,y(t)) \text{ for a.e } t \in J\}$$

is nonempty. Let Γ *be a linear continuous mapping from* $L^1(J,X) \to C(J,X)$. *Then the operator*

$$\Gamma \circ S_F : C(J,X) \to P_{cp,c}(C(J,X)), y \to (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is closed graph operator in $C(J,X) \times C(J,X)$.

Lemma 2. ([7]) *Let* Ω *be bounded and convex set in Banach space* X. $F : \Omega \to P(\Omega)$ *is a u.s.c., condensing multi-valued map. If for every* $x \in \Omega$, $F(x)$ *is a closed and convex set in* Ω, *then* F *has a fixed point in* Ω.

Lemma 3. ([15], *Theorem* 6.1) *Under the assumptions (A1)-(A3), there is a unique evolution system* $U(t,s)$ *on* $0 \leq s \leq t \leq a$, *satisfying:*

(i) $\|U(t,s)\| \leq C \quad \text{for} \quad 0 \leq s \leq t \leq a.$

(ii) *For* $0 \leq s \leq t \leq b, U(t,s) : X \to D$ *and* $t \to U(t,s)$ *is strongly differential in* X. *The derivative* $\frac{\partial}{\partial t}U(t,s) \in L(X)$ *and it is strongly continuous on* $0 \leq s \leq t \leq a$. *Moreover,*

$$\frac{\partial}{\partial t}U(t,s) + A(t)U(t,s) = 0 \quad \text{for} \quad 0 \leq s \leq t \leq a$$

$$\|\frac{\partial}{\partial t}U(t,s)\| = \|A(t)U(t,s)\| \leq \frac{C}{t-s}$$

and $\|A(t)U(t,s)A(s)^{-}\| \leq C \quad \text{for} \quad 0 \leq s \leq t \leq b.$

(iii) *For every* $v \in D$ *and* $t \in [0,a], U(t,s)v$ *is differentiable with respect to* s *on* $0 \leq s \leq t \leq a$ *and* $\frac{\partial}{\partial s}U(t,s)v = U(t,s)A(s)v$.

Lemma 4. ([9]) *Let* $\{A(t) : t \in J\}$ *satisfy conditions (A1)-(A4). If* $\{U(t,s) : 0 \leq s \leq t \leq a\}$ *is the linear evolution system generated by* $\{A(t) : t \in J\}$, *then* $\{U(t,s) : 0 \leq s \leq t \leq a\}$ *is a compact operator whenever* $t - s > 0$.

3. Existence result

In order to define the concept of integral solutions for the problem(1)-(3), we shall consider the following space
$\Omega = \{y : [0,a] \to X : y(t)$ is continuous everywhere except for some t_k at which $y(t_k^-)$ and $y(t_k^+)$, $k = 1,2,\ldots,m$, exist and $y(t_k^-) = y(t_k)\}$.

Obviously, for any $t \in J$ and $y \in \Omega$, Ω is a Banach space with the norm

$$\|y\|_\Omega = \sup\{|y(t)| : t \in [0,a]\}.$$

Let us list the following hypotheses:

(H1) $U(t,s)$ is a compact operator whenever $t - s > 0$ and there exists a constant $M > 0$ such that $\|U(t,s)\| \leq M$ for $0 \leq s \leq t \leq a$.

(H2) There exists a constant $M_1 > 0$ such that $\|A(t)A(0)^{-1}\| \leq M_1$ for $t \in J$.

(H3) The function $G : J \times X \to D$ is continuous and there exists constants $L, L_1 > 0$ such that

$$\|A(0)G(t,u) - A(0)G(t,v)\| \le L(\|u - v\|) for u, v \in X,$$

and

$$\|A(0)G(t,u)\| \le L_1(\|u\| + 1), for u \in X.$$

(H4) The multi-valued map $F : J \times X \to P_{c,cp}(X)$ satisfies the following conditions:

(i) for each $t \in J$, the function $F(t,\cdot) : X \to P_{c,cp}(X)$ is u.s.c. and for each $y \in X$, the function $F(\cdot, y)$ is measurable. And for each fixed $u \in \Omega$ the set

$S_{F,u} = \{f \in L^1(J,X) : f(t) \in F(t,u)$ a.e $t \in J\}$ is nonempty.

(ii) for each positive number $l \in N$, there exists a positive function $w(l)$ dependent on l such that

$$\sup_{\|y\| \le l} \|F(t,y)\| \le w(l) \text{ and } \lim_{l \to \infty} \inf \frac{w(l)}{l} = \gamma < \infty,$$

where

$$\|F(t,y)\| = \sup\{\|f\| : f \in F(t,y), \|y\| = \sup_{0 \le s \le a} \|y(s)\|\}.$$

(H5) $g : \Omega \to X$ satisfies that

(i) there exist positive constants L_2 and L_2' such that

$$\|g(u)\| \le L_2\|u\|_\Omega + L_2' \text{ for all } u \in \Omega.$$

(ii) g is a completely continuous map.

(H6) $I_k \in C(X,X), k = 1, 2, \ldots, m$ are all bounded, that is there exist constant $d_k, k = 1, 2, \ldots, m$ such that $\|I_k(y)\| \le d_k, x \in X$.

Theorem 1. *Let $y_0 \in X$. If the hypotheses (H1)-(H6) are satisfied, then the system(1)-(3) has a mild solution provided that,*

$$L_0 := L[(M+1)\|A(0)^{-1}\| + aMM_1] < 1, \tag{4}$$

and

$$(\|A(0)^{-1}\|L_1 + L_2)M + \|A(0)^{-1}\|L_1 + aMM_1L_1 + aM\gamma < 1. \tag{5}$$

Proof. Consider the operator $N : \Omega \to P(\Omega)$ defined by

$$\begin{aligned} N(y) = u \in \Omega : u(t) \quad = \quad & U(t,0)[y_0 - g(y) - G(0, y(0))] + G(t, y(t)) \\ & + \int_0^t A(s)U(t,s)G(s,y(s))ds + \int_0^t U(t,s)f(s)ds \\ & + \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k^-)), \ f \in S_{F,y}, \ t \in J. \end{aligned}$$

Clearly the fixed points of N are mild solutions to (1)-(3).

We show that N satisfies the hypotheses of lemma 4. For the sake of convenience, we subdivide the proof into several steps.

Step 1. There exists a positive number $l \in N$ such that $N(B_l) \subset B_l$, where $B_l = \{y \in \Omega : \|y(t)\| \leq l, 0 \leq t \leq a\}$.

For each positive number l, B_l is clearly a bounded closed convex set in Ω. We claim that there exists a positive integer l such that $N(B_l) \subset B_l$, where $N(B_l) = \bigcup_{y \in B_l} N(y)$. If it is not true, then for each positive integer l, there exist functions $y_l(\cdot) \in B_l$ and $u_l(\cdot) \in N(y_l)$, but $u_l(\cdot)$ does not belongs to B_l, that is $\|u_l(t)\| > l$ for some $t(l) \in [0, a]$, where $t(l)$ denotes t is dependent on l. However, on the other hand, we have,

$$\begin{aligned} l < \|u_l(t)\| &= \|U(t,0)[y_0 - g(y_l) - G(0, y_l(0))] \\ &\quad + G(t, y_l(t)) + \int_0^t U(t,s)A(s)G(s, y_l(s))ds \\ &\quad + \int_0^t U(t,s)f_l(s)ds \\ &\quad + \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k^-))\| \end{aligned}$$

where $f_l \in S_{F,y_l}$. Hence,

$$\begin{aligned} l &< \|U(t,0)[y_0 - g(y_l) - A(0)A(0)^{-1}G(0, y_l(0))]\| \\ &\quad + \|A(0)A(0)^{-1}G(t, y_l(t))\| + \|\int_0^t A(0)A(0)^{-1}U(t,s)A(s)G(s, y_l(s))ds\| \\ &\quad + \|\int_0^t U(t,s)f_l(s)ds\| \\ &\quad + \sum_{0<t_k<t} \|U(t,t_k)\|\|I_k(y(t_k^-))\| \\ &\leq M[\|y_0\| + L_2 l + L_2' + \|A(0)^{-1}\|L_1(l+1) \\ &\quad + \|A(0)^{-1}\|L_1(l+1) + aMM_1L_1(l+1) + aMw(l) + \sum_{k=1}^m Md_k \\ &\leq M[y_0\| + L_2 l + L_2'] + (M+1)\|A(0)^{-1}\|L_1(l+1) \\ &\quad + aMM_1L_1(l+1) + aMW(l) + \sum_{k=1}^m Md_k. \end{aligned}$$

Dividing on both sides by l and taking the lower limit as $l \to +\infty$, we get

$$(\|A(0)^{-1}\|L_1 + L_2)M + \|A(0)^{-1}\|L_1 + aMM_1L_1 + aM\gamma \geq 1,$$

which contradicts (4). Hence for some positive integer l, $N(B_l) \subseteq B_l$.

Step 2. $N(y)$ is convex for each $y \in \Omega$.

Indeed, if $u_1, u_2 \in N(y)$, then there exist $f_1, f_2 \in S_{F,y}$, such that for each $t \in J$ we have,

$$\begin{aligned} u_i(t) &= U(t,s)[y_0 - g(y) - G(0,y(0))] + G(t,y(t)) \\ &+ \int_0^t U(t,s)A(s)G(s,y(s))ds + \int_0^t U(t,s)f_i(s)ds \\ &+ \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k^-)),\ i=1,2. \end{aligned}$$

Let $0 \le \lambda \le 1$. Then for each $t \in J$ we have,

$$\begin{aligned} (\lambda u_1 + (1-\lambda)u_2)(t) &= U(t,s)[y_0 - g(y) - G(0,y(0))] + G(t,y(t)) \\ &+ \int_0^t U(t,s)A(s)G(s,y(s))ds \\ &+ \int_0^t U(t,s)[\lambda f_1(s) + (1-\lambda f_2(s))]ds \\ &+ \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k^-)). \end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), $\lambda u_1 + (1-\lambda)u_2 \in N(y)$.

Step 3. $N(y)$ is closed for each $y \in \Omega$.

Let $\{x_n\}_{n\ge0} \in N(y)$ such that $x_n \to x$ in Ω. Then $x \in \Omega$ and there exists $f_n \in S_{F,y}$ such that, for every $t \in [0,a]$,

$$\begin{aligned} x_n(t) &= U(t,s)[y_0 - g(y) - G(0,y(0))] + G(t,y(t)) \\ &+ \int_0^t U(t,s)A(s)G(s,y(s))ds + \int_0^t U(t,s)f_n(s)ds \\ &+ \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k^-)). \end{aligned}$$

Using the fact that F has compact values, we may pass to a subsequence if necessary to get that f_n converges to f in $L^1([0,a],X)$ and hence $f \in S_{F,y}$. Then for each $t \in [0,a]$,

$$\begin{aligned} x_n(t) \to x(t) &= U(t,s)[y_0 - g(y) - G(0,y(0))] + G(t,y(t)) \\ &+ \int_0^t U(t,s)A(s)G(s,y(s))ds + \int_0^t U(t,s)f(s)ds \\ &+ \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k^-)),\ t \in J. \end{aligned}$$

Thus, $x \in N(y)$.

Step 4. N is u.s.c. and condensing.

For this purpose, we decompose N as $N_1 + N_2$, where the operators N_1, N_2 are defined on B_l respectively by

$$(N_1y)(t) = G(t,y(t)) - U(t,0)G(0,y(0)) + \int_0^t U(t,s)A(s)G(s,y(s))ds,$$

$$\begin{aligned} N_2 y \quad &= \{u \in \Omega : u(t) = U(t,0)[y_0 - g(y)] + \textstyle\int_0^t U(t,s)f(s)ds \\ &\quad + \textstyle\sum_{0<t_k<t} U(t,t_k)I_k(y(t_k^-)),\ f \in S_{F,y}\}. \end{aligned}$$

We will verify that N_1 is a contraction while N_2 is a completely continuous operator. To prove that N_1 is a contraction, we take $y_1, y_2 \in B_l$ arbitrarily. Then for each $t \in J$ and by condition (H3), we have that,

$$\begin{aligned} &\|(N_1 y_1)(t) - (N_1 y_2)(t)\| \\ \le\ & \|G(t, y_1(t)) - G(t, y_2(t))\| \\ & + \|U(t,0)[G(0, y_1(0)) - G(0, y_2(0))]\| \\ & + \|\int_0^t U(t,s)A(s)[G(s, y_1(s)) - G(s, y_2(s))]ds\| \\ =\ & \|A(0)^{-1}[A(0)G(t, y_1(t)) - A(0)G(t, y_2(t))]\| \\ & + \|U(t,0)A(0)^{-1}[A(0)G(0, y_1(0)) - A(0)G(0, y_2(0))]\| \\ & + \|\int_0^t A(0)^{-1}A(0)U(t,s)A(s)[G(s, y_1(s)) - G(s, y_2(s))]ds\| \\ \le\ & [(M+1)\|A(0)^{-1}\|L + \int_0^t MM_1 L ds] \sup_{0\le s\le a} \|y_1(s) - y_2(s)\| \\ \le\ & L[(M+1)\|A(0)^{-1}\| + aMM_1] \sup_{0\le s\le a} \|y_1(s) - y_2(s)\| \\ =\ & L_0 \sup_{0\le s\le a} \|y_1(s) - y_2(s)\|. \end{aligned}$$

Thus

$$\|N_1 y_1 - N_1 y_2\| \le L_0 \|y_1 - y_2\|.$$

Therefore, by assumption $0 < L_0 < 1$ (see (4)), we obtain that $N_1$is a contraction.

Next, we show that N_2 is u.s.c. and condensing.

(i) $N_2(B_l)$ is clearly bounded.

(ii) $N_2(B_l)$ is equicontinuous.

Let $\tau_1, \tau_2 \in J, \tau_1 < \tau_2$. Let $y \in B_l$ and $u \in N_2(y)$. Then there exists $f \in S_{F,y}$ such that for each $t \in J$, we have,

$$u(t) = U(t,0)[y_0 - g(y)] + \int_0^t U(t,s)f(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k^-)).$$

Then

$$\begin{aligned} \|u(\tau_2) - u(\tau_1)\| \ \le\ & \|[U(\tau_2,0) - U(\tau_1,0)](y_0 - g(y))\| \\ & + \|\int_0^{\tau_1} [U(\tau_2,s) - U(\tau_1,s)]f(s)ds\| \\ & + \|\int_{\tau_1}^{\tau 2} U(\tau_2,s)f(s)ds\| \end{aligned}$$

$$+\sum_{0<t_k<\tau_1}\|U(\tau_2,t_k)-U(\tau_1,t_k)\|d_k$$
$$+\sum_{\tau_1\le t_k<\tau_2} Md_k.$$

The right-hand side tends to zero as $\tau_2-\tau_1\to 0$, since $U(t,s)$ is strongly continuous, and the compactness of $U(t,s), t-s>0$ implies the continuity in the uniform operator topology.

(iii) $(N_2B_l)(t)$ is pre-compact for each $t\in J$, where $N_2B_l(t)=\{u(t):u\in N_2(B_l), t\in J\}$.

Obviously, by conditions (H5)(ii), $(N_2B_l)(t)$ is relatively compact in Ω for $t=0$. Let $0<t\le a$ be fixed and $0<\epsilon<t$. For $y\in B_l$ and $u\in N_2(y)$, there exists a function $f\in S_{F,y}$ such that,

$$\begin{aligned} u(t) &= U(t,0)[y_0-g(y)]+\int_0^{t-\epsilon}U(t,s)f(s)ds \\ &\quad +\int_{t-\epsilon}^{t}U(t,s)f(s)ds \\ &\quad +\sum_{0<t_k<t}U(t,t_k)I_k(y(t_k^-)). \end{aligned}$$

Define

$$\begin{aligned} u_\epsilon(t) &= U(t,0)[y_0-g(y)]+\int_0^{t-\epsilon}U(t,s)f(s)ds \\ &\quad +\sum_{0<t_k<t}U(t,t_k)I_k(y(t_k^-)) \\ &= U(t,0)[y_0-g(y)]+U(\epsilon)\int_0^{t-\epsilon}U(t-\epsilon-s)f(s)ds \\ &\quad +\sum_{0<t_k<t}U(t,t_k)I_k(y(t_k^-)). \end{aligned}$$

Since $U(t,s)(t-s>0)$ is compact, the set $U_\epsilon(t)=\{u_\epsilon(t):u\in N_2(B_l)\|$ is relatively compact in Ω for every ϵ, $0<\epsilon<t$. Moreover, for every $u\in N_2(B_l)$,

$$\|u(t)-u_\epsilon(t)\|=\|\int_{t-\epsilon}^{t}U(t,s)f(s)ds\| \quad \le M\int_{t-\epsilon}^{t}w(l)ds=Mw(l)\epsilon.$$

As $\epsilon\to 0$, we note that there are relatively compact sets arbitrarily close to the set $\{u(t):u\in N_2(B_l)\}$. So the set $\{u(t):u\in N_2(B_l)\}$ is relatively compact in Ω.

As a consequence of (i), (ii) and (iii) together with the Arzela-Ascoli theorem we can conclude that $N_2:B_l\to P(B_l)$ is a completely continuous multi-valued map.

(iv) N_2 has a closed graph.

Let $y_n \to y_*, y_n \in B_l, u_n \in N_2(y_n)$, and $u_n \to u_*$, we prove that $u_* \in N_2(y_*)$. The relation $u_n \in N_2(y_n)$ means that there exists $f_n \in S_{F,y_n}$ such that, for each $t \in J$,

$$\begin{aligned} u_n(t) &= U(t,0)[y_0 - g(y_n)] + \int_0^t U(t,s)f_n(s)ds \\ &\quad + \sum_{0<t_k<t} U(t,t_k)I_k(y_n(t_k^-)). \end{aligned}$$

We must prove that there exists $f_* \in S_{F,y_*}$ such that, for each $t \in J$,

$$\begin{aligned} u_*(t) &= U(t,0)[y_0 - g(y_*)] + \int_0^t U(t,s)f_*(s)ds \\ &\quad + \sum_{0<t_k<t} U(t,t_k)I_k(y_*(t_k^-)). \end{aligned}$$

Since $I_k, k = 1, 2, \ldots, m$, and g are continuous we have that,

$$\begin{aligned} \|u_n - U(t,0)[y_0 - g(y_n)] - \sum_{0<t_k<t} U(t,t_k)I_k(y_n(t_k^-)) - u_* - U(t,0)[y_0 - g(y_*)] \\ - \sum_{0<t_k<t} U(t,t_k)I_k(y_*(t_k^-))\|_\Omega \to 0, \end{aligned}$$

as $n \to \infty$.

Consider the linear continuous operator

$$\Gamma : L^1(J,X) \to C(J,X), \quad f \to \Gamma(f)(t) = \int_0^t U(t,s)f(s)ds.$$

From lemma (1), it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover we have that,

$$u_n(t) - U(t,0)[y_0 - g(y_n)] - \sum_{0<t_k<t} U(t,t_k)I_k(y_n(t_k^-)) \in \Gamma(S_{F,y_n}).$$

In view of $y_n \to y_*$, it follows from lemma 1 that

$$u_*(t) - U(t,0)[y_0 - g(y_*)] - \sum_{0<t_k<t} U(t,t_k)I_k(y_*(t_k^-)) \in \Gamma(S_{F,y_*}),$$

that is, there must exist a $f_*(t) \in S_{F,y_*}$ such that,

$$\begin{aligned} u_*(t)U(t,0)[y_0 - g(y_*)] - \sum_{0<t_k<t} U(t,t_k)I_k(y_*(t_k^-)) \\ = \Gamma(f_*(t)) = \int_0^t U(t,s)f_*(s)ds. \end{aligned}$$

Therefore, N_2 has a closed graph and N_2 is a completely continuous multi-valued map with compact value. So N_2 is u.s.c. On the there hand N_1 is a contraction, hence $N = N_1 + N_2$ is u.s.c. and condensing. By the fixed point theorem lemma 2, there exists a fixed point $y(\cdot)$ for N on B_l, which implies the problem (1)-(3) has a mild solution.

4. Example

As an application of our result, we consider the following partial differential inclusions such as

$$\frac{\partial}{\partial t}[z(t,x)-v(t,z(t,x))]-a(t,x)\frac{\partial^2}{\partial x^2}z(t,x)\in Q(t,z(t,x)),$$

$$z(t,0)=z(t,1)=0,\quad z(t_k^+)-z(t_k^-)=I_k(z(t_k^-)),k=1,\dots,m$$

$$z(0,x)+\sum_{i=0}^{p}\kappa_i(y,x)z(s_i,y)dy=z_0(x),t\in J=[0,1]\quad ,0\le x\le 1 \tag{6}$$

where p is a positive integer, $0<s_0<s_1<\cdots<s_p<1$, and $0<t_1<t_2<\cdots<t_m<1$, $z_0(x)\in X=L^2([0,1])$ and $a(t,x)$ is continuous. We can define respectively that,

$$G(t,w)(x)=v(t,w(x)),x\in[0,1],F(t,w)(x)=Q(t,w(x)),x\in[0,1]$$

$$g(w(t))=\sum_{i=0}^{p}K_iw(t_i),w\in\Omega(\ \Omega\text{ is defined as in section 3}),$$

where

$$K_i(z)(x)=\int_0^1\kappa_i(y,x)z(y)dy.$$

And $I_k\in C(X,X),k=1,\dots,m$, satisfying (H6).

Take $X=L^2[0,1]$ and also define $A(t):X\to X$ by $(A(t)w)(x)=a(t,x)w''$ with domain $[D(A)=\{w\in X: w,w'\text{are absolutely continuous},w''\in X,w(0)=w(1)=0\}]$.

Let $A(t)$ generate an evolution system $U(t,s)$([9],[15]) such that (H1) and (H2) hold.

$$\|A(0)G(t,u)-A(0)G(t,v)\|\le L(\|u-v\|)foru,v\in X.$$

and

$$\|A(0)G(t,u)\|\le L_1(\|u\|+1),foru\in X.$$

Then from Theorem 1, system (6) admits a mild solution on [0,1] under the above assumptions.

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