J. Appl. Math. & Informatics Vol. 27(2009), No. 5 - 6, pp. 1119 - 1132
Website: http://www.kcam.biz

EXISTENCE OF BOUNDARY BLOW-UP SOLUTIONS FOR A CLASS OF QUASILINEAR ELLIPTIC SYSTEMS

MINGZHU WU AND ZUODONG YANG*

ABSTRACT. In this paper, we consider the quasilinear elliptic system

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) = u(a_1 u^{m_1} + b_1(x)u^m + \delta_1 v^n),$$
$$\mathrm{div}(|\nabla v|^{q-2}\nabla v) = v(a_2 v^{r_1} + b_2(x)v^r + \delta_2 u^s), \text{ in } \Omega$$

where $m > m_1 > p-2, r > r_1 > q-2, p, q \geq 2$, and $\Omega \subset \mathbf{R}^\mathbf{N}$ is a smooth bounded domain. By constructing certain super and subsolutions, we show the existence of positive blow-up solutions and give a global estimate.

AMS Mathematics Subject Classification : 35J25, 35J60.
Key words and phrases : Boundary blow-up, positive solution, quasilinear elliptic system, existence.

1. Introduction

In this paper, we are concerned with a system of quasilinear elliptic equations:

$$\begin{cases} \mathrm{div}(|\nabla u|^{p-2}\nabla u) = u(a_1 u^{m_1} + b_1(x)u^m + \delta_1 v^n), & x \in \Omega \\ \mathrm{div}(|\nabla v|^{q-2}\nabla v) = v(a_2 v^{r_1} + b_2(x)v^r + \delta_2 u^s) & x \in \Omega \\ u = v = +\infty, & x \in \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^N$ is a smooth bounded domain, and constants $a_i \geq 0 (i = 1, 2)$. Functions $b_1, b_2 \in C^\eta(\Omega)$ is positive weight function, which is singular on $\partial\Omega$ and $0 < \eta < 1$. The boundary condition is interpreted as $u(x), v(x) \to +\infty$ as $d(x) = \mathrm{dist}(x, \partial\Omega) \to 0^+$. Assume that $m > m_1 > p-2, r > r_1 > q-2, p, q \geq 2, n > 0, s > 0$.

By positive boundary blow-up solutions (u, v) of (1.1) we mean that $(u, v) \in W^{1,p}_{loc}(\Omega) \cap C^1_{loc}(\Omega)$ and (u, v) satisfies

$$-\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\psi dx = \int_\Omega u(a_1 u^{m_1} + b_1(x)u^m + \delta_1 v^n)\psi dx \quad \forall \psi \in C_0^\infty(\Omega)$$

Received August 20, 2008. Revised September 20, 2008. Accepted October 14, 2008.
*Corresponding author. Project Supported by the National Natural Science Foundation of China(Grant No.10571022). Project Supported by the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (Grant No.08KJB).

$$-\int_\Omega |\nabla v|^{q-2}\nabla v\cdot\nabla\psi dx=\int_\Omega v(a_2v^{r_1}+b_2(x)v^r+\delta_2u^s)\psi dx \quad \forall\psi\in C_0^\infty(\Omega)$$

and $u,v>0$ in Ω, $u(x)\to\infty, v(x)\to\infty$ as $d(x)\to 0$, where $d(x)=\text{dist}(x,\partial\Omega)$.

Recently, existence and non-existence of solutions of the elliptic system

$$\begin{cases}\Delta u+f(u,v)=0, & x\in\Omega\\ \Delta v+g(u,v)=0, & x\in\Omega\end{cases} \tag{1.2}$$

has received much attention for $\Omega\subset\mathbf{R^N}$ or $\Omega=\mathbf{R^N}$. We list here, for example, [1,3,4,8,11-13,15].

When $f=-a(|x|)v^\alpha, g=-b(|x|)u^\beta$, system (1.2) becomes

$$\begin{cases}\Delta u=a(|x|)v^\alpha, & x\in\mathbf{R^N}\\ \Delta v=b(|x|)u^\beta, & x\in\mathbf{R^N}\end{cases} \tag{1.3}$$

for which existence results for positive boundary blow-up solutions can be found in a recent paper by Lair and Wood [8]. Lair and Wood established that all positive entire radial solutions of (1.3) are boundary blow-up provided that

$$\int_0^\infty ta(t)dt=\infty, \quad \int_0^\infty tb(t)dt=\infty.$$

If, on the other hand

$$\int_0^\infty ta(t)dt<\infty, \quad \int_0^\infty tb(t)<\infty$$

then all positive entire radial solutions of (1.3) are bounded.

F. Cirstea and V.D. Radulescu [1], extended the above results to a larger class of systems

$$\begin{cases}\Delta u=a(|x|)g(v), & x\in\mathbf{R^N}\\ \Delta v=b(|x|)f(u), & x\in\mathbf{R^N}\end{cases}$$

Z.D. Yang [17], extended the above results to a class of systems

$$\begin{cases}\text{div}(|\nabla u|^{p-2}\nabla u)=a(|x|)g(v), & x\in\mathbf{R^N}\\ \text{div}(|\nabla v|^{q-2}\nabla v)=b(|x|)f(u), & x\in\mathbf{R^N}\end{cases}$$

Recently, J.Garcia-Melian and J.D. Rossi [20] for which existence and nonexistence results for boundary blow-up solutions can be obtained to the elliptic system

$$\begin{cases}\Delta u=u^{m_1}v^{n_1}, & x\in\Omega\subset\mathbf{R^N}\\ \Delta v=u^{m_2}v^{n_2}, & x\in\Omega\subset\mathbf{R^N}\end{cases}$$

Z.D.Yang and M.Z.Wu [27] extended the above results to the following quasilinear elliptic system:

$$\begin{cases}\text{div}(|\nabla u|^{p-2}\nabla u)=u^{m_1}v^{n_1}, & x\in\Omega\subset\mathbf{R^N}\\ \text{div}(|\nabla v|^{q-2}\nabla v)=u^{m_2}v^{n_2}, & x\in\Omega\subset\mathbf{R^N}\end{cases} \tag{1.4}$$

where $N\geq 3, p>1, q>1, m_1>p-1, n_2>q-1, n_1, m_2>0$ are real numbers and $\Omega\subset\mathbf{R^N}$ is a bounded domain of class $C^{2,\eta}$ for some $\eta, 0<\eta<1$, subject to three different types of Dirichlet boundary conditions: $u=\lambda, v=\mu$ or $u=v=+\infty$ or $u=+\infty, v=\mu$ on $\partial\Omega$, where $\lambda,\mu>0$. Under several hypotheses on the parameters m_1, n_1, m_2, n_2 which is a critical case, they show that the existence of positive solutions.

In the paper [9], the authors studied the following elliptic systems:

$$\begin{cases} -\triangle u = u(a_1 - b_1 u^m - c_1 v^n), & x \in \Omega \\ -\triangle v = v(a_2 - b_2 v^p - c_2 v^q), & x \in \Omega \\ u = v = +\infty, & x \in \partial\Omega, \end{cases}$$

where $a_i \geq 0, b_i, c_i (i = 1, 2)$ are positive constants, and $m, q > 0, n, p \geq 0$.

In the paper [7], the authors considered existence of positive solutions for the following elliptic systems

$$\begin{cases} \triangle u = u(a_1 u^{m_1} + b_1(x) u^m + \delta_1 v^n), & x \in \Omega \\ \triangle v = v(a_2 v^{p_1} + b_2(x) v^p + \delta_2 u^q), & x \in \Omega \\ u = v = +\infty, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where $a_i \geq 0, (i = 1, 2)$, $m > m_1 > 0$ and $p > p_1 > 0$. Assume that $n(2 - \gamma_2) < p\gamma_1$, $q(2 - \gamma_1) < m\gamma_2$, $\delta_1, \delta_2 > 0$, and b_1, b_2 are positive functions satisfying

$$\lim_{x \to \partial\Omega} b_1(x) d(x)^{\gamma_1} = A, \ \lim_{x \to \partial\Omega} b_2(x) d(x)^{\gamma_2} = B$$

where A, B are positive constants. Then for any positive solution (u, v) to the system (1.5), it holds that

$$\lim_{x \to \partial\Omega} d(x)^\alpha u(x) = (\frac{\alpha(\alpha + 1)}{A})^{1/m}, \ \lim_{x \to \partial\Omega} d(x)^\beta v(x) = (\frac{\beta(\beta + 1)}{B})^{1/p}$$

In the paper [35], Z.D.Yang and C.Liu consider the boundary blow-up quasilinear elliptic problems,

$$\mathrm{div}(|\nabla u|^{m-2} \nabla u) \pm \lambda |\nabla u|^{q(m-1)} = k(x) g(u)$$

in a C^2 bounded domain with boundary condition $u|_{\partial\Omega} = +\infty$, where $m > 1$, $q \in [0, m/(m-1)]$ and $\lambda \geq 0$. Under suitable growth assumptions on $k(x)$ near the boundary and on g both at zero and at infinity, they show the existence of at least one solution in $C^1(\Omega)$. The proof is based on the method of explosive sub-supersolutions, which permits positive weights $k(x)$ which are unbounded and/or oscillatory near the boundary.

In the paper [36], by a perturbation method and by constructing comparison functions, the authors show the exact asymptotic behavior of solutions near the boundary of the quasilinear elliptic problem

$$\begin{cases} \mathrm{div}(|\nabla u|^{m-2} \nabla u) \pm |\nabla u|^{q(m-1)} = b(x) e^{u(x)}, & x \in \Omega \\ u = +\infty, & x \in \partial\Omega \end{cases}$$

where Ω is a C^2 bounded domain with a smooth boundary in $R^N (N \geq 2)$, $m > 1$, $q \geq 0$, b is nonnegative and nontrivial in Ω, which may vanish on the boundary.

Motivated by the results of the above cited papers, we further study the existence of positive solutions for (1.1), the results of the semilinear equations systems are extended to the quasilinear ones. We can find the related results for $p = 2$ in [7]. Using an argument inspired by J.Garcia-Melian & J.D. Rossi

[20], Lei Wei & Mingxin Wang [7] and Z.D. Yang [17,18,19,27] and other authors [2,5,6,10,14,16,26,29-31], we obtain the following main results which complement corresponding results in [17-19, 27], and extended to results in [7,20].

Before stating our results, we firstly give two hypotheses:

$$D_2 d(x)^{-\gamma_1} \le b_1(x) \le D_1 d(x)^{-\gamma_1}, \ \ x \in \Omega \tag{1.5}$$

$$K_2 d(x)^{-\gamma_2} \le b_2(x) \le K_1 d(x)^{-\gamma_2}, \ \ x \in \Omega \tag{1.6}$$

where $D_i, K_i > 0, 0 < \gamma_1 < \min\{p, \frac{pm}{p-2}\}, 0 < \gamma_2 < \min\{q, \frac{qr}{q-2}\}$

By a modification of the method given in [7,17-20,27], we obtain the following results.

Theorem 1. *Assume that* $\frac{(p-\gamma_1)s}{qr+(2-q)\gamma_2} < \frac{m+2-p}{r+2-q} < \frac{pm+(2-p)\gamma_1}{(q-\gamma_2)n}$, *and (1.5)-(1.6) are satisfied. If* $\delta_1, \delta_2 > 0$ *are sufficient small, then elliptic system (1.1) has at least a positive solution. If* $\delta_1, \delta_2 < 0$ *and* $m > n, r > s$, *then elliptic system (1.1) has at least a positive solution.*

Theorem 2. *Assume that* $\frac{(p-\gamma_1)s}{qr+(2-q)\gamma_2} < \frac{m+2-p}{r+2-q} < \frac{pm+(2-p)\gamma_1}{(q-\gamma_2)n}$, *and (1.5)-(1.6) are satisfied. If* $(\delta_1, \delta_2) \to (0^+, 0^+)$, *then there exist a subsequence* $\{(u_k, v_k)\}$ *of solution sequence corresponding to (1.1) such that* $(u_k, v_k) \to (U_{m,\gamma_1}, U_{r,\gamma_2})$ *in* $C^1(\Omega)$, *where* U_{m,γ_1} *and* U_{r,γ_2} *are the unique positive solutions of the problems*

$$\begin{cases} div(|\nabla u|^{p-2}\nabla u) = a_1 u^{m_1+1} + b_1(x) u^{m+1}, \ \ x \in \Omega \\ u = +\infty, \ \ x \in \partial\Omega \end{cases} \tag{1.7}$$

and

$$\begin{cases} div(|\nabla v|^{q-2}\nabla v) = a_2 v^{r_1+1} + b_2(x) v^{r+1}, \ \ x \in \Omega \\ v = +\infty, \ \ x \in \partial\Omega \end{cases} \tag{1.8}$$

respectively.

Theorem 3. *Assume that* $s(p-\gamma_1) \le (m+2-p)\gamma_2, n(q-\gamma_2) \le (r+2-q)\gamma_1, \delta_1, \delta_2 > 0$, *and (1.5)-(1.6) are satisfied. If* (u, v) *is a positive solution of elliptic system (1.1), then there exist* $C_1, C_2 > 0$ *such that*

$$C_2 d(x)^{-\alpha} \le u(x) \le C_1 d(x)^{-\alpha}, \ \ x \in \Omega$$

$$C_2 d(x)^{-\beta} \le v(x) \le C_1 d(x)^{-\beta}, \ \ x \in \Omega$$

where $\alpha = \frac{p-\gamma_1}{m+2-p}, \beta = \frac{q-\gamma_2}{r+2-q}$.

2. Preliminaries

In this section we collect some results concerning the solutions to the problem

$$\begin{cases} \mathrm{div}(|\nabla u|^{p-2}\nabla u) = d(x)^{-\gamma} u^{m+1}, \ \ x \in \Omega \\ u = +\infty, \ \ x \in \partial\Omega, \end{cases} \tag{2.1}$$

that will be used in the next sections. Here $d(x)$ stands for the distance of a point $x \in \Omega$ to the boundary $\partial\Omega$. Since Ω is C^2, it is well known(cf.Lemma 14.16 in [21]) that $d(x)$ is C^2 in a neighourhood of $\partial\Omega$. Redefining $d(x)$ outside this neighourhood if necessary, we can always assume that $d(x) \in C^2(\overline{\Omega})$.

From reference [28], we give the following lemma

Lemma 1. (Weak Comparison Principle) *Let Ω be a bounded domain in $\mathbf{R^N}(N \geq 2)$ with smooth boundary $\partial\Omega$ and $\theta : (0,\infty) \to (0,\infty)$ is continuous and nondecreasing. Let $u_1, u_2 \in W^{1,p}_{loc}(\Omega) \cap C(\Omega)$ satisfy*

$$\int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \nabla\psi dx + \int_\Omega \theta(u_1)\psi dx \leq \int_\Omega |\nabla u_2|^{p-2}\nabla u_2 \nabla\psi dx + \int_\Omega \theta(u_2)\psi dx$$

for all non-negative $\psi \in W^{1,p}_{loc}(\Omega)$. Then the inequality

$$\lim_{x\to\partial\Omega} \sup(u_1(x) - u_2(x)) \leq 0$$

implies that $u_1 \leq u_2$ in Ω.

The following lemma, part of Lemma 3 in [17]

Lemma 2. *Assume $m > p-2$ and $\gamma < p$. Then problem (2.1) has a positive solution, which will be denoted by $U_{m,\gamma}$ and*

$$C_1 d(x)^{-\rho_1} \leq U_{m,\gamma}(x) \leq C_1 d(x)^{-\rho_1},$$

where $\rho_1 = (p-\gamma)/(m+2-p)$. This solution is obtained as the limit as $n \to +\infty$ of the solutions U_n to the problem

$$\begin{cases} div(|\nabla u|^{p-2}\nabla u) = d(x)^{-\gamma}u^{m+1}, & x \in \Omega \\ u = n, & x \in \partial\Omega. \end{cases}$$

Lemma 3. *Let u satisfies $div(|\nabla u|^{p-2}\nabla u) \leq Cd(x)^{-\gamma}u^{m+1}$ in Ω for some positive constant C, and $u = +\infty$ on $\partial\Omega$. Then $u(x) \geq C^{-\frac{1}{m+2-p}}U_{m,\gamma}(x)$. Similarly, if $div(|\nabla u|^{p-2}\nabla u) \geq Cd(x)^{-\gamma}u^{m+1}$ in Ω, then $u(x) \leq C^{-\frac{1}{m+2-p}}U_{m,\gamma}(x)$, regardless of the value of u on the boundary.*

Proof. Let $w = C^{\frac{1}{m-p+2}}u$. We have

$$\begin{aligned} \operatorname{div}(|\nabla w|^{p-2}\nabla w) &= C^{\frac{p-1}{m+2-p}}\operatorname{div}(|\nabla u|^{p-2}\nabla u) \\ &\leq C^{\frac{p-1}{m+2-p}}Cd(x)^{-\gamma}u^{m+1} = d(x)^{-\gamma}w^{m+1}, \end{aligned}$$

that is,

$$\operatorname{div}(|\nabla w|^{p-2}\nabla w) - d(x)^{-\gamma}w^{m+1} \leq \operatorname{div}(|\nabla U_{m,\gamma}|^{p-2}\nabla U_{m,\gamma}) - d(x)^{-\gamma}U_{m,\gamma}^{m+1} \quad \text{in } \Omega$$

Because $\lim_{x\to\partial\Omega} w = \lim_{x\to\partial\Omega} U_{m,\gamma} = +\infty$, so we have $\lim_{x\to\partial\Omega}\sup(w - U_{m,\gamma}) = 0$. From Lemma 1, it follows that

$$C^{\frac{1}{m+2-p}}u \geq U_{m,\gamma} \quad \Rightarrow \quad u \geq C^{-\frac{1}{m+2-p}}U_{m,\gamma}(x) \quad \text{in } \Omega.$$

□

Lemma 4. *Assume $b \in C^\eta(\Omega)$ and there exist $C'', C' > 0, \gamma < 2$ such that $C''d(x)^{-\gamma} \leq b(x) \leq C'd(x)^{-\gamma}$ in Ω. Then there exist positive constants M_1 and M_2, such that $M_2 d(x)^{-\frac{p-\gamma}{(1+l)-(p-1)}} \leq u(x) \leq M_1 d(x)^{-\frac{p-\gamma}{(1+l)-(p-1)}}$ and $u(x)$ is solution of the following equation*

$$\begin{cases} div(|\nabla u|^{p-2}\nabla u) = b(x)u^{l+1}, & x \in \Omega \\ u = +\infty, & x \in \partial\Omega, \end{cases} \tag{2.2}$$

It can be proofed easily by use the above lemma. So we skip the proof of it.

We give in the following some results about method of sub- and supersolutions for the system (1.1). Assume $\delta_1, \delta_2 > 0$. By a positive sub-solutions (u, v) of (1.1) we mean that super solution $(\overline{u}, \overline{v}) \in W^{1,p}(K) \cap C^1(K)$ for any compact set $K \subset\subset \Omega$ and sub solution $(\underline{u}, \underline{v})$ satisfies

$$\begin{cases} \int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u} \cdot \nabla\psi dx \leq -\int_\Omega \underline{u}(a_1\underline{u}^{m_1} + b_1(x)\underline{u}^m + \delta_1\overline{v}^n)\psi dx \quad \forall\psi \in C_0^\infty(\Omega) \\ \int_\Omega |\nabla \underline{v}|^{q-2}\nabla \underline{v} \cdot \nabla\psi dx \leq -\int_\Omega \underline{v}(a_2\underline{v}^{r_1} + b_2(x)\underline{v}^r + \delta_2\overline{u}^s)\psi dx \quad \forall\psi \in C_0^\infty(\Omega) \\ \int_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u} \cdot \nabla\psi dx \geq -\int_\Omega \overline{u}(a_1\overline{u}^{m_1} + b_1(x)\overline{u}^m + \delta_1\underline{v}^n)\psi dx \quad \forall\psi \in C_0^\infty(\Omega) \\ \int_\Omega |\nabla \overline{v}|^{q-2}\nabla \overline{v} \cdot \nabla\psi dx \geq -\int_\Omega \overline{v}(a_2\overline{v}^{r_1} + b_2(x)\overline{v}^r + \delta_2\underline{u}^s)\psi dx \quad \forall\psi \in C_0^\infty(\Omega) \end{cases}$$

Similarly, assume $\delta_1, \delta_2 < 0$, then provided that

$$\begin{cases} \int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u} \cdot \nabla\psi dx \leq -\int_\Omega \underline{u}(a_1\underline{u}^{m_1} + b_1(x)\underline{u}^m + \delta_1\underline{v}^n)\psi dx \quad \forall\psi \in C_0^\infty(\Omega) \\ \int_\Omega |\nabla \underline{v}|^{q-2}\nabla \underline{v} \cdot \nabla\psi dx \leq -\int_\Omega \underline{v}(a_2\underline{v}^{r_1} + b_2(x)\underline{v}^r + \delta_2\underline{u}^s)\psi dx \quad \forall\psi \in C_0^\infty(\Omega) \\ \int_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u} \cdot \nabla\psi dx \geq -\int_\Omega \overline{u}(a_1\overline{u}^{m_1} + b_1(x)\overline{u}^m + \delta_1\overline{v}^n)\psi dx \quad \forall\psi \in C_0^\infty(\Omega) \\ \int_\Omega |\nabla \overline{v}|^{q-2}\nabla \overline{v} \cdot \nabla\psi dx \geq -\int_\Omega \overline{v}(a_2\overline{v}^{r_1} + b_2(x)\overline{v}^r + \delta_2\overline{u}^s)\psi dx \quad \forall\psi \in C_0^\infty(\Omega) \end{cases}$$

Lemma 5. *Assume $(\underline{u}, \underline{v})$ is a subsolution and $(\overline{u}, \overline{v})$ is a supersolution to (1.1) with $\underline{u} = \overline{u} = \underline{v} = \overline{v} = +\infty$ on $\partial\Omega$ and $\underline{u} \leq \overline{u}, \underline{v} \leq \overline{v}$ in Ω. Then problem (1.1) has at least a solution (u, v) with $\underline{u} \leq u \leq \overline{u}, \underline{v} \leq v \leq \overline{v}$ in Ω and $u = v = +\infty$ on $\partial\Omega$.*

Proof. Since Ω is a C^2 bounded domain, from [32,Theorem 4.2] we know that there exist a series of C^∞ domains $\{\Omega_k\}_1^\infty$, such that $\overline{\Omega_k} \subset \Omega_{k+1} \subset \Omega$, $\cup_{k=1}^\infty \Omega_k = \Omega$. Assume $\delta_1, \delta_2 > 0$. Since $\underline{v}$ and $b_1(x)$ is a bounded and positive function in $\overline{\Omega_k}$, the problem

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = u(a_1u^{m_1} + b_1(x)u^m + \delta_1\underline{v}^n), & x \in \Omega_k \\ u = \underline{u}, \quad x \in \partial\Omega_k, \end{cases} \tag{2.3}$$

has a positive solution, which we denote by u_1. Moreover,

$$\operatorname{div}(|\nabla \underline{u}|^{p-2}\nabla \underline{u}) \geq \underline{u}(a_1\underline{u}^{m_1} + b_1(x)\underline{u}^m + \delta_1\overline{v}^n) \geq \underline{u}(a_1\underline{u}^{m_1} + b_1(x)\underline{u}^m + \delta_1\underline{v}^n) \text{ in } \Omega_k.$$

From Lemma 1, we have $\underline{u} \leq u_1$. Likewise, $\operatorname{div}(|\nabla \overline{u}|^{p-2}\nabla \overline{u}) \leq \overline{u}(a_1\overline{u}^{m_1} + b_1(x)\overline{u}^m + \delta_1\underline{v}^n)$ in Ω_k, and so $\overline{u} \geq u_1$. We now define v_1 as the solution to

$$\begin{cases} \operatorname{div}(|\nabla v|^{q-2}\nabla v) = v(a_2v^{r_1} + b_2(x)v^r + \delta_2u_1^s), & x \in \Omega_k \\ v = \overline{v}, \quad x \in \partial\Omega_k. \end{cases} \tag{2.4}$$

It is not hard to see that $\underline{v} \leq v_1 \leq \overline{v}$ in Ω_k. We continue this procedure and define u_2 as the solution to

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = u(a_1u^{m_1} + b_1(x)u^m + \delta_1{v_1}^n), & x \in \Omega_k \\ u = \underline{u}, \quad x \in \partial\Omega_k. \end{cases} \tag{2.5}$$

Then it follows as before that $\underline{u} \leq u_2 \leq \overline{u}$ in Ω. In addition,

$$\operatorname{div}(|\nabla u_1|^{p-2}\nabla u_1) \leq u_1(a_1{u_1}^{m_1} + b_1(x){u_1}^m + \delta_1{v_1}^n),$$

and hence $u_1 \geq u_2$.

We can recursively define v_n as the unique solution to (2.4) replacing u_1 by u_n, and u_n as the unique solution to (2.5) replacing v_1 by v_{n-1}. In this way, we obtain two sequences $\{u_n\}$ and $\{v_n\}$, such that u_n is decreasing, v_n is increasing, $\underline{u} \le u_n \le \overline{u}$ and $\underline{v} \le v_n \le \overline{v}$ in Ω_k. By $C^{1,\alpha}(\overline{\Omega})$ estimates in [22] and monotonic iterations , one concludes that there exists a $(u_k, v_k) \in \overline{\Omega_k}$, which is a weak solution of (1.1) with Ω_k and $k \ge 1$.

Now, we want to apply elliptic interior estimates together with a diagonal process to conclude:$\{(u_k, v_k) : k \ge 1\}$ has a subsequence $\{(u_{k_i}, v_{k_i}) : k_i \uparrow \infty\}$ such that $\{(u_{k_i}, v_{k_i})\}$ converges to a function (u, v) in Ω (pointwise), and this convergence is in $C^1 \times C^1$ on every compact set in Ω.

Step 1. On Ω_2, $\{(u_k, v_k) : k \ge 2\}$ is uniformly bounded by $(\underline{u}(x), \underline{v}(x))$ and $(\overline{u}(x), \overline{v}(x))$. Since both $\underline{u}, \underline{v}$ and $\overline{u}(x), \overline{v}(x)$ are bounded functions on Ω_2, there exists an $M > 0$ such that

$$\|u_k(x)\|_{L^\infty(\Omega_2)} \le M \quad \text{and} \quad \|v_k(x)\|_{L^\infty(\Omega_2)} \le M \quad \text{for all } \ k \ge 2.$$

Using Theorem 1.1 in [33, pages 107], we see that u_k and v_k belongs to $C^\alpha(\overline{\Omega}_2)$ for some $0 < \alpha < 1$, and

$$\|u_k(x)\|_{C^\alpha} \le C_1 \quad \text{and} \quad \|v_k(x)\|_{C^\alpha} \le C_1$$

Here, C_1 is determined by M. By Proposition 3.7 in [34] we also know that u_k and v_k belong to $C^{1,\alpha}(\overline{\Omega_2})$ and

$$\|u_k(x)\|_{C^{1,\alpha}} \le C_2 \quad \text{and} \quad \|v_k(x)\|_{C^{1,\alpha}} \le C_2$$

Here, C_2 is determined by C_1. Form the arguments above, we see that there exists a $C > 0$ such that

$$\|u_k(x)\|_{C^{1+\alpha}} \le C \quad \text{and} \quad \|v_k(x)\|_{C^{1+\alpha}} \le C, \quad \text{for all } \ k \ge 2.$$

Since the embedding $C^{1+\alpha}(\Omega_1) \to C^1(\Omega_1)$ is compact, there exists a sequence denoted by $\{(u_{k_{1j}}, v_{k_{1j}})\}_{j=1,2\ldots}$ (where $k_{1j} \uparrow \infty$) that converges in $C^1 \times C^1(\Omega_1)$. Let $u_1(x) = \lim_{j\to\infty} u_{k_{1j}}(x)$, $v_1(x) = \lim_{j\to\infty} v_{k_{1j}}(x)$, for $x \in \Omega_1$; then (u_1, v_1) is a solution of (1.1) with $\underline{u} \le u_1 \le \overline{u}$ and $\underline{v} \le v_1 \le \overline{v}$.

Step 2. Repeat Step 1 up to the existence of the sequence $\{(u_{k_{1j}}, v_{k_{1j}})\}_{j=1,2\ldots}$ to get a subsequence $\{(u_{k_{2i}}, v_{k_{2i}})\}_{i=1,2\ldots}$ converging in $C^1 \times C^1(\Omega_2)$ to a limit (u_2, v_2). Then likewise, (u_2, v_2) is a solution on Ω_2 and $(u_2, v_2)|\Omega_1 = (u_1, v_1)$. Repeat Step 1 again on Ω_3,..., etc.

In this way, we obtain a sequence $\{(u_{k_{nj}}, v_{k_{nj}})\}_{j=1,2\ldots}$ which converges in $C^1 \times C^1(\Omega_k)$ and is a subsequence of $\{(u_{k_{(n-1)j}}, v_{k_{(n-1)j}})\}_{j=1,2\ldots}$. Let $u_k(x) = \lim_{j\to\infty} u_{k_{nj}}(x)$, $v_k(x) = \lim_{j\to\infty} v_{k_{nj}}(x)$; then, (u_k, v_k) is a solution in Ω_k and $(u_k, v_k)|\Omega_{k-1} = (u_{k-1}, v_{k-1})$.

Step 3. By a diagonal process, $\{(u_{k_{nn}}, v_{k_{nn}})\}_{n=1,2\ldots}$ is a subsequence of $\{(u_{k_{nj}}, v_{k_{nj}})\}_{j=1,2\ldots}$ for every n. Thus, on Ω_k for each k, we have

$$\lim_{n\to\infty} u_{k_{nn}} = u_k \quad \text{and} \quad \lim_{n\to\infty} v_{k_{nn}} = v_k.$$

So, if we define $u(x) = \lim_{n\to\infty} u_{k_{nn}}(x)$, $v(x) = \lim_{n\to\infty} v_{k_{nn}}(x)$, then $(u(x), v(x))$ satisfies (1.1) and $\underline{u} \le u \le \overline{u}$, $\underline{v} \le v \le \overline{v}$ (since $\underline{u} \le u_k \le \overline{u}$ and $\underline{v} \le v_k \le \overline{v}$ for every k). When $\delta_1, \delta_2 < 0$, the lemma can be proved similarly. This complete the proof of it. □

Lemma 6. *Assume $a_1 \ge 0$, $m > m_1 > 0$, and b_1 satisfies (1.5). Then (1.7) has at least a positive solution u, and there exist $C_1, C_2 > 0$ such that*

$$C_2 d(x)^{-\frac{p-\gamma_1}{m+2-p}} \le u(x) \le C_1 d(x)^{-\frac{p-\gamma_1}{m+2-p}}, \quad x \in \Omega.$$

Proof. Let U denote a positive solution to the equation

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = b_1(x)u^{m+1}, & x \in \Omega \\ u = +\infty, \quad x \in \partial\Omega. \end{cases}$$

Set $\overline{u} = U, \underline{u} = \varepsilon U$, where $\varepsilon > 0$. Now we consider the following equation:

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = a_1 u^{m_1+1} + b_1(x)u^{m+1}, & x \in \Omega_k \\ u = \underline{u}, \quad x \in \partial\Omega_k, \end{cases} \tag{2.6}$$

where the definition of Ω_k is as same as the lemma 5. Clearly, we have

$$\operatorname{div}(|\nabla \overline{u}|^{p-2}\nabla \overline{u}) = b_1(x)\overline{u}^{m+1} \le a_1\overline{u}^{m_1+1} + b_1(x)\overline{u}^{m+1}, \quad x \in \Omega_k.$$

By $0 < \gamma_1 < p$ and condition (1.5), we have $b_0 = \inf_\Omega b_1(x) > 0$. U is a positive function with boundary blow up, thus $m_0 = \inf_\Omega U(x) > 0$. If ε is sufficient small, it follows from $m > m_1 > p - 2$ that

$$\frac{U^{m+1}}{2} \ge \frac{a_1\varepsilon^{m_1+2-p}U^{m_1+1}}{b_0}, \quad x \in \Omega.$$

If $\varepsilon^{m+2-p} < \frac{1}{2}$, we have

$$U^{m+1} \ge \frac{a_1\varepsilon^{m_1+2-p}U^{m_1+1}}{b_1} + \varepsilon^{m+2-p}U^{m+1}, \quad x \in \Omega_k.$$

It is easy to see that

$$b_1(x)U^{m+1} \ge a_1\varepsilon^{m_1+2-p}U^{m_1+1} + b_1(x)\varepsilon^{m+2-p}U^{m+1}, \quad x \in \Omega_k.$$

Furthermore, we have

$$\operatorname{div}(|\nabla \underline{u}|^{p-2}\nabla \underline{u}) = \varepsilon^{p-1}b_1(x)U^{m+1} \ge a_1(\varepsilon U)^{m_1+1} + b_1(x)(\varepsilon U)^{m+1}, \quad x \in \Omega_k.$$

By the standard super and subsolutions argument, there exists a solution u_k such that $\underline{u} \le u_k \le \overline{u}$. We concludes that there exists a $u_k \in \Omega_k$, which is a weak solution of (2.6) with Ω_k and $k \ge 1$.

Now, we want to apply elliptic interior estimates together with a diagonal process to conclude:$\{u_k : k \ge 1\}$ has a subsequence $\{u_{k_i} : k_i \uparrow \infty\}$ such that $\{u_{k_i}\}$ converges to a function u in Ω (pointwise), and this convergence is in C^1 on every compact set in Ω.

Step 1. On Ω_2, $\{u_k : k \geq 2\}$ is uniformly bounded by $\underline{u}(x)$ and $\overline{u}(x)$. Since both $\underline{u}$ and $\overline{u}(x)$ are bounded functions on Ω_2, there exists an $M > 0$ such that

$$\|u_k(x)\|_{L^\infty(\Omega_2)} \leq M \quad \text{for all} \quad k \geq 2.$$

Using Theorem 1.1 in [33, pages 107], we see that u_k belongs to $C^\alpha(\overline{\Omega_2})$ for some $0 < \alpha < 1$, and $\|u_k(x)\|_{C^\alpha} \leq C_1$

Here, C_1 is determined by M. By Proposition 3.7 in [34] we also know that u_k belong to $C^{1,\alpha}(\overline{\Omega_2})$ and $\|u_k(x)\|_{C^{1,\alpha}} \leq C_2$

Here, C_2 is determined by C_1. Form the arguments above, we see that there exists a $C > 0$ such that

$$\|u_k(x)\|_{C^{1+\alpha}} \leq C, \quad \text{for all} \quad k \geq 2.$$

Since the embedding $C^{1+\alpha}(\Omega_1) \to C^1(\Omega_1)$ is compact, there exists a sequence denoted by $\{u_{k_{1j}}\}_{j=1,2...}$ (where $k_{1j} \uparrow \infty$) that converges in $C^1(\Omega_1)$. Let $u_1(x) = \lim_{j\to\infty} u_{k_{1j}}(x)$, for $x \in \Omega_1$; then u_1 is a solution of (1.7) with $\underline{u} \leq u_1 \leq \overline{u}$.

Step 2. Repeat Step 1 up to the existence of the sequence $\{u_{k_{1j}}\}_{j=1,2...}$ to get a subsequence $\{u_{k_{2i}}\}_{i=1,2...}$ converging in $C^1(\Omega_2)$ to a limit u_2. Then likewise, u_2 is a solution on Ω_2 and $u_2|\Omega_1 = u_1$. Repeat Step 1 again on Ω_3,..., etc. In this way, we obtain a sequence $\{u_{k_{nj}}\}_{j=1,2...}$ which converges in $C^1(\Omega_k)$ and is a subsequence of $\{u_{k_{(n-1)j}}\}_{j=1,2...}$. Let $u_k(x) = \lim_{j\to\infty} u_{k_{nj}}(x)$; then, u_k is a solution in Ω_k and $u_k|\Omega_{k-1} = u_{k-1}$.

Step 3. By a diagonal process, $\{u_{k_{nn}}\}_{n=1,2...}$ is a subsequence of $\{u_{k_{nj}}\}_{j=1,2...}$ for every n. Thus, on Ω_k for each k, we have

$$\lim_{n\to\infty} u_{k_{nn}} = u_k$$

So, if we define $u(x) = \lim_{n\to\infty} u_{k_{nn}}(x)$, then $u(x)$ satisfies (1.7) and $\underline{u} \leq u \leq \overline{u}$ (since $\underline{u} \leq u_k \leq \overline{u}$ for every k). From lemma 3 and lemma 4 , it follows that there exist $C_1, C_2 > 0$ such that

$$C_2 d(x)^{-\frac{p-\gamma_1}{m+2-p}} \leq u(x) \leq C_1 d(x)^{-\frac{p-\gamma_1}{m+2-p}}, \quad x \in \Omega.$$

This complete the proof of it. □

3. Proof of main theorems

Proof of Theorem 1.

We can find a ε_0 in $(0,1)$ such that $\max\{\varepsilon_0^{m+2-p}, \varepsilon_0^{r+2-q}\} \leq \frac{1}{2}$. The assumptions $\frac{(p-\gamma_1)s}{qr+(2-q)\gamma_2} < \frac{m+2-p}{r+2-q} < \frac{pm+(2-p)\gamma_1}{(q-\gamma_2)n}$ imply

$$\delta_0 = \min\{\varepsilon_0^{p-2}\frac{C_1^{-n}C_2^m D_2}{2} \inf d(x)^{\frac{(q-\gamma_2)n}{r+2-q} - \frac{pm+(2-p)\gamma_1}{m+2-p}},$$

$$\varepsilon_0^{q-2}\frac{C_1^{-s}C_2^r K_2}{2} \inf d(x)^{\frac{(p-\gamma_1)l}{m+2-p} - \frac{qr+(2-q)\gamma_2}{r+2-q}}\} > 0$$

Firstly, we will show that system (1.1) has at least a positive solution when $0 < \delta_1, \delta_2 < \delta_0$. Our aim is to look for an ordered super and subsolutions. Let $(\overline{u}, \overline{v}) = (U_{m,\gamma_1}, U_{r,\gamma_2})$ and $(\underline{u}, \underline{v}) = (\varepsilon_0 U_{m,\gamma_1}, \varepsilon_0 U_{r,\gamma_2})$ where $U_{m,\gamma_1}, U_{r,\gamma_2}$ are unique positive solutions of (1.7) and (1.8) respectively. By Lemma 6, there exist $C_1, C_2 > 0$ such that

$$C_2 d(x)^{-\frac{p-\gamma_1}{m+2-p}} \le U_{m,\gamma_1}(x) \le C_1 d(x)^{-\frac{p-\gamma_1}{m+2-p}}, \quad x \in \Omega$$

$$C_2 d(x)^{-\frac{q-\gamma_2}{r+2-q}} \le U_{r,\gamma_2}(x) \le C_1 d(x)^{-\frac{q-\gamma_2}{r+2-q}}, \quad x \in \Omega$$

It is clear that

$$\begin{aligned}\operatorname{div}(|\nabla \overline{u}|^{p-2}\nabla \overline{u}) &= a_1 U_{m,\gamma_1}^{m_1+1} + b_1(x) U_{m,\gamma_1}^{m+1} \\ &\le U_{m,\gamma_1}(a_1 U_{m,\gamma_1}^{m_1} + b_1(x) U_{m,\gamma_1}^{m} + \delta_1(\varepsilon U_{p,\gamma_2})^n) = \overline{u}(a_1 \overline{u}^{m_1} + b_1(x)\overline{u}^m + \delta_1 \underline{v}^n).\end{aligned}$$

From $\delta_1 \le \delta_0$, it follows that

$$\delta_1 \le \varepsilon_0^{p-2} \frac{C_1^{-n} C_2^m D_2}{2} d(x)^{\frac{(q-\gamma_2)n}{r+2-q} - \frac{pm+(2-p)\gamma_1}{m+2-p}},$$

i.e.

$$\varepsilon_0^{p-2} \frac{C_2^m D_2}{2} d(x)^{-\frac{pm+(2-p)\gamma_1}{m+2-p}} \ge \delta_1 C_1^n d(x)^{-\frac{(q-\gamma_2)n}{r+2-q}}, \quad x \in \Omega$$

From

$$\frac{b_1(x)}{2} U_{m,\gamma_1}^m \ge \frac{C_2^m D_2}{2} d(x)^{-\frac{pm+(2-p)\gamma_1}{m+2-p}}, U_{r,\gamma_2}^n \le C_1^n d(x)^{-\frac{(q-\gamma_2)n}{r+2-q}},$$

we have

$$\varepsilon_0{}^{p-2} \frac{b_1(x)}{2} U_{m,\gamma_1}^m \ge \delta_1 U_{r,\gamma_2}^n$$

Thus, we have

$$\begin{aligned}&a_1 U_{m,\gamma_1}^{m_1} + b_1(x) U_{m,\gamma_1}^m \\ &\ge \varepsilon_0{}^{m_1+2-p} a_1 U_{m,\gamma_1}^{m_1} + \varepsilon_0{}^{m+2-p} b_1(x) U_{m,\gamma_1}^m + \varepsilon_0{}^{2-p} \delta_1 U_{r,\gamma_2}^n\end{aligned}$$

Furthermore, we conclude

$$\operatorname{div}(|\nabla \underline{u}|^{p-2}\nabla \underline{u}) \ge \underline{u}(a_1 \underline{u}^{m_1} + b_1(x)\underline{u}^m + \delta_1 \overline{v}^n)$$

It is clear that

$$\begin{aligned}\operatorname{div}(|\nabla \overline{v}|^{q-2}\nabla \overline{v}) &= a_2 U_{r,\gamma_2}^{r_1+1} + b_2(x) U_{r,\gamma_2}^{r+1} \\ &\le U_{k,\gamma_2}(a_2 U_{r,\gamma_2}^{r_1} + b_2(x) U_{r,\gamma_2}^{r} + \delta_2(\varepsilon U_{m,\gamma_1})^s) = \overline{v}(a_2 \overline{v}^{r_1} + b_2(x)\overline{v}^r + \delta_2 \underline{u}^s).\end{aligned}$$

From $\delta_2 \le \delta_0$, it follows that

$$\delta_2 \le \varepsilon_0^{q-2} \frac{C_1^{-l} C_2^r K_2}{2} d(x)^{\frac{(p-\gamma_1)s}{m+2-p} - \frac{qr+(2-q)\gamma_2}{r+2-q}}$$

$$\varepsilon_0^{q-2} \frac{C_2^r K_2}{2} d(x)^{-\frac{qr+(2-q)\gamma_2}{r+2-q}} \ge \delta_2 C_1^s d(x)^{-\frac{(p-\gamma_1)s}{m+2-p}}, \quad x \in \Omega$$

Since

$$\frac{b_2(x)}{2} U_{r,\gamma_2}^r \ge \frac{C_2^r K_2}{2} d(x)^{-\frac{qr+(2-q)\gamma_2}{r+2-q}}, U_{m,\gamma_1}^s \le C_1^s d(x)^{-\frac{(p-\gamma_1)s}{m+2-p}},$$

we have

$$\varepsilon_0{}^{q-2} \frac{b_2(x)}{2} U_{r,\gamma_2}^r \ge \delta_2 U_{m,\gamma_1}^s$$

Thus, we have

$$a_2 U_{r,\gamma_2}^{r_1} + b_2(x) U_{r,\gamma_2}^r \ge \varepsilon_0^{r_1+2-q} a_2 U_{r,\gamma_2}^{r_1} + \varepsilon_0^{r+2-q} b_2(x) U_{r,\gamma_2}^r + \varepsilon_0^{2-q} \delta_2 U_{r,\gamma_2}^s$$

Futhermore, we conclude

$$\operatorname{div}(|\nabla \underline{v}|^{q-2}\nabla \underline{v}) \geq \underline{v}(a_2\underline{v}^{r_1} + b_2(x)\underline{v}^r + \delta_1\overline{u}^s)$$

By lemma 5, system (1.1) at least has a positive solution (u, v).
Assume $\delta_1, \delta_2 < 0$. Set $(\underline{u}, \underline{v}) = (U_{m,\gamma_1}, U_{r,\gamma_2}), (\overline{u}, \overline{v}) = (MU_{m,\gamma_1}, MU_{r,\gamma_2})$. Clearly, we have

$$\operatorname{div}(|\nabla \underline{u}|^{p-2}\nabla \underline{u}) = a_1\underline{u}^{m_1+1} + b_1(x)\underline{u}^{m+1} \geq \underline{u}(a_1\underline{u}^{m_1} + b_1(x)\underline{u}^m + \delta_1\underline{v}^n)$$

$$\operatorname{div}(|\nabla \underline{v}|^{q-2}\nabla \underline{v}) = a_2\underline{v}^{r_1+1} + b_2(x)\underline{v}^{r+1} \geq \underline{v}(a_2\underline{v}^{r_1} + b_2(x)\underline{v}^r + \delta_1\underline{u}^s)$$

Denote

$$M = \max\{2^{\frac{1}{m+2-p}}, 2^{\frac{1}{r+2-q}}, (\frac{-2\delta_1 C_1^n}{D_2 C_2^m} \sup d(x)^{\frac{pm+(2-p)\gamma_1}{m+2-p} - \frac{(q-\gamma_2)n}{r+2-q}})^{\frac{1}{m-n}},$$

$$(\frac{-2\delta_2 C_1^l}{K_2 C_2^r} \sup d(x)^{\frac{rq+(2-q)\gamma_1}{r+2-q} - \frac{(p-\gamma_1)s}{m+2-q}})^{\frac{1}{r-s}}\}$$

Then we have

$$\frac{M^{m-n}D_2C_2^m}{2}d(x)^{-\frac{pm+(2-p)\gamma_1}{m+2-p}} \geq -\delta_1 C_1^n d(x)^{-\frac{(q-\gamma_2)n}{r+2-q}}$$

$$\frac{b_1(x)M^{m-n}U_{m,\gamma_1}^m}{2} \geq -\delta_1 U_{r,\gamma_2}^n$$

Therefore, it holds that

$$\frac{M^{m+2-p}b_1(x)U_{m,\gamma_1}^m}{2} \geq -\delta_1 M^{n+2-p}U_{r,\gamma_2}^n$$

It follows that

$$\begin{aligned}
\operatorname{div}(|\nabla \overline{u}|^{p-2}\nabla \overline{u}) &= M^{p-1}(a_1U_{m,\gamma_1}^{m_1+1} + b_1(x)U_{m,\gamma_1}^{m+1}) \\
&\leq MU_{m,\gamma_1}(a_1M^{m_1}U_{m,\gamma_1}^{m_1} + b_1(x)M^mU_{m,\gamma_1}^m + \delta_1U_{k,\gamma_2}^n) \\
&= \overline{u}(a_1\overline{u}^{m_1} + b_1(x)\overline{u}^m + \delta_1\overline{v}^n).
\end{aligned}$$

Similarly, we have

$$\operatorname{div}(|\nabla \overline{v}|^{q-2}\nabla \overline{v}) \leq \overline{v}(a_2\overline{v}^{r_1} + b_2(x)\overline{v}^r + \delta_2\overline{u}^s)$$

By lemma 5, system (1.1) at least has a positive solution (u, v). □

Proof of Theorem 2.

Proof. Let $(\overline{u}, \overline{v}) = (U_{m,\gamma_1}, U_{r,\gamma_2})$ and $(\underline{u}, \underline{v}) = (\varepsilon U_{m,\gamma_1}, \varepsilon U_{r,\gamma_2})$, where $U_{m,\gamma_1}, U_{r,\gamma_2}$ are unique positive solutions of (1.7) and (1.8) respectively, and $\max\{\varepsilon^{m+2-p}, \varepsilon^{r+2-q}\} < \frac{1}{2}$. If $\lim_{k\to\infty}\delta_1^k = 0, \lim_{k\to\infty}\delta_2^k = 0$ and $0 < \delta_1^k, \delta_2^k \leq \delta_0$, we may similarly prove that $(\overline{u}, \overline{v})$ and $(\underline{u}, \underline{v})$ are still a pair super and subsolutions corresponding to the elliptic system (1.1). (ε in the pair super and subsolutions can have no relation to δ_1^k, δ_2^k). It follows from theorem 1 that there exists a positive solution denoted by (u_k, v_k) such that

$$\underline{u} \leq u_k \leq \overline{u}, \underline{v} \leq v_k \leq \overline{v}$$

Thus $\{(u_k, v_k)\}$ has a convergent subsequence in $C^1(\overline{\Omega})$ which is still denoted by itself and the limits are denoted by (u_0, v_0). For any compact $E \subset \Omega$, it follows that u_k, v_k are uniformly bounded in E. Let

$$\begin{cases} \operatorname{div}(|\nabla u_k|^{p-2}\nabla u_k) = u_k(a_1 u_k^{m_1} + b_1(x)u_k^m + \delta_1^k {v_k}^n), & x \in E \\ \operatorname{div}(|\nabla v_k|^{q-2}\nabla v_k) = v_k(a_2 v_k^{r_1} + b_2(x)v_k^r + \delta_2^k u_k^s), & x \in E \end{cases}$$

pass to the limit as $k \to \infty$, we have that

$$\begin{cases} \operatorname{div}(|\nabla u_0|^{p-2}\nabla u_0) = u_0(a_1 u_0^{m_1} + b_1(x)u_0^m), & x \in E \\ \operatorname{div}(|\nabla v_0|^{q-2}\nabla v_0) = v_0(a_2 v_0^{r_1} + b_2(x)v_0^r), & x \in E \end{cases}$$

Thus it follows from the arbitrary property of $E \subset \Omega$ that

$$\begin{cases} \operatorname{div}(|\nabla u_0|^{p-2}\nabla u_0) = u_0(a_1 u_0^{m_1} + b_1(x)u_0^m), & x \in \Omega \\ \operatorname{div}(|\nabla v_0|^{q-2}\nabla v_0) = v_0(a_2 v_0^{r_1} + b_2(x)v_0^r), & x \in \Omega \\ u_0 = v_0 = +\infty, & x \in \partial\Omega \end{cases}$$

By lemma 6, we have $u_0 = U_{m,\gamma_1}, v_0 = U_{r,\gamma_2}$. □

Proof of Theorem 3.

Proof. Since (u, v) is a positive solution of elliptic system (1.1), without loss of generality, we can assume $u(x), v(x) \geq 1$ in Ω. From the elliptic system, it follows that

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) \geq b_1(x)u^{m+1}, \operatorname{div}(|\nabla v|^{q-2}\nabla v) \geq b_2(x)v^{r+1}, \quad x \in \Omega.$$

Thus, there exists $C > 0$ such that

$$u(x) \leq Cd(x)^{-\frac{p-\gamma_1}{m+2-p}} = Cd(x)^{-\alpha}, v(x) \leq Cd(x)^{-\frac{q-\gamma_2}{r+2-q}} = Cd(x)^{-\beta}, \quad x \in \Omega.$$

Hence, we have

$$\begin{aligned} \operatorname{div}(|\nabla u|^{p-2}\nabla u) &= u(a_1 u^{m_1} + b_1(x)u^m + \delta_1 v^n) \\ &\leq u(a_1 u^{m_1} + D_1 d(x)^{-\gamma_1} u^m + \delta_1 C^n d(x)^{-n\beta}) \end{aligned}$$

$$\operatorname{div}(|\nabla v|^{q-2}\nabla v) = v(a_2 v^{r_1} + b_2(x)v^r + \delta_2 u^s) \leq v(a_2 v^{r_1} + K_1 d(x)^{-\gamma_2} v^r + \delta_2 C^q d(x)^{-l\alpha})$$

From $0 < n\beta \leq \gamma_1 < p, 0 < q\alpha \leq \gamma_2 < q$, it follows that there exists $C_* > 0$ such that in Ω

$$D_1 d(x)^{-\gamma_1} u^m + \delta_1 C^n d(x)^{-n\beta} \leq C_* d(x)^{-\gamma_1} u^m$$

$$K_1 d(x)^{-\gamma_2} v^r + \delta_2 C^s d(x)^{-s\alpha} \leq C_* d(x)^{-\gamma_2} v^r$$

Furthermore, we have

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) \leq u(a_1 u^{m_1} + C_* d(x)^{-\gamma_1} u^m), \quad x \in \Omega$$

$$\operatorname{div}(|\nabla v|^{q-2}\nabla v) \leq v(a_2 v^{r_1} + C_* d(x)^{-\gamma_2} v^r), \quad x \in \Omega$$

Since u, v are positive and boundary blow-up functions, $\inf_\Omega u(x) > 0$ and $\inf_\Omega v(x) > 0$ hold. By $m > m_1, r > r_1$, $\inf_\Omega d(x)^{-\gamma_1} > 0$ and $\inf_\Omega d(x)^{-\gamma_2} > 0$, we can take sufficiently large constant $C > 0$ such that

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) \leq Cd(x)^{-\gamma_1} u^{m+1}, \operatorname{div}(|\nabla v|^{q-2}\nabla v) \leq Cd(x)^{-\gamma_2} v^{r+1}, \quad x \in \Omega$$

That implies that there exists $C_2 > 0$ such that

$$u(x) \geq C_2 d(x)^{-\alpha}, v(x) \geq C_2 d(x)^{-\beta}, \quad x \in \Omega$$

Thus, we may take suitable $C_1, C_2 > 0$ such that in Ω

$$C_2 d(x)^{-\alpha} \leq u(x) \leq C_1 d(x)^{-\alpha}, \quad C_2 d(x)^{-\beta} \leq v(x) \leq C_1 d(x)^{-\beta}.$$

□

References

1. F. Cirstea and V. Radulescu,*Entire solutions blowing up at infinity for semilinear elliptic systems*, J. Math. Pures Appl. **81** (2002), 827-846.
2. Ph. Clement, R. Manasevich and E. Mitidieri,*Positive solutions for a quasilinear system via blow up*, Comm. in Partial Diff. Eqns. **18**, No.12 (1993), 2071-2106.
3. Ph. Clement, D.G. de Figueiredo and E. Mitidieri, *Positive solutions of semilinear elliptic systems*, Comm. in P.D.E. **17**, No.5/6 (1992), 923-940.
4. Z.Zhang, *Existence of large solutions for a semilinear elliptic problem via explosive sub-supersolutions*, Electrons.J.Differential Equations.**2**(2006),1-8.
5. P. L. Felmer, R. Manasevich and F. de Thelin,*Existence and uniqueness of positive solutions for certain quasilinear elliptic system*, Comm.in P.D.E. **17** (1992), 2013-2029.
6. Z. M. Guo, *Existence of positive radial solutions for a class of quasilinear elliptic systems in annular domains*, Chinese Journal of Contemporary Math. **17**, No.4 (1996), 337-350
7. Lei Wei and Mingxin Wang, *Existence and extimate of large solutions for an elliptic system*, Nonlinear Analysis. in press
8. A. V. Lair and A. W. Wood, *Existence of entire large positive solutions of semilinear elliptic systems*, J. Differential Eqns **164**, No.2 (2000), 380-394.
9. Huiling Li and Mingxin Wang, *Existence and uniqueness of positive solutions to the boundary blow-up problem for an elliptic system*, J. Differential Eqns **234**,(2007), 246-266.
10 E. Mitidieri, G. Sweers and R. van der Vorst, *Nonexistence theorems for systems of quasilinear partial differential equations*, Differential Integral Equations **8** (1995), 1331-1354.
11 E. Mitidieri, *Nonexistence of positive solutions of semilinear elliptic system in* $\mathbf{R^N}$, Diff. Integral Equations **9** (1996), 465-479.
12 E. Mitidieri, *A Rellich type identity and applications*, Comm. in Partial Diff. Equations **18** (1993), 125-171.
13 L. A. Peletier and R. van der Vorst, *Existence and non-existence of positive solutions of non-linear elliptic systems and the biharmonic equations*, Diff. Integral Eqns. **54** (1992), 747-767.
14 J.Lopez-Gomez, *Optimal uniqueness theorems and exact blow-up rates of large solutions*, J. Differential Equations, 224(2006), 385-439.
15 J. Serrin and H. Zou, *Non-existence of positive solutions of Lane Emden systems*, Differential Integral Equations **9**, No.4 (1996), 635-653.
16 Z. D. Yang and Q. S. Lu, *Non-existence of positive radial solutions for a class of quasilinear elliptic system*, Comm. Nonlinear Sci. Numer. Simul. **5**, No.4 (2000), 184-187.
17 Z.D. Yang, *Existence of entire explosive positive radial solutions for a class of quasilinear elliptic systems*, J. Math. Anal Appl., 288(2003), 768-783
18 Z.D. Yang and Q.S. Lu,*Blow-up estimates for a quasilinear reaction-diffusion system*, Math. Methods in the Appl. Sci., 26(2003), 1005-1023.
19 Z.D. Yang and Q.S. Lu,*Nonexistence of positive solutions to a quasilinear elliptic system and blow-up estimates for a quasilinear reaction-diffusion system*, J. Computational and Appl. Math., 150(2003), 37-56.

20 J. Grrcia-Melian and J.D. Rossi,*Boundary blow-up solutions to elliptic systems of competitive type*, J. Differential Equations, 206(2004), 156-181.
21 D. Gibarg, N.S. Trudinger,*Elliptic Partial Differential Equations of second Order*, Springer, Berlin, 1983.
22 G.M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal., 12(1988), 1203-1219.
23 Z. M. Guo and J. R. L. Webb, *Uniqueness of positive solutions for quasilinear elliptic equations when a parameter is large.* Proc .Roy. Soc. Edinburgh. 124A(1994), 189-198.
24 J. Diaz, *Nonlinear Partial Differential Equations and Free Boundaries*, vol I, Elliptic Equations, in: Pitman Research Notes in Mathematics Series, vol. 106, Longman, 1985.
25 C.V.Pao, *Nonlinear Parabolic and Elliptic Equations, Plenum*, New York, 1992.
26 J.Garcia-Melian, *Nondegeneracy and uniqueness for boundary blow-up elliptic problems*, J. Differential Equations, 223(2006), 208-227.
27 Z.D. Yang and Mingzhu Wu,*Existence of boundary blow-up solutions for a class of quasilinear elliptic systems with subcritical case*, Comm.Pure Appl. Anal., 6(2)(2007), 531-540.
28 G.Diaz and R.Letelier, *Explosive solutions of quasilinear elliptic equations:Existence and uniquness*, Nonlinear Anal., 20(1993), 97-125.
29 D.D.Hai and R.Shivaji, *An existence result on positive solutions for a class of p-Laplacian systems*, Nonlinear Anal., 56(2004), 1007-1010.
30 Caisheng Chen, *On positive weak solutions for a class of quasilinear elliptic systems*, Nonlinear Anal., 62(2005), 751-756.
31 M. Chhetri, D.D.Hai and R.Shivaji, *On positive solutions for a class of p-Laplacian semipositone systems*, Discrete Continuous Dyn. Syst., 9(4)(2003), 1063-1071.
32 A.C.Lazer, P.J.McKenna, *On a problem of Bieberbach and Rademacher*, Nonlinear Anal. 21(1993), 327-335.
33 O.A.Ladyzhenskaya, N.N. Ural'tseva,*Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.
34 P.Tolksdorf, *On the Dirichlet problem for quasilinear equations in domains with conical boundary point*, Comm. Partial Differential Equations 8(7)(1983) 773-817.
35 C.L.Liu, Z.D.Yang, *Existence of large solutions for a quasilinear elliptic problem via explosive sub-supersolutions*, Appl.Math.Comput 199(2008), 414-424.
36 C.L.Liu, Z.D.Yang, *Boundary blow-up quasilinear elliptic problems of the Bieberbach type with nonlinear gradient terms*, Nonlinear Anal. (in press).

Zuodong Yang: His research interests focus on the structure of solution for a class of quasilinear elliptic equations(systems) and parabolic equations(systems).

Institute of Mathematics, School of Mathematics and Computer Science, Nanjing Normal University, Jiangsu Nanjing 210097, China;

College of Zhongbei, Nanjing Normal University, Jiangsu Nanjing 210046, China.

`e-mail: zdyang_jin@263.net`

Mingzhu Wu

Institute of Mathematics, School of Mathematics and Computer Science, Nanjing Normal University, Jiangsu Nanjing 210097, China.