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VARIANTS OF NEWTON'S METHOD USING FIFTH-ORDER QUADRATURE FORMULAS: REVISITED

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ABSTRACT. In this paper, we point out some errors in a recent paper by Cordero and Torregrosa [7]. We prove the convergence of the variants of Newton's method for solving the system of nonlinear equations using two different approaches. Several examples are given, which illustrate the cubic convergence of these methods and verify the theoretical results.

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1. Introduction

Consider the problem of finding a real zero of a function $\mathbf{F}:\mathbf{D}\subseteq\mathbf{R^n}\longrightarrow\mathbf{R^n}$ that is, a real solution $\alpha\in R^n$ of the system of nonlinear equations

$$\mathbf{F}(\mathbf{X})=\mathbf{0}, \tag{1}$$

of n equations with n variables. In recent years, several iterative methods have been developed to solve the system of nonlinear equations (1), essentially by using the Taylor's polynomial [5,19], Adomian decomposition [1,4,8,10], homotopy perturbation method [13], quadrature formulas [3,6,7,9,10,12,18] and other techniques [2,11,14-17,19-21]. It is well known that the quadrature rules play an important and significant role in the evaluation of the integrals. It has been shown [3,6,7,9,10,12,18] that the quadrature formulas have been used to develop some iterative methods for solving the system of nonlinear equations (1). Cordero and Torregrosa [7] derived and analyzed two new iterative methods for solving the system of nonlinear equations (1) by using the open and closed Simpson quadrature formulas. They showed that both methods converge quadratically. Further more, they have shown that, if the coordinate functions f_i of $\mathbf{F}$ verify

$$\frac{\partial f_i(\alpha)}{\partial x_j\partial x_k}=0,\ \text{for}\ \ i,j,k\in\{1,2,\dots,n\}, \tag{2}$$

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then both iterative methods have cubic convergence. Here, we note that the convergence procedure provided by the authors has some errors, particularly in equation (20) and equation (24) of their paper [7]. We provide correct results after removing these errors, which is in section 3 of this paper. In this paper, we correct their errors and prove the cubic convergence of their methods without any condition by using the fixed point technique as well as the Taylor series technique. In section 4 of this paper, we provide numerical examples to verify the theoretical results. We note that there is no need of condition (2) for the cubic convergence of the methods. Our results can be considered as a refinement and improvement of the previously known results of [7] and others.

2. Iterative methods

In this section, we develop the iterative methods for solving the system of nonlinear equations (1) and this is the main motivation.

Let $\mathbf{F}:\mathbf{K}\subseteq\mathbf{R^n}\longrightarrow\mathbf{R^n}$ be r-times Fréchet-differentiable on a convex set $\mathbf{K}\subseteq\mathbf{R^n}$ and α be a real zero of the system of nonlinear equations (1), of n equations with n variables. For any $\mathbf{X},\mathbf{X_k}\in\mathbf{K}$, we may write (see [19]) the Taylor's expansion for $\mathbf{F}$ as follows.

$$\begin{aligned}\mathbf{F(X)} &= \mathbf{F(X_k)}+\mathbf{F'(X_k)(X-X_k)}+\frac{\mathbf{1}}{\mathbf{2!}}\mathbf{F''(X_k)(X-X_k)^2}\\ &+\frac{1}{3!}\mathbf{F'''(X_k)(X-X_k)^3}+\cdots+\frac{\mathbf{1}}{\mathbf{(r-1)!}}\mathbf{F^{(r-1)}(X_k)(X-X_k)^{r-1}}\\ &+\int_0^1\frac{(1-t)^{r-1}}{(r-1)!}\mathbf{F^{(r)}(X_k+t(X-X_k))(X-X_k)^r dt},\end{aligned}\tag{3}$$

For $r=1$, we have

$$\mathbf{F(X)}=\mathbf{F(X_k)}+\int_{\mathbf{0}}^{\mathbf{1}}\mathbf{F'(X_k+t(X-X_k))(X-X_k)dt}.\tag{4}$$

Approximating the integral in (4), we can obtain

$$\int_0^1\mathbf{F'(X_k+t(X-X_k))(X-X_k)dt}\cong\mathbf{F'(X_k)(X-X_k)}.\tag{5}$$

From (1), (4) and (5), we have

$$\mathbf{X}=\mathbf{X_k}-(\mathbf{F'(X_k)})^{-1}\mathbf{F(X_k)}.$$

This allows us to suggest the following one-step iterative method for solving the system of nonlinear equations (1).

Algorithm 2.1. For a given $\mathbf{X_0}$, compute the approximate solution $\mathbf{X_{k+1}}$ by the following iterative scheme:

$$\mathbf{X_{k+1}}=\mathbf{X_k}-(\mathbf{F'(X_k)})^{-1}\mathbf{F(X_k)},\quad \mathbf{k=0,1,2,\dots},\tag{6}$$

where $\mathbf{F'(X_k)}$ is the Jacobian matrix at the point $\mathbf{X_k}$. This method is known as Newton's method for solving the system of nonlinear equations (1). It is well known that the Newton's method has quadratic convergence (see [19]).

If we approximate the integral in (4) by using the Simpson quadrature formula, then we have

$$\int_0^1 \mathbf{F'(X_k + t(X - X_k))(X - X_k)dt} \cong \frac{1}{6}\left[\mathbf{F'(X_k) + 4F'(\frac{X_k + X}{2}) + F'(X)}\right]\mathbf{(X - X_k)}. \quad (7)$$

From (1), (4) and (7), we obtain

$$\mathbf{X = X_k - 6}\left[\mathbf{F'(X_k) + 4F'(\frac{X_k + X}{2}) + F'(X)}\right]^{-1}\mathbf{F(X_k)}.$$

Using this relation, we can suggest the following iterative method for solving the system of nonlinear equations (1).

Algorithm 2.2 [7]. For a given $\mathbf{X_0}$, compute the approximate solution $\mathbf{X_{k+1}}$ by the following iterative schemes:

$$\mathbf{X_{k+1} = X_k - 6}\left[\mathbf{F'(X_k) + 4F'(\frac{X_k + \lambda_k}{2}) + F'(\lambda_k)}\right]^{-1}\mathbf{F(X_k)}, \quad \mathbf{k = 0, 1, 2, \ldots}, \quad (8)$$

where $\lambda_k = \mathbf{X_k - (F'(X_k))^{-1}F(X_k)}$.

In a similar way, approximating the integral in (4) by using open quadrature formula, we have

$$\int_0^1 \mathbf{F'(X_k + t(X - X_k))(X - X_k)dt} \cong \frac{1}{3}\left[\mathbf{2F'(\frac{3X_k + X}{4}) - F'(\frac{X_k + X}{2}) + 2F'(\frac{X_k + 3X}{4})}\right]\mathbf{(X - X_k)}. \quad (9)$$

From (1), (4) and (9), we obtain

$$\mathbf{X = X_k - 3}\left[\mathbf{2F'(\frac{3X_k + X}{4}) - F'(\frac{X_k + X}{2}) + 2F'(\frac{X_k + 3X}{4})}\right]^{-1}\mathbf{F(X_k)}.$$

Using this relation, one can suggest the following iterative method for solving the system of nonlinear equations (1).

Algorithm 2.3 [7]. For a given $\mathbf{X_0}$, compute the approximate solution $\mathbf{X_{k+1}}$ by the following iterative schemes:

$$\mathbf{X_{k+1} = X_k - 3}\left[\mathbf{2F'(\frac{3X_k + \lambda_k}{4}) - F'(\frac{X_k + \lambda_k}{2}) + 2F'(\frac{X_k + 3\lambda_k}{4})}\right]^{-1}\mathbf{F(X_k)}, \quad k = 0, 1, 2, \ldots, \quad (10)$$

where $\lambda_k = \mathbf{X_k - (F'(X_k))^{-1}F(X_k)}$.

3. Convergence analysis

For the investigation of the convergence criteria of the iterative methods developed in section 2, we need the following results.

Consider $\mathbf{X} = (\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n})^{\mathbf{t}} \in \mathbf{R^n}$, $n > 1$, and denote by $J_{ij}(\mathbf{X})$ the (i, j)-entry of the Jacobian matrix $J(\mathbf{X})$, and by $H_{ij}(\mathbf{X})$ the corresponding entry of its inverse, then

$$\sum_{j=1}^{n} H_{ij}(\mathbf{X})\mathbf{J_{jk}}(\mathbf{X}) = \delta_{\mathbf{ik}}, \tag{11}$$

where $\delta_{\mathbf{ik}}$ is the Kronecker symbol.

Lemma 1[7]. *Let $\lambda(\mathbf{X})$ be the iteration function of classical Newton's method with*

$$\lambda_i(\mathbf{X}) = \mathbf{x_i} - \sum_{\mathbf{j=1}}^{\mathbf{n}} \mathbf{H_{ij}}(\mathbf{X})\mathbf{f_j}(\mathbf{X}), \quad \textbf{for} \;\; \mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots \mathbf{n},$$

then

$$\frac{\partial \lambda_i(\alpha)}{\partial x_k} = 0, \tag{12}$$

and

$$\frac{\partial^2 \lambda_i(\alpha)}{\partial x_k \partial x_l} = \sum_{j=1}^{n} H_{ij}(\alpha)\frac{\partial^2 f_j(\alpha)}{\partial x_k \partial x_l}, \quad for \;\; i, k, l \in \{1, 2, \ldots n\}. \tag{13}$$

Lemma 2 [7]. *Let $\mu(\mathbf{X})$ be the iteration function such that,*

$$\mu_i(\mathbf{X}) = \frac{\mathbf{x_i} + \lambda_\mathbf{i}(\mathbf{X})}{\mathbf{2}}, \quad \textbf{for} \;\; \mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots \mathbf{n},$$

where $\lambda(\mathbf{X})$ be the iteration function of classical Newton's method, then

$$\frac{\partial \mu_i(\alpha)}{\partial x_k} = \frac{1}{2}\delta_{ik}, \tag{14}$$

and

$$\frac{\partial^2 \mu_i(\alpha)}{\partial x_k \partial x_l} = \frac{1}{2}\sum_{j=1}^{n} H_{ij}(\alpha)\frac{\partial^2 f_j(\alpha)}{\partial x_k \partial x_l}, \quad for \;\; i, k, l \in \{1, 2, \ldots n\}. \tag{15}$$

Theorem 3.1 [7]. *Let $\mathbf{F} : \mathbf{D} \subseteq \mathbf{R^n} \longrightarrow \mathbf{R^n}$ be sufficiently differentiable at each point of an open neighborhood D of $\alpha \in R^n$, that is a solution of the system of nonlinear equations* (1). *Let us suppose that $J(\mathbf{X})$ is continuous and nonsingular at α, then the sequence $\{\mathbf{X_k}\}_{k\geq 0}$ obtained by using the iterative methods* (8) *and* (10) *converge quadratically to α. Moreover, if the coordinate functions f_i of $\mathbf{F}$ verify that $\frac{\partial f_i(\alpha)}{\partial x_j \partial x_k} = 0$, for $i, j, k \in \{1, 2, \ldots, n\}$, then both methods have convergence of order three.*

Remark 3.1. In the proof of the theorem 3.1, authors [7] have made the following technical mistakes:

Consider the equation (19) of the paper [7], which is:

$$g_i(\mathbf{X}) = \lambda_\mathbf{i}(\mathbf{X}) + \sum_{\mathbf{j=1}}^{\mathbf{n}} \mathbf{H_{ij}(X)f_j(X)} - \mathbf{6}\sum_{\mathbf{j=1}}^{\mathbf{n}} \mathbf{M_{ij}(X)f_j(X)}. \tag{16}$$

The authors have derived equation (20) in their paper [7] from the above equation

$$L_{ij}(\mathbf{X}) \left(\mathbf{g_j(X)} - \lambda_\mathbf{j}(\mathbf{X}) - \sum_{\mathbf{j=1}}^{\mathbf{n}} \mathbf{H_{ji}(X)f_i(X)} \right) + \mathbf{6}\sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{f_i(X)} = \mathbf{0}. \tag{17}$$

We note that, the derivation of equation (17) from equation (16) is wrong. There is no relation between equation (16) and equation (17). Actually in [7], authors did not take into account that the $L_{ij}(\mathbf{X})$ and $M_{ij}(\mathbf{X})$, are only the elements of a matrix $L(\mathbf{X})$ and its inverse matrix $(L(\mathbf{X}))^{-1} = M(\mathbf{X})$, respectively, but are not the inverse of each other. We derive the correct relation from equation (16), which is equation (27) of this paper.

Also, another error was found in equation (24) of [7], is due to differentiation of the equation

$$L_{ij}(\mathbf{X}) = \mathbf{J_{ij}(X)} + \mathbf{4J_{ij}}\left(\mu(\mathbf{X})\right) + \mathbf{J_{ij}}\left(\lambda(\mathbf{X})\right), \tag{18}$$

in a wrong way, where $\lambda(\mathbf{X})$ be the iteration function of classical Newton's method with

$$\lambda_i(\mathbf{X}) = \mathbf{x_i} - \sum_{\mathbf{j=1}}^{\mathbf{n}} \mathbf{H_{ij}(X)f_j(X)}, \quad \text{for } \mathbf{i = 1, 2, \ldots n},$$

and

$$\mu(\mathbf{X}) = \frac{\mathbf{X}+\lambda(\mathbf{X})}{\mathbf{2}}, \quad \text{with} \quad \mu_\mathbf{i}(\mathbf{X}) = \frac{\mathbf{x_i}+\lambda_\mathbf{i}(\mathbf{X})}{\mathbf{2}}, \quad \text{for } \mathbf{i = 1, 2, \ldots n}.$$

Using these facts, equation (18) can be written as

$$L_{ij}(\mathbf{X}) = \frac{\partial \mathbf{f_i(X)}}{\partial \mathbf{x_j}} + \mathbf{4}\frac{\partial \mathbf{f_i}\left(\mu(\mathbf{X})\right)}{\partial \mathbf{x_j}} + \frac{\partial \mathbf{f_i}\left(\lambda(\mathbf{X})\right)}{\partial \mathbf{x_j}}. \tag{19}$$

By differentiating this equation with respect to x_k, authors [7] have obtained

$$\frac{\partial L_{ij}(\mathbf{X})}{\partial \mathbf{x_k}} = \frac{\partial^2 f_i(\mathbf{X})}{\partial \mathbf{x_j}\partial \mathbf{x_k}} + 4\left[\sum_{q=1}^{n} \frac{\partial^2 f_i\left(\mu(\mathbf{X})\right)}{\partial \mu_q \partial x_k}\frac{\partial \mu_p(\mathbf{X})}{\partial \mathbf{x_j}} + \sum_{q=1}^{n} \frac{\partial f_i\left(\mu(\mathbf{X})\right)}{\partial \mu_q}\frac{\partial^2 \mu_p(\mathbf{X})}{\partial \mathbf{x_j}\partial \mathbf{x_k}}\right]$$
$$+ \sum_{q=1}^{n} \frac{\partial^2 f_i\left(\lambda(\mathbf{X})\right)}{\partial \lambda_q \partial x_k}\frac{\partial \lambda_p(\mathbf{X})}{\partial \mathbf{x_j}} + \sum_{q=1}^{n} \frac{\partial f_i\left(\lambda(\mathbf{X})\right)}{\partial \lambda_q}\frac{\partial^2 \lambda_p(\mathbf{X})}{\partial \mathbf{x_j}\partial \mathbf{x_k}}, \tag{20}$$

in equation (24) of [7].

Note that, equation (19) shows that $\frac{\partial f_i(\mu(\mathbf{X}))}{\partial x_j}$ and $\frac{\partial f_i(\lambda(\mathbf{X}))}{\partial x_j}$ are functions of $\mu(\mathbf{X})$ and $\lambda(\mathbf{X})$, respectively, where as $\mu(\mathbf{X})$ and $\lambda(\mathbf{X})$ both are functions of $\mathbf{X}$. But, the differentiation did by the authors in equation (20) is incorrect. We remove their errors by using the chain rule of differentiation in a correct way (see equation (35)).

We now discuss the convergence criteria of the iterative methods (8) and (10) for solving the system of nonlinear equations (1), using the fixed point approach.

Theorem 3.2. *Let* $\mathbf{F}:\mathbf{D}\subseteq\mathbf{R^n}\longrightarrow\mathbf{R^n}$ *be sufficiently differentiable at each point of an open neighborhood* D *of* $\alpha\in R^n$*, that is a solution of the system of nonlinear equations* (1)*. Let us suppose that* $J(\mathbf{X})$ *is continuous and nonsingular at* α*, then the sequence* $\{\mathbf{X_k}\}_{k\geq 0}$ *obtained by using the iterative methods* (8) *and* (10) *converge cubically to* α*.*

Proof. The iterative scheme defined in (8), is given by

$$\lambda_k=\mathbf{X_k}-(\mathbf{J}(\mathbf{X_k}))^{-1}\mathbf{F}(\mathbf{X_k}), \tag{21}$$

$$\mathbf{X_{k+1}}=\mathbf{X_k}-\mathbf{6}\left[\mathbf{J}(\mathbf{X_k})+\mathbf{4J}(\frac{\mathbf{X_k}+\lambda_\mathbf{k}}{\mathbf{2}})+\mathbf{J}(\lambda_\mathbf{k})\right]^{-\mathbf{1}}\mathbf{F}(\mathbf{X_k}). \tag{22}$$

Let us consider the solution $\alpha\in R^n$ of the system of nonlinear equations (1) as a fixed point of some iteration function $\mathbf{G}:\mathbf{R^n}\longrightarrow\mathbf{R^n}$, by means of the fixed-point iteration method

$$\mathbf{X_{k+1}}=\mathbf{G}(\mathbf{X_k}),\ \ \mathbf{k}=\mathbf{0},\mathbf{1},\mathbf{2},\ldots,$$

described in (22). Let us denote by $g_i:R^n\longrightarrow R$, $i=0,1,2,\ldots,n$, the coordinate functions of $\mathbf{G}$ and expanding $g_i(\mathbf{X})$, $\mathbf{X}=(\mathbf{x_1},\mathbf{x_2},\ldots,\mathbf{x_n})^{\,t}\in R^n$, in a Taylor series about α yields

$$\begin{aligned}g_i(\mathbf{X})=\mathbf{g_i}(\alpha)+\sum_{\mathbf{j_1=1}}^{\mathbf{n}}\frac{\partial\mathbf{g_i}(\alpha)}{\partial\mathbf{x_{j_1}}}\mathbf{e_{j_1}}+\frac{\mathbf{1}}{\mathbf{2}}\sum_{\mathbf{j_1=1}}^{\mathbf{n}}\sum_{\mathbf{j_2=1}}^{\mathbf{n}}\frac{\partial^2\mathbf{g_i}(\alpha)}{\partial\mathbf{x_{j_1}}\partial\mathbf{x_{j_2}}}\mathbf{e_{j_1}}\mathbf{e_{j_2}}\\+\frac{1}{6}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\sum_{j_3=1}^{n}\frac{\partial^3 g_i(\alpha)}{\partial x_{j_1}\partial x_{j_2}\partial x_{j_3}}e_{j_1}e_{j_2}e_{j_3}+\cdots,\end{aligned} \tag{23}$$

where $e_{j_k}=x_{j_k}-\alpha_{j_k}$.

We denote L_{ij} the (i,j)-entry of the matrix

$$L(\mathbf{X})=\mathbf{J}(\mathbf{X})+\mathbf{4J}\left(\mu(\mathbf{X})\right)+\mathbf{J}\left(\lambda(\mathbf{X})\right),$$

and M_{ij} is the (i,j)-entry of $(L(\mathbf{X}))^{-1}$. Thus, the ith component of the iteration function corresponding to the method (22) is

$$g_i(\mathbf{X})=\lambda_\mathbf{i}(\mathbf{X})+\sum_{\mathbf{j=1}}^{\mathbf{n}}\mathbf{H_{ij}}(\mathbf{X})\mathbf{f_j}(\mathbf{X})-\mathbf{6}\sum_{\mathbf{j=1}}^{\mathbf{n}}\mathbf{M_{ij}}(\mathbf{X})\mathbf{f_j}(\mathbf{X}). \tag{24}$$

For $i=1,2,\ldots n$, in (24), we have

$$g_1(\mathbf{X})=\lambda_\mathbf{1}(\mathbf{X})+\sum_{\mathbf{j=1}}^{\mathbf{n}}\mathbf{H_{1j}}(\mathbf{X})\mathbf{f_j}(\mathbf{X})-\mathbf{6}\sum_{\mathbf{j=1}}^{\mathbf{n}}\mathbf{M_{1j}}(\mathbf{X})\mathbf{f_j}(\mathbf{X}),$$

$$g_2(\mathbf{X}) = \lambda_{\mathbf{2}}(\mathbf{X}) + \sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} \mathbf{H_{2j}}(\mathbf{X})\mathbf{f_j}(\mathbf{X}) - \mathbf{6}\sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} \mathbf{M_{2j}}(\mathbf{X})\mathbf{f_j}(\mathbf{X}),$$

$$\vdots \tag{25}$$

$$g_n(\mathbf{X}) = \lambda_{\mathbf{n}}(\mathbf{X}) + \sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} \mathbf{H_{nj}}(\mathbf{X})\mathbf{f_j}(\mathbf{X}) - \mathbf{6}\sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} \mathbf{M_{nj}}(\mathbf{X})\mathbf{f_j}(\mathbf{X}).$$

Pre-multiplying 1st, 2nd, ... , nth equations in (25) by $L_{i1}(\mathbf{X})$, $L_{i2}(\mathbf{X})$, ... , $L_{in}(\mathbf{X})$, respectively, we have

$$\begin{aligned}
L_{i1}(\mathbf{X})\mathbf{g_1}(\mathbf{X}) &= \mathbf{L_{i1}}(\mathbf{X})\lambda_{\mathbf{1}}(\mathbf{X}) + \mathbf{L_{i1}}(\mathbf{X})\sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} \mathbf{H_{1j}}(\mathbf{X})\mathbf{f_j}(\mathbf{X}) \\
&\quad -6L_{i1}(\mathbf{X})\sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} \mathbf{M_{1j}}(\mathbf{X})\mathbf{f_j}(\mathbf{X}), \\
L_{i2}(\mathbf{X})\mathbf{g_2}(\mathbf{X}) &= \mathbf{L_{i2}}(\mathbf{X})\lambda_{\mathbf{2}}(\mathbf{X}) + \mathbf{L_{i2}}(\mathbf{X})\sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} \mathbf{H_{2j}}(\mathbf{X})\mathbf{f_j}(\mathbf{X}) \\
&\quad -6L_{i2}(\mathbf{X})\sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} \mathbf{M_{2j}}(\mathbf{X})\mathbf{f_j}(\mathbf{X}), \\
&\vdots \\
L_{in}(\mathbf{X})\mathbf{g_n}(\mathbf{X}) &= \mathbf{L_{in}}(\mathbf{X})\lambda_{\mathbf{n}}(\mathbf{X}) + \mathbf{L_{in}}(\mathbf{X})\sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} \mathbf{H_{nj}}(\mathbf{X})\mathbf{f_j}(\mathbf{X}) \\
&\quad -6L_{in}(\mathbf{X})\sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} \mathbf{M_{nj}}(\mathbf{X})\mathbf{f_j}(\mathbf{X}).
\end{aligned} \tag{26}$$

Adding all equations in (26), we have

$$\begin{aligned}
\sum_{j=1}^{n} L_{ij}(\mathbf{X})\mathbf{g_j}(\mathbf{X}) &= \sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} \mathbf{L_{ij}}(\mathbf{X})\lambda_{\mathbf{j}}(\mathbf{X}) + \sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} \mathbf{L_{ij}}(\mathbf{X})\left(\sum_{\mathbf{q}=\mathbf{1}}^{\mathbf{n}} \mathbf{H_{jq}}(\mathbf{X})\mathbf{f_q}(\mathbf{X})\right) \\
&\quad - 6\sum_{j=1}^{n} L_{ij}(\mathbf{X})\left(\sum_{\mathbf{q}=\mathbf{1}}^{\mathbf{n}} \mathbf{M_{jq}}(\mathbf{X})\mathbf{f_q}(\mathbf{X})\right).
\end{aligned}$$

By using (11), the above equation can be manipulated as

$$\begin{aligned}
\sum_{j=1}^{n} L_{ij}(\mathbf{X})\mathbf{g_j}(\mathbf{X}) &= \sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} \mathbf{L_{ij}}(\mathbf{X})\lambda_{\mathbf{j}}(\mathbf{X}) + \sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} \mathbf{L_{ij}}(\mathbf{X})\left(\sum_{\mathbf{q}=\mathbf{1}}^{\mathbf{n}} \mathbf{H_{jq}}(\mathbf{X})\mathbf{f_q}(\mathbf{X})\right) \\
&\quad - 6\sum_{q=1}^{n} \delta_{iq} f_q(\mathbf{X}).
\end{aligned}$$

Through which, we have

$$\sum_{j=1}^{n} L_{ij}(\mathbf{X})\left(\mathbf{g_j}(\mathbf{X}) - \lambda_{\mathbf{j}}(\mathbf{X}) - \sum_{\mathbf{q}=\mathbf{1}}^{\mathbf{n}} \mathbf{H_{jq}}(\mathbf{X})\mathbf{f_q}(\mathbf{X})\right) + \mathbf{6f_i}(\mathbf{X}) = \mathbf{0}. \quad (27)$$

By direct differentiation of equation (27) with respect to x_k, we have

$$\sum_{j=1}^{n} \frac{\partial L_{ij}(\mathbf{X})}{\partial \mathbf{x_k}}\left(g_j(\mathbf{X}) - \lambda_{\mathbf{j}}(\mathbf{X}) - \sum_{\mathbf{q}=\mathbf{1}}^{\mathbf{n}} \mathbf{H_{jq}}(\mathbf{X})\mathbf{f_q}(\mathbf{X})\right)$$
$$+ \sum_{j=1}^{n} L_{ij}(\mathbf{X})\left(\frac{\partial \mathbf{g_j}(\mathbf{X})}{\partial \mathbf{x_k}} - \frac{\partial \lambda_{\mathbf{j}}(\mathbf{X})}{\partial \mathbf{x_k}} - \sum_{\mathbf{q}=\mathbf{1}}^{\mathbf{n}} \frac{\partial \mathbf{H_{jq}}(\mathbf{X})}{\partial \mathbf{x_k}}\mathbf{f_q}(\mathbf{X})\right.$$
$$\left. - \sum_{q=1}^{n} H_{jq}(\mathbf{X})\frac{\partial \mathbf{f_q}(\mathbf{X})}{\partial \mathbf{x_k}}\right) + 6\frac{\partial f_i(\mathbf{X})}{\partial \mathbf{x_k}} = 0. \quad (28)$$

Evaluating (28) at $\mathbf{X} = \alpha$ and by using (12) and (18), we obtain

$$\sum_{j=1}^{n} \frac{\partial L_{ij}(\alpha)}{\partial x_k}(\alpha_j - \alpha_j) + 6\sum_{j=1}^{n} J_{ij}(\alpha)\left(\frac{\partial g_j(\alpha)}{\partial x_k} - \sum_{q=1}^{n} H_{jq}(\alpha)\frac{\partial f_q(\alpha)}{\partial x_k}\right)$$
$$+ 6\frac{\partial f_i(\alpha)}{\partial x_k} = 0.$$

Through which, we get

$$\sum_{j=1}^{n} J_{ij}(\alpha)\frac{\partial g_j(\alpha)}{\partial x_k} - \sum_{j=1}^{n} J_{ij}(\alpha)\left(\sum_{q=1}^{n} H_{jq}(\alpha)\frac{\partial f_q(\alpha)}{\partial x_k}\right) + \frac{\partial f_i(\alpha)}{\partial x_k} = 0.$$

By using (11) in this equation, we have

$$\sum_{j=1}^{n} J_{ij}(\alpha)\frac{\partial g_j(\alpha)}{\partial x_k} - \sum_{q=1}^{n} \delta_{iq}\frac{\partial f_q(\alpha)}{\partial x_k} + \frac{\partial f_i(\alpha)}{\partial x_k} = 0.$$

Thus, we obtain

$$\sum_{j=1}^{n} J_{ij}(\alpha)\frac{\partial g_j(\alpha)}{\partial x_k} - \frac{\partial f_i(\alpha)}{\partial x_k} + \frac{\partial f_i(\alpha)}{\partial x_k} = 0.$$

From the above equation, we have

$$J_{i1}(\alpha)\frac{\partial g_1(\alpha)}{\partial x_k} + J_{i2}(\alpha)\frac{\partial g_2(\alpha)}{\partial x_k} + \cdots + J_{in}(\alpha)\frac{\partial g_n(\alpha)}{\partial x_k} = 0. \quad (29)$$

Since the Jacobian matrix $J(\mathbf{X})$ is assumed to be nonsingular in a neighborhood of α, so from (29) it can be concluded that

$$\frac{\partial g_j(\alpha)}{\partial x_k} = 0, \text{ for } j,k \in \{1,2,\ldots,n\}. \quad (30)$$

Let us differentiate equation (28) with respect to x_l, we obtain

$$\sum_{j=1}^{n}\frac{\partial^2 L_{ij}(\mathbf{X})}{\partial \mathbf{x_k}\partial \mathbf{x_l}}\left(g_j(\mathbf{X})-\lambda_{\mathbf{j}}(\mathbf{X})-\sum_{\mathbf{q=1}}^{\mathbf{n}}\mathbf{H_{jq}(X)f_q(X)}\right)$$
$$+\sum_{j=1}^{n}\frac{\partial L_{ij}(\mathbf{X})}{\partial \mathbf{x_k}}\left(\frac{\partial g_j(\mathbf{X})}{\partial \mathbf{x_l}}-\frac{\partial \lambda_j(\mathbf{X})}{\partial \mathbf{x_l}}-\sum_{q=1}^{n}\frac{\partial H_{jq}(\mathbf{X})}{\partial \mathbf{x_l}}f_q(\mathbf{X})-\sum_{\mathbf{q=1}}^{\mathbf{n}}\mathbf{H_{jq}(X)}\frac{\partial \mathbf{f_q(X)}}{\partial \mathbf{x_l}}\right)$$
$$+\sum_{j=1}^{n}\frac{\partial L_{ij}(\mathbf{X})}{\partial \mathbf{x_l}}\left(\frac{\partial g_j(\mathbf{X})}{\partial \mathbf{x_k}}-\frac{\partial \lambda_j(\mathbf{X})}{\partial \mathbf{x_k}}-\sum_{q=1}^{n}\frac{\partial H_{jq}(\mathbf{X})}{\partial \mathbf{x_k}}f_q(\mathbf{X})-\sum_{\mathbf{q=1}}^{\mathbf{n}}\mathbf{H_{jq}(X)}\frac{\partial \mathbf{f_q(X)}}{\partial \mathbf{x_k}}\right)$$
$$+\sum_{j=1}^{n}L_{ij}(\mathbf{X})\left(\frac{\partial^2 \mathbf{g_j(X)}}{\partial \mathbf{x_k}\partial \mathbf{x_l}}-\frac{\partial^2 \lambda_{\mathbf{j}}(\mathbf{X})}{\partial \mathbf{x_k}\partial \mathbf{x_l}}-\sum_{\mathbf{q=1}}^{\mathbf{n}}\frac{\partial^2 \mathbf{H_{jq}(X)}}{\partial \mathbf{x_k}\partial \mathbf{x_l}}\mathbf{f_q(X)}-\sum_{\mathbf{q=1}}^{\mathbf{n}}\frac{\partial \mathbf{H_{jq}(X)}}{\partial \mathbf{x_k}}\frac{\partial \mathbf{f_q(X)}}{\partial \mathbf{x_l}}\right)$$
$$+\sum_{j=1}^{n}L_{ij}(\mathbf{X})\left(-\sum_{\mathbf{q=1}}^{\mathbf{n}}\frac{\partial \mathbf{H_{jq}(X)}}{\partial \mathbf{x_l}}\frac{\partial \mathbf{f_q(X)}}{\partial \mathbf{x_k}}-\sum_{\mathbf{q=1}}^{\mathbf{n}}\mathbf{H_{jq}(X)}\frac{\partial^2 \mathbf{f_q(X)}}{\partial \mathbf{x_k}\partial \mathbf{x_l}}\right)+\mathbf{6}\frac{\partial^2 \mathbf{f_i(X)}}{\partial \mathbf{x_k}\partial \mathbf{x_l}}=\mathbf{0}. \tag{31}$$

From equation (11), we have $\sum_{j=1}^{n}H_{ij}(\mathbf{X})\frac{\partial \mathbf{f_j(X)}}{\partial \mathbf{x_k}}=\delta_{\mathbf{ik}}$, By differentiating this relation w. r. to x_l, we get

$$\sum_{j=1}^{n}\frac{\partial H_{ij}(\mathbf{X})}{\partial \mathbf{x_l}}\frac{\partial f_j(\mathbf{X})}{\partial \mathbf{x_k}}=-\sum_{j=1}^{n}H_{ij}(\mathbf{X})\frac{\partial^2 \mathbf{f_j(X)}}{\partial \mathbf{x_k}\partial \mathbf{x_l}}. \tag{32}$$

By using equations (12), (13), (18), (30), (32) and evaluating (31) at $\mathbf{X}=\alpha$, we obtain

$$\sum_{j=1}^{n}\frac{\partial L_{ij}(\alpha)}{\partial x_k}\left(-\sum_{q=1}^{n}H_{jq}(\alpha)\frac{\partial f_q(\alpha)}{\partial x_l}\right)+\sum_{j=1}^{n}\frac{\partial L_{ij}(\alpha)}{\partial x_l}\left(-\sum_{q=1}^{n}H_{jq}(\alpha)\frac{\partial f_q(\alpha)}{\partial x_k}\right)$$
$$+\sum_{j=1}^{n}6J_{ij}(\alpha)\left(\frac{\partial^2 g_j(\alpha)}{\partial x_k\partial x_l}-\sum_{q=1}^{n}H_{jq}(\alpha)\frac{\partial^2 f_q(\alpha)}{\partial x_k\partial x_l}+\sum_{q=1}^{n}H_{jq}(\alpha)\frac{\partial^2 f_q(\alpha)}{\partial x_l\partial x_k}\right)$$
$$+\sum_{j=1}^{n}6J_{ij}(\alpha)\left(+\sum_{q=1}^{n}H_{jq}(\alpha)\frac{\partial^2 f_q(\alpha)}{\partial x_k\partial x_l}-\sum_{q=1}^{n}H_{jq}(\alpha)\frac{\partial^2 f_q(\alpha)}{\partial x_k\partial x_l}\right)+6\frac{\partial^2 f_i(\alpha)}{\partial x_k\partial x_l}=0. \tag{33}$$

Thus, we have

$$-\sum_{j=1}^{n}\frac{\partial L_{ij}(\alpha)}{\partial x_k}\left(\sum_{q=1}^{n}H_{jq}(\alpha)\frac{\partial f_q(\alpha)}{\partial x_l}\right)-\sum_{j=1}^{n}\frac{\partial L_{ij}(\alpha)}{\partial x_l}\left(\sum_{q=1}^{n}H_{jq}(\alpha)\frac{\partial f_q(\alpha)}{\partial x_k}\right)$$
$$+\sum_{j=1}^{n}6J_{ij}(\alpha)\left(\frac{\partial^2 g_j(\alpha)}{\partial x_k\partial x_l}\right)+6\frac{\partial^2 f_i(\alpha)}{\partial x_k\partial x_l}=0. \tag{34}$$

Differentiating equation (19) with respect to x_k, we get

$$\frac{\partial L_{ij}(\mathbf{X})}{\partial \mathbf{x_k}}=\frac{\partial^2 f_i(\mathbf{X})}{\partial \mathbf{x_j}\partial \mathbf{x_k}}+4\sum_{p=1}^{n}\frac{\partial^2 f_i(\mu(\mathbf{X}))}{\partial x_j\partial \mu_p}\frac{\partial \mu_p(\mathbf{X})}{\partial \mathbf{x_k}}+\sum_{p=1}^{n}\frac{\partial^2 f_i(\lambda(\mathbf{X}))}{\partial x_j\partial \lambda_p}\frac{\partial \lambda_p(\mathbf{X})}{\partial \mathbf{x_k}}. \tag{35}$$

Evaluating (35) at $\mathbf{X} = \alpha$ and by using (12) and (14), we have

$$\frac{\partial L_{ij}(\alpha)}{\partial x_k} = 3\frac{\partial^2 f_i(\alpha)}{\partial x_j \partial x_k}. \tag{36}$$

By using (36) in (34), we obtain

$$-\sum_{j=1}^{n} 3\frac{\partial^2 f_i(\alpha)}{\partial x_j \partial x_k}\left(\sum_{q=1}^{n} H_{jq}(\alpha)\frac{\partial f_q(\alpha)}{\partial x_l}\right) - \sum_{j=1}^{n} 3\frac{\partial^2 f_i(\alpha)}{\partial x_j \partial x_l}\left(\sum_{q=1}^{n} H_{jq}(\alpha)\frac{\partial f_q(\alpha)}{\partial x_k}\right)$$
$$+ 6\sum_{j=1}^{n} J_{ij}(\alpha)\left(\frac{\partial^2 g_j(\alpha)}{\partial x_k \partial x_l}\right) + 6\frac{\partial^2 f_i(\alpha)}{\partial x_k \partial x_l} = 0. \tag{37}$$

By using equation (11) in the above equation, we obtain

$$-3\frac{\partial^2 f_i(\alpha)}{\partial x_l \partial x_k} - 3\frac{\partial^2 f_i(\alpha)}{\partial x_k \partial x_l} + 6\sum_{j=1}^{n} J_{ij}(\alpha)\left(\frac{\partial^2 g_j(\alpha)}{\partial x_k \partial x_l}\right) + 6\frac{\partial^2 f_i(\alpha)}{\partial x_k \partial x_l} = 0.$$

Thus, we get $\sum_{j=1}^{n} J_{ij}(\alpha)\left(\frac{\partial^2 g_j(\alpha)}{\partial x_k \partial x_l}\right) = 0$. This relation can be written as

$$J_{i1}(\alpha)\left(\frac{\partial^2 g_1(\alpha)}{\partial x_k \partial x_l}\right) + J_{i2}(\alpha)\left(\frac{\partial^2 g_2(\alpha)}{\partial x_k \partial x_l}\right) + \cdots + J_{in}(\alpha)\left(\frac{\partial^2 g_n(\alpha)}{\partial x_k \partial x_l}\right) = 0. \tag{38}$$

Since the Jacobian matrix $J(\mathbf{X})$ is assumed to be nonsingular in a neighborhood of α, Thus from (38), we can conclude that

$$\frac{\partial^2 g_j(\alpha)}{\partial x_k \partial x_l} = 0, \text{ for } j, k, l \in \{1, 2, \ldots, n\}, \tag{39}$$

By using (30) and (39) in Taylor series (23), we conclude that the iterative method (8) is at least cubically convergent. In a similar way, one can prove, the cubic convergence of the iterative method (10). □

Now, we use Taylor's expansion technique to prove that the iterative methods (8) and (10) have cubic convergence. This technique is mainly due to Darvishi and Barati [8-10] and Frontini and Sormani [12].

Theorem 3.3. *Let* $\mathbf{F} : \mathbf{K} \subseteq \mathbf{R^n} \longrightarrow \mathbf{R^n}$ *be r-times Fréchet-differentiable on a convex set* $\mathbf{K}$ *containing the root* α *of the system of nonlinear equations* (1). *The iterative methods* (8) *and* (10) *have cubic convergence.*

Proof. The iterative scheme defined in (8), is given by

$$\lambda_k = \mathbf{X_k} - (\mathbf{F}'(\mathbf{X_k}))^{-1}\mathbf{F}(\mathbf{X_k}), \tag{40}$$

$$\mathbf{X_{k+1}} = \mathbf{X_k} - \mathbf{6}\left[\mathbf{F}'(\mathbf{X_k}) + \mathbf{4F}'(\frac{\mathbf{X_k} + \lambda_\mathbf{k}}{\mathbf{2}}) + \mathbf{F}'(\lambda_\mathbf{k})\right]^{-1}\mathbf{F}(\mathbf{X_k}). \tag{41}$$

Consider $e_k = \mathbf{X_k} - \alpha$, then equation (41), we obtain

$$e_{k+1} - e_k = -6\left[\mathbf{F}'(\mathbf{X_k}) + \mathbf{4F}'(\frac{\mathbf{X_k} + \lambda_\mathbf{k}}{\mathbf{2}}) + \mathbf{F}'(\lambda_\mathbf{k})\right]^{-1}\mathbf{F}(\mathbf{X_k}),$$

through which, we have

$$\left[\mathbf{F}'(\mathbf{X_k}) + 4\mathbf{F}'(\frac{\mathbf{X_k}+\lambda_\mathbf{k}}{\mathbf{2}}) + \mathbf{F}'(\lambda_\mathbf{k})\right] e_{k+1} = \left[\mathbf{F}'(\mathbf{X_k}) + 4\mathbf{F}'(\frac{\mathbf{X_k}+\lambda_\mathbf{k}}{\mathbf{2}}) + \mathbf{F}'(\lambda_\mathbf{k})\right] e_k - 6\mathbf{F}(\mathbf{X_k}). \quad (42)$$

From equation (3) with $\mathbf{X} = \alpha$, we obtain

$$\mathbf{F}(\alpha) = \mathbf{F}(\mathbf{X_k}) + \mathbf{F}'(\mathbf{X_k})(\alpha - \mathbf{X_k}) + \frac{\mathbf{1}}{\mathbf{2!}}\mathbf{F}''(\mathbf{X_k})(\alpha - \mathbf{X_k})^\mathbf{2} + \frac{1}{3!}\mathbf{F}'''(\mathbf{X_k})(\alpha - \mathbf{X_k})^\mathbf{3} + \mathbf{O}(||(\alpha - \mathbf{X_k})^\mathbf{4}||),$$

since α be the root of the system of nonlinear equations (1), we get

$$\mathbf{F}(\mathbf{X_k}) = \mathbf{F}'(\mathbf{X_k})\mathbf{e_k} - \frac{\mathbf{1}}{\mathbf{2!}}\mathbf{F}''(\mathbf{X_k})\mathbf{e_k^2} + \frac{\mathbf{1}}{\mathbf{3!}}\mathbf{F}'''(\mathbf{X_k})\mathbf{e_k^3} + \mathbf{O}(||\mathbf{e_k^4}||). \quad (43)$$

Pre-multiplying equation (43) with $(\mathbf{F}'(\mathbf{X_k}))^{-\mathbf{1}}$, we obtain

$$(\mathbf{F}'(\mathbf{X_k}))^{-1}\mathbf{F}(\mathbf{X_k}) = \mathbf{e_k} - \frac{\mathbf{1}}{\mathbf{2!}}(\mathbf{F}'(\mathbf{X_k}))^{-\mathbf{1}}\mathbf{F}''(\mathbf{X_k})\mathbf{e_k^2} + \frac{1}{3!}(\mathbf{F}'(\mathbf{X_k}))^{-1}\mathbf{F}'''(\mathbf{X_k})\mathbf{e_k^3} + \mathbf{O}\left(||\mathbf{e_k^4}||\right). \quad (44)$$

Applying Taylor's expansion for $\mathbf{F}'(\frac{\mathbf{X_k}+\lambda_\mathbf{k}}{\mathbf{2}})$ and $\mathbf{F}'(\lambda_\mathbf{k})$ at the point $\mathbf{X_k}$, we have

$$\mathbf{F}'(\frac{\mathbf{X_k}+\lambda_\mathbf{k}}{\mathbf{2}}) = \mathbf{F}'(\mathbf{X_k}) - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{F}''(\mathbf{X_k})\left((\mathbf{F}'(\mathbf{X_k}))^{-\mathbf{1}}\mathbf{F}(\mathbf{X_k})\right) + \frac{1}{8}\mathbf{F}'''(\mathbf{X_k})\left((\mathbf{F}'(\mathbf{X_k}))^{-\mathbf{1}}\mathbf{F}(\mathbf{X_k})\right)^\mathbf{2} + \cdots \quad (45)$$

From equations (44) and (45), we get

$$\mathbf{F}'(\frac{\mathbf{X_k}+\lambda_\mathbf{k}}{\mathbf{2}}) = \mathbf{F}'(\mathbf{X_k}) - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{F}''(\mathbf{X_k})\Big[\mathbf{e_k} - \frac{\mathbf{1}}{\mathbf{2!}}(\mathbf{F}'(\mathbf{X_k}))^{-\mathbf{1}}\mathbf{F}''(\mathbf{X_k})\mathbf{e_k^2} + \frac{1}{3!}(\mathbf{F}'(\mathbf{X_k}))^{-1}\mathbf{F}'''(\mathbf{X_k})\mathbf{e_k^3} + \mathbf{O}\left(||\mathbf{e_k^4}||\right)\Big] + \frac{1}{8}\mathbf{F}'''(\mathbf{X_k})$$
$$\left[e_k - \frac{1}{2!}(\mathbf{F}'(\mathbf{X_k}))^{-1}\mathbf{F}''(\mathbf{X_k})\mathbf{e_k^2} + \frac{\mathbf{1}}{\mathbf{3!}}(\mathbf{F}'(\mathbf{X_k}))^{-\mathbf{1}}\mathbf{F}'''(\mathbf{X_k})\mathbf{e_k^3} + \mathbf{O}\left(||\mathbf{e_k^4}||\right)\right]^2 + \cdots$$

Thus, we have

$$\mathbf{F}'(\frac{\mathbf{X_k}+\lambda_\mathbf{k}}{\mathbf{2}}) = \mathbf{F}'(\mathbf{X_k}) - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{F}''(\mathbf{X_k})\mathbf{e_k} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{F}''(\mathbf{X_k})(\mathbf{F}'(\mathbf{X_k}))^{-\mathbf{1}}\mathbf{F}''(\mathbf{X_k})\mathbf{e_k^2} + \frac{1}{8}\mathbf{F}'''(\mathbf{X_k})\mathbf{e_k^2} + \mathbf{O}\left(||\mathbf{e_k^3}||\right). \quad (46)$$

Similarly, we can calculate

$$\mathbf{F}'(\lambda_\mathbf{k}) = \mathbf{F}'(\mathbf{X_k}) - \mathbf{F}''(\mathbf{X_k})\mathbf{e_k} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{F}''(\mathbf{X_k})(\mathbf{F}'(\mathbf{X_k}))^{-\mathbf{1}}\mathbf{F}''(\mathbf{X_k})\mathbf{e_k^2} + \frac{1}{2}\mathbf{F}'''(\mathbf{X_k})\mathbf{e_k^2} + \mathbf{O}\left(||\mathbf{e_k^3}||\right). \quad (47)$$

From equations (42), (43), (46) and (47), we obtain

$$\left[\mathbf{F}'(\mathbf{X_k}) + 4\mathbf{F}'(\frac{\mathbf{X_k}+\lambda_\mathbf{k}}{\mathbf{2}}) + \mathbf{F}'(\lambda_\mathbf{k})\right] e_{k+1} = \mathbf{F}'(\mathbf{X_k})\mathbf{e_k}$$

$$+ 4\left[\mathbf{F}'(\mathbf{X_k}) - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{F}''(\mathbf{X_k})\mathbf{e_k} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{F}''(\mathbf{X_k})\left(\mathbf{F}'(\mathbf{X_k})\right)^{-1}\mathbf{F}''(\mathbf{X_k})\mathbf{e_k^2} + \frac{\mathbf{1}}{\mathbf{8}}\mathbf{F}'''(\mathbf{X_k})\mathbf{e_k^2}\right.$$

$$\left.+\mathbf{O}\left(||\mathbf{e_k^3}||\right)\right] e_k + \left[\mathbf{F}'(\mathbf{X_k}) - \mathbf{F}''(\mathbf{X_k})\mathbf{e_k} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{F}''(\mathbf{X_k})\left(\mathbf{F}'(\mathbf{X_k})\right)^{-1}\mathbf{F}''(\mathbf{X_k})\mathbf{e_k^2}\right.$$

$$\left.+\frac{1}{2}\mathbf{F}'''(\mathbf{X_k})\mathbf{e_k^2} + \mathbf{O}\left(||\mathbf{e_k^3}||\right)\right] e_k - 6\left[\mathbf{F}'(\mathbf{X_k})\mathbf{e_k} - \frac{\mathbf{1}}{\mathbf{2!}}\mathbf{F}''(\mathbf{X_k})\mathbf{e_k^2}\right.$$

$$\left.+\frac{1}{3!}\mathbf{F}'''(\mathbf{X_k})\mathbf{e_k^3} + \mathbf{O}\left(||\mathbf{e_k^4}||\right)\right].$$

Thus, from the above equation, we have

$$\left[\mathbf{F}'(\mathbf{X_k}) + 4\mathbf{F}'(\frac{\mathbf{X_k}+\lambda_\mathbf{k}}{\mathbf{2}}) + \mathbf{F}'(\lambda_\mathbf{k})\right] e_{k+1}$$

$$= \left[\frac{3}{2}\mathbf{F}''(\mathbf{X_k})\left(\mathbf{F}'(\mathbf{X_k})\right)^{-1}\mathbf{F}''(\mathbf{X_k})\right] e_k^3 + \mathbf{O}\left(||\mathbf{e_k^4}||\right). \quad (48)$$

Error equation (48) shows that the iterative method (8) has cubic convergence. Similarly, one can show that the iterative scheme (10) is also cubically convergent. □

4. Numerical examples

We present some numerical examples to verify the theoretical results of the methods defined in (8) (CT1) and (10) (CT2), see Table 4.1. We compare the Newton's method (NM) with these methods. All computations were done on MAPLE. We consider 30 digits floating point arithmetic (Digits := 30) and $\varepsilon = 10^{-14}$. The following stopping criteria is used for computer programs.

$$(i).\ ||\mathbf{X_{k+1}} - \mathbf{X_k}||_\infty < \varepsilon, \qquad (\mathbf{ii}).\ ||\mathbf{F}(\mathbf{X_k})||_\infty < \varepsilon.$$

The computational order of convergence p approximated (see[22]) by means of

$$p \approx \frac{\ln\left(||\mathbf{X_{k+1}} - \mathbf{X_k}||_\infty / ||\mathbf{X_k} - \mathbf{X_{k-1}}||_\infty\right)}{\ln\left(||\mathbf{X_k} - \mathbf{X_{k-1}}||_\infty / ||\mathbf{X_{k-1}} - \mathbf{X_{k-2}}||_\infty\right)}.$$

Example 4.1[7]. Consider the following system of nonlinear equations

$$x_2x_3 + x_4(x_2 + x_3) = 0,$$
$$x_1x_3 + x_4(x_1 + x_3) = 0,$$
$$x_1x_2 + x_4(x_1 + x_2) = 0,$$
$$x_1x_2 + x_1x_3 + x_2x_3 = 1.$$

Example 4.2[7]. Consider the following system of nonlinear equations

$$e^{x_1^2} + 8x_1\sin(x_2) = 0,$$
$$x_1 + x_2 = 1.$$

Example 4.3[8]. Consider the following system of nonlinear equations

$$x_1^2 + x_2^2 + x_3^2 = 1,$$
$$2x_1^2 + x_2^2 - 4x_3 = 0,$$
$$3x_1^2 - 4x_2^2 + x_3^2 = 0.$$

Example 4.4[7]. Consider the following system of nonlinear equations

$$x_1^2 - 2x_1 - x_2 + 0.5 = 0, \quad x_1^2 + 4x_2^2 - 4 = 0.$$

Example 4.5 [7]. Consider the following system of nonlinear equations

$$e^{x_1^2} - e^{\sqrt{2}x_1} = 0, \quad x_1 - x_2 = 0.$$

Table 4.1. (Numerical Examples and Comparison)

Exp.	Initial Value	Method	IT	Approximate solution	p
4.1	$(0.6, 0.6, 0.6, -0.2)^t$	NM	5	$(0.57735, 0.57735, 0.57735, -0.28868)^t$	2.0
		CT1	3	$(0.57735, 0.57735, 0.57735, -0.28868)^t$	3.3
		CT2	3	$(0.57735, 0.57735, 0.57735, -0.28868)^t$	3.3
4.2	$(0.2, 0.8)^t$	NM	5	$(-0.14028501081, 0.1402850108)^t$	2.0
		CT1	4	$(-0.14028501081, 0.1402850108)^t$	3.0
		CT2	4	$(-0.14028501081, 0.1402850108)^t$	3.0
4.3	$(0.5, 0.5, 0.5)^t$	NM	6	$(0.69828861, 0.62852430, 0.34256419)^t$	2.0
		CT1	4	$(0.69828861, 0.62852430, 0.34256419)^t$	3.0
		CT2	4	$(0.69828861, 0.62852430, 0.34256419)^t$	3.0
4.4	$(0.5, 0.5)^t$	NM	7	$(-0.2222145551, 0.9938084186)^t$	2.0
		CT1	5	$(-0.2222145551, 0.9938084186)^t$	3.0
		CT2	5	$(-0.2222145551, 0.9938084186)^t$	3.0
	(-0.3, 1)t	NM	5	$(-0.2222145551, 0.9938084186)^t$	2.0
		CT1	3	$(-0.2222145551, 0.9938084186)^t$	3.0
		CT2	3	$(-0.2222145551, 0.9938084186)^t$	3.0
4.5	$(-0.5, 0.5)^t$	NM	5	$(0, 0)^t$	3.0
		CT1	3	$(0, 0)^t$	5.1
		CT2	3	$(0, 0)^t$	5.1

From the Table 4.1, we see that the computational order of convergence p of the methods defined in (8) and (10) is at least 3.

Remark 4.1. We would like to emphasize that if the condition (2) holds, then the Newton's Method has cubic convergence and the methods defined in (8) and (10) have fifth order convergence. This fact is illustrated by example 4.5 (see Table 4.1).

5. Conclusion.

In this paper, we have shown that the methods proposed by Cordero and Torregrosa [7] for solving the system of nonlinear equations (1) are cubically convergent without any condition. WE also rectified the fundamental errors in their convergence results.

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