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APPROXIMATION METHOD FOR SCATTERED DATA FROM SHIFTS OF A RADIAL BASIS FUNCTION

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ABSTRACT. In this paper, we study approximation method from scattered data to the derivatives of a function f by a radial basis function ϕ. For a given function f, we define a nearly interpolating function and discuss its accuracy. In particular, we are interested in using smooth functions ϕ which are (conditionally) positive definite. We estimate accuracy of approximation for the Sobolev space while the classical radial basis function interpolation applies to the so-called native space. We observe that our approximant provides spectral convergence order, as the density of the given data is getting smaller.

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1. Introduction

A function $\phi : \mathbb{R}^d \to \mathbb{R}$ is *radial* in the sense that $\phi(x) = \Phi(|x|)$, where $|\cdot|$ is the usual Euclidean norm. Given a set X of scattered points in $\mathbb{R}^d$ with $d \geq 2$ and values $f|_X$, $X := \{x_1, \cdots, x_N\}$, sampled from an underlying function $f : \mathbb{R}^d \to \mathbb{R}$, we need to construct a function $s : \mathbb{R}^d \to \mathbb{R}$ from the space

$$S(\phi) = \text{span}\Big\{\phi(\cdot - x_j) : x_j \in X\Big\}$$

such that s approximates f in some sense. Radial basis functions (RBFs) provide well-established tools for solving this scattered data approximation or interpolation problem. The starting point of radial basis function method is choosing

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a suitable radial function ϕ which is positive or conditionally positive definite function.

Definition 1. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a continuous function. We say that ϕ is conditionally positive definite of order $m \in \mathbb{N} := \{1, 2, \cdots\}$ if for every finite set of pairwise distinct points $X := \{x_1, \cdots, x_N\} \subset \mathbb{R}^d$ and for every $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{R}^N \setminus 0$ satisfying

$$\sum_{j=1}^{N} \alpha_j p(x_j) = 0, \quad p \in \Pi_{<m},$$

the quadratic form

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \phi(x_i - x_j)$$

is positive. Here, $\Pi_{<m}$ denotes the subspace of $C(\mathbb{R}^d)$ consisting of all algebraic polynomials of degree less than m on $\mathbb{R}^d$.

The following notations are used throughout this paper. Put

$$\mathbb{Z}_+^d := \Big\{(\gamma_1, \ldots, \gamma_d) \in \mathbb{Z}^d \,:\, \gamma_k \geq 0\Big\}.$$

For $\alpha, \beta \in \mathbb{Z}_+^d$, we set $\alpha! := \alpha_1! \cdots \alpha_d!$, $|\alpha|_1 := \sum_{k=1}^{d} \alpha_k$, and $\alpha^\beta = \alpha_1^{\beta_1} \cdots \alpha_d^{\beta_d}$. For any $\alpha \in \mathbb{Z}_+^d$, D^α indicates the differential operator and $f^{(\alpha)} := D^\alpha f$. The Fourier transform of $f \in L_1(\mathbb{R}^d)$ is defined as

$$\hat{f}(\theta) := \int_{\mathbb{R}^d} f(t)\, e^{-i\theta \cdot t} dt.$$

Also, for a function $f \in L_1(\mathbb{R}^d)$, we use the notation $f^\vee$ for the inverse Fourier transform. The Fourier transform can be uniquely extended to the space of tempered distributions on $\mathbb{R}^d$. Finally, throughout this paper, we use c (or c_j) to represent an arbitrary constant that may change value.

Suppose that a continuous function $f : \mathbb{R}^d \to \mathbb{R}$ is known only at a set of discrete points X in open bounded subset $\Omega \subset \mathbb{R}^d$. Then, we construct a nearly interpolating approximation to the underlying function f by using $f|_X$ based on RBF interpolation. Specifically, our implementation is is given as follows:

Step 1: Construct an 'admissible coefficients' $(A(\cdot, x_j))_{x_j \in X}$ annihilate polynomials up to a certain degree.
Step 2: Define a function f_X by

$$f_X := \sum_{j=1}^{N} A(\cdot, x_j) f(x_j), \quad t \in \Omega,$$

Step 3: Define an RBF interpolation to $f^* := \sigma_\omega^\vee * (\chi_\Omega f_X)$ where χ_Ω is the characteristic function of Ω and $\omega = \omega(h)$ and σ_ω is a suitable C^∞-cutoff function.

As observed above, our aim is to find a suitable approximant which approximation the derivative of the original underlying function. The function f belongs to the Sobolev space. For $k > 0$, the Sobolev space is defined by

$$W_p^n(\Omega) := \left\{ f \in L_p(\Omega) : |f|_{k,p} := \sum_{|\alpha|_1 \le k} \|D^\alpha f\|_{L_p(\Omega)} < \infty \right\}$$

with $1 \le p \le \infty$. In order to discuss the accuracy of an approximation method, let us define the 'density' of $X \subset \Omega$:

$$h := h(X;\Omega) := \sup_{x\in\Omega} \min_{x_j \in X} |x - x_j|. \tag{1.1}$$

We also assume that $\Omega \subset \mathbb{R}^d$ is an open bounded domain with cone property. Moreover, without great loss, we assume that N is bounded by ch^{-d} with $c > 0$ independent of X and Ω.

In this study, we assume that the function ϕ has a generalized Fourier transform in the sense of tempered distribution, and we require that this distribution coincides on $\mathbb{R}^d \setminus 0$ with some continuous function while having a certain type of singularity (necessarily of finite order) at the origin. Hence, here and in the sequel, we assume that $\hat{\phi}$ satisfies the following properties

$$|\cdot|^{2m}\hat{\phi} = F \neq 0, \quad m > d/2, \quad \text{and } F \in L_\infty(\mathbb{R}^d). \tag{1.2}$$

For a given conditinally positive definie RBF ϕ and a center set X, we choose a dilated basis function $\phi_\omega = \phi(\cdot/\omega)$. Then, our approximant is given by the interpolation to f^*, that is

$$S_{f^*,X}(x) := \sum_{i=1}^{L} \beta_i p_i(x) + \sum_{j=1}^{N} \alpha_j \phi_\omega(x - x_j), \quad p \in \Pi_{<m}, \tag{1.3}$$

where $p_1, \ldots, p_L$ is a basis of $\Pi_{<m}$. The coefficients α_j $(j = 1, \cdots, N)$ and β_i $(i = 1, \cdots, L)$ are required to satisfy the $(N + \ell) \times (N + \ell)$ system of linear equations:

$$\begin{aligned} &S_{f^*,X}(x_j) = f^*(x_j), \quad j = 1, \ldots, N, \\ &\sum_{j=1}^{N} \alpha_j p_i(x_j) = 0, \quad i = 1, \ldots, L. \end{aligned} \tag{1.4}$$

Here, for $m > 0$, we require X to have the nondegeneracy property for Π_m and the nonsingularity of the system (1.3) is guaranteed when ϕ is conditionally positive definite [5]. For more details, the reader is referred to the papers [6, 7, 8], and the survey papers [2, 3, 9].

The prototype of our results is as follows:

Theorem 1. *Let ϕ be a smooth basis function and $S_{f^*,X}$ be an interpolant to f^* on X employing ϕ_ω. Assume that $\omega(h) = h^r$. If $f \in W_\infty^n(\Omega)$ and K is a compact subset of Ω, we have*

$$\left| D^\beta f(x) - S_{f^*,X}(x) \right| \le ch^{r(n-|\beta|_1)}, \quad x \in K$$

with a constant $c >$ depending on Ω and f.

2. Construction of f^*

For a given $\beta \in \mathbb{Z}_+^d$, the purpose of this section is to construct a β-mollification f^* from $f|_X$ such that

$$D^\beta f^* \sim D^\beta f, \quad |\beta|_1 < k.$$

For this, we first introduce the following definition, which is the extension of the one in [14]:

Definition 2. Let $\beta \in \mathbb{Z}_+^d$. The coefficients $(A(\cdot, x_j))_{x_j \in X}$ are said to be β-'admissible' on Ω if the following conditions hold:

(a) There exists $c_1 > 0$ such that, for any $x \in \Omega$, $A(t, x_j) = 0$ whenever $|x - x_j| > c_1 h$, with h the density of X as in (1.1).

(b) The set $\left\{ (A(x, x_j))_{x_j \in X} \,:\, x \in \Omega \right\}$ is bounded in $\ell_1(X)$ by $h^{-|\beta|_1}$ up to some constant.

(c) They satisfy the polynomial annihilation property:

$$\sum_{j=1}^{N} A(x, x_j)\, p(x_j) = D^\beta p(x), \qquad \forall p \in \Pi_n,$$

For a given β and $f|_X$, let us construct a mollification f^* from f as the following three steps:

Step 1: With a given set of β-admissible coefficients, define f_X by

$$f_X(x) := \sum_{j=1}^{N} A(t, x_j) f(x_j), \quad x \in \Omega. \tag{2.1}$$

Step 2: Define σ_Ω to be a C^∞-cutoff function such that $\sigma_\Omega(x) = 1$ for $x \in \Omega$ and $\sigma_\Omega(x) = 0$ for $|x| > r$ with a sufficiently large $r > 0$.

Step 3: Finally, the β- mollification f^* is given by

$$f^* := \sigma_\omega^\vee * (\chi_\Omega f_X)$$

where χ_Ω is the characteristic function of Ω and $\sigma_\omega^\vee = \sigma^\vee(\cdot/\omega)$.

We describe an approximation property of f_X to f on Ω:

Lemma 1. *Let $\beta \in \mathbb{Z}_+^d$, and assume that the coefficients $(A(\cdot, x_j))_{x_j \in X}$ is β-admissible on Ω. Let the function f_X be defined as in (2.1). Suppose that the approximand $f \in W_\infty^n(\mathbb{R}^d)$. Then, we have the following property*

$$\left\| D^\beta f - f_X \right\|_{L_\infty(\Omega)} \le ch^{n-|\beta|_1},$$

where c is independent of X and Ω.

Proof. Let

$$T_{n,x} f = \sum_{|\alpha|_1 < n} (\cdot - x) D^\alpha f(x)$$

be the Taylor polynomial of f around x of degree $n \in \mathbb{N}$. Then, since $|\beta| < n$ and the set $(A(\cdot, x_j))_{x_j \in X}$ is β-admissible, we have

$$\sum_{x_j \in X} A(x, x_j)(x - x_j)^\beta / \beta! = 1$$

Also, by construction, $\sum_{x_j \in X} |A(x, x_j)| \le ch^{-|\beta|_1}$. Thus, due to the fact that $f \in W_\infty^n(\mathbb{R}^d)$, we can obtain

$$\begin{aligned} |D^\beta f(x) - f_X(x)| &\le \sum_{x_j \in X} \sum_{|\alpha|_1 = n} |A(x, x_j)||x - x_j|^\alpha / \alpha! \\ &\quad + \sum_{|\alpha|_1 = n+1} |A(x, x_j)||x - x_j|^\alpha D^\alpha f(\xi_j) / \alpha!. \\ &\le ch^{n-|\beta|_1} \end{aligned}$$

for some constant $c > 0$ and ξ_j between x and ξ_j. It completes the proof. □

Lemma 2. *Let $\beta \in \mathbb{Z}_+^d$, and assume that the coefficients $(A(\cdot, x_j))_{x_j \in X}$ for f_X are β-admissible for $\Pi_{<n}$ on Ω. Let K be a compact subset of Ω. Let $f \in W_\infty^n(\mathbb{R}^d)$. Then, there exists a constant $c > 0$ depending on n such that*

$$\left| D^\beta f(x) - f^*(x) \right| \le c\omega^{n-|\beta|_1}, \quad x \in K.$$

Proof. We start to prove this lemma by dividing the error $f(x) - f^*(x)$ as follows:

$$\begin{aligned} \left| D^\beta f(x) - f^*(x) \right| &\le |D^\beta f(x) - \sigma_\omega^\vee * D^\beta f(x)| \\ &\quad + |\sigma_\omega^\vee * (D^\beta - \chi_\Omega D^\beta f)(x)| \\ &\quad + |\sigma_\omega^\vee * (\chi_\Omega D^\beta f - \chi_\Omega f_X)(x)|. \end{aligned} \tag{2.2}$$

It is observed from the literature (e.g., see [15]) that for every $f \in W_\infty^n(\mathbb{R}^d)$,

$$\left\| D^\beta f - \sigma_\omega^\vee * D^\beta f \right\|_{L_\infty(\mathbb{R}^d)} = o(\omega^{n-|\beta|_1}).$$

Now let us turn to the estimate of the second term in the above bound (2.2). Let

$$\delta := \sup\Big\{r > 0 : K + r \subset \Omega\Big\}.$$

By using the fact that $\displaystyle\int_{\mathbb{R}^d} \sigma_\omega^\vee(y)dy = \sigma_\omega(0) = 1$ for any $\omega > 0$, we get the relations

$$\begin{aligned} \sigma_\omega^\vee * (D^\beta f - \chi_\Omega D^\beta f)(x) &= \int_{\Omega'} \sigma_\omega^\vee(x-y) D^\beta f(y) dy \\ &\le \|D^\beta f\|_{L_\infty(\mathbb{R}^d)} \int_{\Omega'/\omega} |\sigma^\vee(x/\omega - y)| dy \end{aligned}$$

where Ω' indicates the complement set of Ω in $\mathbb{R}^d$. Since $\sigma^\vee$ decays faster than any polynomial degree, there exists a constant $c_n > 0$ such that for $y \in \Omega'$ and $x \in K$,

$$\int_{\Omega'/\omega} |\sigma^\vee(x/\omega - y)| dy \le c_n \omega^n \int_{B'_\delta} (\omega + |y|)^{-n-d} dy \le c_n \omega^n.$$

Finally, to obtain a bound of the third term in (2.2), we find the inequality

$$\begin{aligned} &\Big|\sigma_\omega^\vee * \chi_\Omega D^\beta f(x) - \sigma_\omega^\vee * \chi_\Omega f_X(x)\Big| \\ &\qquad \le \int_{\Omega/\omega} |\sigma^\vee(x/\omega - y)| |f(\omega y) - f_X(\omega y)| dy. \end{aligned} \tag{2.3}$$

By Lemma 1,

$$\Big\|f - f_X\Big\|_{L_\infty(\Omega)} \le c h^{n - |\beta|_1} |f|_{k,\infty}.$$

Inserting this relation into (2.3), we prove

$$\Big|\sigma_\omega^\vee * \chi_\Omega f(x) - \sigma_\omega^\vee * \chi_\Omega f_X(x)\Big| \le c\, h^{n-|\beta|_1}$$

with c independent of X. □

3. Spectral approximation order

In this section, we provide the error estimate of $f - S_{f^*,X}$. Among many radial functions currently in use, we first choose the basis function to be the 'shifted thin-plate spline'

$$\phi_c(x) := \begin{cases} \Big(|x|^2 + c^2\Big)^{\lambda/2}, & \lambda \in \mathbb{Z}_+,\ \lambda,\ d \text{ odd}, \\ \Big(|x|^2 + c^2\Big)^{\lambda/2} \log(|x|^2 + c^2)^{1/2}, & \lambda \in \mathbb{Z}_+,\ \lambda,\ d \text{ even}, \end{cases}$$

whose properties are quite well understood, both theoretically as well as practically. We find from [1] that the Fourier transform of ϕ is of the form

$$\hat\phi = c(m,d) \tilde K_m(\gamma\cdot) | \cdot |^{2m} \neq 0$$

where $c(m,d)$ is a constant depending on m and d.

For the further analysis, we introduce the following function space:

$$\mathcal{F}_\phi := \left\{ f : |f|_\phi^2 := \int_{\mathbb{R}^d} \frac{|\hat{f}(\theta)|^2}{\hat{\phi}(\theta)} d\theta < \infty \right\} \tag{3.1}$$

From the papers (see, e.g., [14], [7]), we cite

Lemma 3. *Let $S_{X,f}$ in (1.3) be an interpolant to f on $X = \{x_1, \cdots, x_N\}$. Given ϕ and m, for all functions f in the native space $\mathcal{F}_\phi$, there is an error bound of the form*

$$\left| f(x) - S_{f,X}(x) \right| \le |f|_\phi P_{\phi,X}(x)$$

where $P_{\phi,X}(x)$ is the norm of the error functional, i.e.,

$$P_{\phi,X}(x) = \sup_{|f|_\phi \neq 0} \frac{|f(x) - S_{f,X}(x)|}{|f|_\phi}. \tag{3.2}$$

Now, we are ready to provide the main result of this paper.

Theorem 2. *Let ϕ be the 'shifted' thin-plate spline and $S_{f^*,X}$ be an interpolant to f^* on X employing ϕ_ω as in (1.3), where*

$$\omega(h) = h^r, \quad r \in (0,1).$$

For a given $\beta \in \mathbb{Z}_+^d$, assume that the coefficients $(A(\cdot, x_j))_{x_j \in X}$ are β-admissible for $\Pi_{<n}$ on Ω. Let K be a compact subset of Ω. If $f \in W_\infty^n(\mathbb{R}^d)$, then we have the error bound

$$\left| D^\beta f(x) - S_{f^*,X}(x) \right| \le c h^{r(n-|\beta|_1)}, \quad x \in K$$

with a constant $c >$ depending on Ω and f.

Proof. Let $x \in K$. By virtue of Lemma 2, we get

$$\begin{aligned} \left| D^\beta f(x) - S_{f^*,X}(x) \right| &\le |D^\beta f(x) - f^*(x)| + |f^*(x) - S_{f^*,X}(x)| \\ &\le c_1 h^{n-|\beta|_1} + |f^*(x) - S_{f^*,X}(x)| \end{aligned}$$

Hence, it remains to estimate only the error $f^* - S_{f^*,X}$. In fact, the function $S_{f^*,X}(\omega\cdot)$ can be understood as an interpolant employing the scattered shifts of ϕ to the function $f^*(\omega\cdot)$ on X/ω in Ω/ω, i.e.,

$$S_{f^*,X}(\omega\cdot) = S_{f^*(\omega\cdot),X/\omega}.$$

Then, by using Lemma 3, it gives the bound

$$\left| f^*(\omega x) - S_{f^*(\omega x),X/\omega}(x) \right| \le P_{X/\omega}(x), |f^*(\omega\cdot)|_{\phi_\omega}, \quad x \in \Omega/\omega.$$

Moreover, invoking the explicit formula of $|\cdot|_{\phi_\omega}$ in (3.1), we have the inequality

$$\begin{aligned} |f^*(\omega\cdot)|_{\phi_\omega} &\le \omega^{-d/2} |\sigma^2/\hat{\phi}\|_{L_\infty(\mathbb{R}^d)} \left\| \chi_\Omega f_X \right\|_{L_2(\mathbb{R}^d)} \\ &\le c \|\sigma^2/\hat{\phi}\|_{L_\infty(\mathbb{R}^d)} \end{aligned}$$

with a constant depending on f and Ω. When $\omega(h) = h^r$ with $r \in (0,1)$, the error bound of $P_{X/\omega}$ is

$$P_{X/\omega}(x) \le c \exp(-\epsilon/h^{(1-r)}) \tag{3.3}$$

for some $\epsilon > 0$ (see [8]). It implies that

$$\left\| f^*(x) - S_{f^*,X}(x) \right\| \le c \exp(-\epsilon/h^{(1-r)})$$

with a constant $c > 0$ independent of $x \in K$. It finishes the proof. □

Now consider the Gaussian function

$$\phi = \exp(-|x|^2/\lambda^2), \quad \lambda > 0.$$

It is known that the Fourier transform of the Gaussian is also Gaussian-type function. If $\omega(h) = h^r$ with $r \in (0,1)$, it has been shown in [8] that

$$P_{X/\omega}(x) \le c \exp(-\epsilon/h^{(1-r)})$$

for some $\epsilon > 0$. Thus, it yields the following result.

Theorem 3. *Let ϕ be the Gaussian function and $\beta \in \mathbb{Z}_+^d$. Then, under the same assumptions and notations of Theorem 2, we have an error bound*

$$\left\| D^\beta f - S_{f^*,X} \right\|_{L_\infty(K)} \le ch^{r(n-|\beta|_1)}.$$

with a constant $c > 0$.

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