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A NOTE FOR RESTRICTED INFORMATION MARKETS

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ABSTRACT. This paper considers the problems of martingale measures and risk-minimizing hedging strategies in the market with restricted information. By constructing a general restricted information market model, the explicit relation of arbitrage and the minimal martingale measure between two different information markets are discussed. Also a link among all equivalent martingale measures under restricted information market is given. As an example of restricted information markets, this paper constitutes a jump-diffusion process model and presents a risk minimizing problem under different information. Through Itô formula and projection results in Schweizer[13], the explicit optimal strategy for different market information are given.

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1. Introduction

Pricing contingent claims is one of the most important problems in mathematical finance. The formula of Black and Scholes for the valuation of options has led to the great development of mathematical finance. Mathematical finance is attracting more and more attention of researchers. Initiated by Cox and Ross[7] and Harrison and Kreps[11], the "martingale method" of pricing derivative is one of two approached to the pricing of derivative securities. This approach consists of writing the value of the security as the expected value of the discounted payoff under a martingale measure. If the market is incomplete, then there are many equivalent martingale measures. It may be reasonable to suppose that there should be a special martingale measure which determines the prices of contingent claims. As the candidates of such measures, several martingale measures are proposed; minimal martingale measure (Fölmer and Schweizer[4]),

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variance-optimal martingale measure (Schweizer [14] or Delbaen and Schachermayer [2]), canonical martingale measure (Miyahara, [21]), etc. The examples which were given by Schachermayer [22] are useful for the understanding and the investigation of the relations among the measures above. The importance of minimal martingale measure was described in Miyahara [21], etc. Recently, it is mentioned that minimal martingale measure is related to the exponential utility function and to the fair prices of options (see Davis [10] and Frittelli [9]). Without question martingale measures play an important role in pricing and related items. Hedging contingent claims is also one of the most important problems in mathematical finance. In incomplete finance markets, there are non-attainable contingent claims and always exists more or less risk for hedging the non-attainable claims. For sake of comparing hedging those strategies, many optimal standards is proposed. The concept of risk-minimization was introduce by Föllmer and Sondermann[3] and consists in comparing strategies by means of a risk measure in terms of a conditional mean square error process. In the case where the price process is a (local)martingale under P(a situation called martingale case), it was shown that a unique risk-minimizing strategy exists and it can be computed using the Kunita-Watanabe(K-W) projection theorem. The case of semimartingale price process is more delicate and Schweizer[12] introduced the concept of risk-minimization in a local sense. Existence of a locally risk - minimizing strategy is then related to the existence of the $F-S$ decomposition, which can be viewed as an extension of the K-W decomposition. Characterization of the solution can then be expressed by means of the minimal martingale measure introduced by Föllmer& Schweizer[4]. Some useful work has been done, but majority of these discusses are based on perfect markets. Real financial markets are imperfect markets. In fact there are some investors different to general investors in the financial market. Because of their conditions, for example, they live in the country, the investors can't know all market information such as some invest policies, construction plans and so on, which are known by general investors. They might only know price information of risky assets. These make the investor's information incomplete. It is well known that hedging market risk and capturing arbitrage opportunity are closed to market information. So it conforms to financial application to discuss financial markets under different information. There are several recent papers dealing with invest strategies under restricted information in finance. Schweizer[13] presented risk-minimizing hedging strategies of contingent claims under restricted information, Pham [6] researched the problem of mean-variance hedging for partially observed drift processes, Frey and Runggaldier[19] focused on the computation of the optimal hedging strategies when asset price processes is observed at discrete random times. Frey [18] researched the risk minimization with incomplete information in a model for high-frequency data model. The utility maximization problem when only stock prices are observed was studied by Lakner[16]. Different from those above, the paper focuses on the relation of market completeness, arbitrage and minimal martingale measure between markets with different information, which is important to hedging contingent claims. The explicit relation of arbitrage and the minimal martingale measure between two different information markets are discussed by constructing a general restricted information markets. As an example, we constitute a jump-diffusion model and present the problem of risk-minimizing hedging. Moreover we give under the restricted information model the explicit risk-minimizing strategies for complete and restricted information.

The paper is organized as follows. In section2 we present a restricted information model. In Section 3 the explicit relation of arbitrage and the minimal martingale measure between two different information markets are discussed. Also a link among all equivalent martingale measures under restricted information market is given. In section 4 a jump-diffusion model is constituted. Explicit risk-minimizing hedging strategies under different information are presented in section 5 and 6 respectively.

2. A restricted information model

Assume $(\Omega, F, \mathcal{F}, \mathcal{P})$ is a probability space with filtration, P is the natural probability measure. The filtration $\mathcal{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ satisfies assumptions: 1) $\mathcal{F}$ is right continuous, i.e. $\mathcal{F}_t = \bigcap\limits_{s>t} \mathcal{F}_s$; 2)$\mathcal{F}_0$ contains all $P-$null sets in F. On $(\Omega, F, \mathcal{F}, \mathcal{P})$, define a financial market as below: Assume S is a local bounded $d-$ dimensional semi-martingale. With S we denote the movement of d risky assets. Also assume there is a risk-less asset denoted by B. For simplicity we assume $B \equiv 1$ (i.e. S is the discounted asset price). Assume market participants's investment behaviors are based on their valid market information. We denote by $\mathcal{F}_t$ the valid market information that general investors know up to t. Assume in the above market, besides general information investors there are another investors who know less market information than general investors. We call them restricted information investors or incomplete information investors. More explicitly, we assume that the restricted information investors only acquire market information denoted by minor $\sigma-$ filtration $\mathcal{G}$ rather than $\mathcal{F}$. where

$$\mathcal{G} = (\mathcal{G}_t)_{0\leq t\leq T}, \ \mathcal{F}_t^S \subset \mathcal{G}_t \subset \mathcal{F}_t,$$

$\mathcal{F}^S = (\mathcal{F}_t^S)_{0\leq t\leq T}$ denotes the $\sigma-$ filtration generated by price processes S . The market for restricted information investors,i.e. the market with information set $\mathcal{G}$, is called restricted information markets.

Remark 1. Obviously, the assumption for $\mathcal{G}$ is reasonable. On the one hand, there are some investors who cannot achieve all the market information $\mathcal{F}$ actually and only know the restricted information $\mathcal{G}_t \subset \mathcal{F}_t$. On the other hand, if the investor don't know the past price information, he wouldn't throw his money . So the assumption $\mathcal{F}_t^S \subset \mathcal{G}_t$ is also reasonable.

3. Martingale measures for markets with different information

It is well known that market information set is an element of finance markets. Finance markets vary with the market information set $\mathcal{F}$. Obviously, the problem that market information has what influence to the market completeness and arbitrage is worth studying. In the section we will discuss it. For $H \in \{\mathcal{F}, \mathcal{G}\}$, we recall from Grorud&Pontier[1], Pham[6] some definitions and notations.

Definition 1. A probability measure Q is called H-equivalent martingale(local martingale) measure for S, if S is a (H, Q) martingale(local martingale), and $\frac{dQ}{dP} \in L^2(P, H_T)$. Define

$M_e^2(P, H) = \{Q \sim P, \frac{dQ}{dP} \in L^2(P, H_T),\ S \text{ is a } (H, Q) \text{ martingale}\}$

$M_{loc}^2(P, H) = \{Q \sim P, \frac{dQ}{dP} \in L^2(P, H_T),\ S \text{ is a } (H, Q) \text{ local martingale}\}$

denote equivalent martingale(local martingale) measure set for S respectively.

Definition 2. Let $R \in M_{loc}^2(P, \mathcal{F})$. If for every $\mathbf{H} \in L^2(\Omega, \mathcal{F}_T, P)$, there exists a $\mathcal{F}_0-$ random variable a and a portfolio $\mathcal{F}-$predictable $\vartheta, \vartheta \in L^2(\Omega \times [0, T], dR \times d[S, S])$, such that

$$(C_1): \qquad \mathbf{H} = a + \sum_{i=1}^{d} \int_0^T \vartheta^i dS^i,$$

then the market is complete for general information investors. Also, we can define market completeness for the restricted information market.

Definition 3. Let $R \in M_e^2(P, \mathcal{G})$, If for every $\mathbf{H} \in L^2(\Omega, \mathcal{G}_T, P)$, there exists a $\mathcal{G}_0-$random variable a and a portfolio $\mathcal{G}-$predictable $\vartheta, \vartheta \in L^2(\Omega \times [0, T], dR \times d[S, S])$, such that

$$(C_2): \qquad \mathbf{H} = a + \sum_{i=1}^{d} \int_0^T \vartheta^i dS^i,$$

then the market is complete for restricted information investors.

Remark 2. By the capital assets pricing basic theorem, we know: 1) If H_0 is trivial and $M_{loc}^2(P, H)$ is singleton, the associated market is complete.
2) If $M_{loc}^2(P, H)$ is non-empty there is no arbitrage in the associated market; the converse result is false. But $M_{loc}^2(P, H)$ is non-empty is equivalent to a weaker property:the "no free lunch with vanish risk".

The theorem below gives the relation of arbitrage between the markets with different information.

Theorem 4. *If $M_e^2(P, \mathcal{F}) \neq \emptyset$, then $M_e^2(P, \mathcal{G}) \neq \emptyset$.*

Proof. Since $M_e^2(P, \mathcal{F}) \neq \emptyset$, let $Q \in M_e^2(P, \mathcal{F})$, then we know by definition $\frac{dQ}{dP} \in L^2(P, \mathcal{F}_T)$, S is $(\mathcal{F}, Q)$ martingale. Let $Z_T = \frac{dQ}{dP}$, $\quad Z_t = E(Z_T/\mathcal{F}_t)$, then $Z = (Z_t)_{0 \leq t \leq T}$ is a strictly positive square integrable martingale on $(P, \mathcal{F})$. Let $\zeta_t = E[Z_T/\mathcal{G}_t]$, obviously $\zeta_t \in \mathcal{G}_t$ and

$$E[\zeta_t/\mathcal{G}_s] = E[E[Z_t/\mathcal{G}_t]/\mathcal{G}_s] = E[Z_t/\mathcal{G}_s] = E[E[Z_t/\mathcal{F}_s]/\mathcal{G}_s] = E[Z_s/\mathcal{G}_s] = \zeta_s.$$

So $\zeta = (\zeta_t)_{0 \le t \le T}$ is $(P, \mathcal{G})$ martingale. Thus we can define a probability measure $\tilde{Q}$ as below

$$\frac{d\tilde{Q}}{dP} = \zeta_T. \tag{1}$$

Next, we prove $\tilde{Q} \in M_e^2(P, \mathcal{G})$. By definition we only prove three points as below

1) $\tilde{Q} \sim P$. Since $Q \sim P$, from the definition of $\tilde{Q}$, $\tilde{Q} \sim P$ holds obviously.

2) $\frac{d\tilde{Q}}{dP} \in L^2(P, \mathcal{G})$. In fact, only note that $\zeta_T = E[Z_T/\mathcal{G}_T]$, $\zeta_T \in \mathcal{G}_T$, then we have

$$E(\frac{d\tilde{Q}}{dP})^2 = E[E(Z_T/\mathcal{G}_T)]^2 \le E[E(Z_T^2/\mathcal{G}_T)] = EZ_T^2 < \infty.$$

$\frac{d\tilde{Q}}{dP} \in L^2(P, \mathcal{G})$ is proved.

3) S is a $(\tilde{Q}, \mathcal{G})$ martingale. By Protter[17], If $Q \sim P, N_t = E[\frac{dQ}{dP}/H_t]$, then S is (Q, H) martingale if and only if SN is (P, H) martingale. Thus

$$\begin{aligned} E[S_t\zeta_t/\mathcal{G}_s] &= E[S_t E(Z_t/\mathcal{G}_t)/\mathcal{G}_s] = E[E(S_t Z_t/\mathcal{G}_t)/\mathcal{G}_s] \\ &= E[S_t Z_t/\mathcal{G}_s] = E[E(S_t Z_t/\mathcal{F}_s)/\mathcal{G}_s] \\ &= E[S_s Z_s/\mathcal{G}_s] = S_s E[Z_s/\mathcal{G}_s] = S_s\zeta_s. \end{aligned}$$

So $S\zeta$ is a martingale, and S is a $(\tilde{Q}, \mathcal{G})$ martingale. □

Remark 3. 1) General speaking, because $\mathcal{F}-$ stopping times aren't always $\mathcal{G}-$ stopping times, $M_{loc}^2(P, \mathcal{F}) \ne \emptyset$ doesn't imply $M_{loc}^2(P, \mathcal{G}) \ne \emptyset$.

2) Under the condition that S is a bounded semi-martingale, because $M_e^2(P, H) = M_{loc}^2(P, H)$, the non-arbitrage market for general information investor is also non-arbitrage for restricted information investor. That is a result in accord with the market fact that the investors with more information can easy gain more arbitrage opportunities than those with less information. The converse result is false certainly.

For the further discussion , we recall from Schweizer[15] the definition of minimal martingale measure.

Let price process S is a P semi-martingale with canonical decomposition $S_t = S_0 + M_t + A_t$, M is a local martingale, A is a predictable finite variation process.

Definition 5. If S satisfies structure condition (SC): $S = S_0 + M + \lambda d\langle M, M\rangle$. Moreover, $\hat{Z} = \varepsilon(-\int \lambda dM)$ is a $P-$ martingale, then call $\hat{P}$ defined by $\hat{Z}_T = \frac{d\hat{P}}{dP}$ minimal signal martingale measure for S, H-minimal martingale measure if in addition $\hat{P} \in M_e^2(P, H)$.

Remark 4. If $H-$minimal martingale measure for S exists, then it is unique. For discussing the relation of minimal martingale measure between markets with different information, we introduce a property which is an equivalent definition of minimal martingale measure in essence(see for detail in Pham[5]).

Lemma 6. *An equivalent martingale $\hat{P}$ is a minimal martingale measure for S if and only if any square integrable martingale under P and orthogonal to M remains a martingale under $\hat{P}$.*

Theorem 7. *If every $\mathcal{G}$ martingale is also a $\mathcal{F}$ martingale, $\hat{P}^{\mathcal{F}}$ is a $\mathcal{F}-$ minimal martingale measure, Z is $\hat{P}^{\mathcal{F}}$'s density process with respect to P , then $\hat{P}^{\mathcal{G}}$ defined by (1) is also a $\mathcal{G}-$ minimal martingale measure.*

Proof. Let $S = S_0+M+A$, K is a square integrable $(P, \mathcal{G})$ martingale orthogonal to M. Thus K is also a square integrable $(P, \mathcal{F})$ martingale. Using lemma 6, we only prove K is a $(\hat{P}^{\mathcal{G}}, \mathcal{G})$ martingale.
By the property of conditional expectation and the definition of martingale, $\forall\ t > s$

$$\begin{aligned} E[K_t\zeta_t/\mathcal{G}_s] &= E[K_tE(Z_t/\mathcal{G}_t)/\mathcal{G}_s] = E[E(K_tZ_t/\mathcal{G}_t)/\mathcal{G}_s] \\ &= E[K_tZ_t/\mathcal{G}_s] = E[E(K_tZ_t/\mathcal{F}_s)/\mathcal{G}_s] \\ &= E[K_sZ_s/\mathcal{G}_s] = K_sE[Z_s/\mathcal{G}_s] = K_s\zeta_s. \end{aligned}$$

So $K\zeta$ is $P-$ martingale, By lemma in Protter[17,pp.109], K is $(\hat{P}^{\mathcal{G}}, \mathcal{G})$ martingale. Theorem 7 follows immediately from Lemma 6. □

Obviously, from the definitions of completeness, we know that under the condition $\mathcal{F}_T = \mathcal{G}_T$, if restricted information market is complete, then general information market is also complete. Generally speaking, there isn't close relation of completeness between markets with different information. But there really are a link among all the equivalent martingale in $M^2_{loc}(P, \mathcal{G})$.

Theorem 8. *In case of a complete market for the restricted information investor (i.e. verifying (C_2)) such that exists $Q \in M^2_{loc}(P, \mathcal{G})$ for which the discounted prices S are $(Q, \mathcal{F}^S)-$martingales, then every $R \in M^2_{loc}(P, \mathcal{G})$ is equal to $f \cdot Q$, where $f \in L^1(\mathcal{F}^S_0, Q)$.*

Proof. $\forall R \in M^2_{loc}(P, \mathcal{G})$, let $\frac{dR}{dQ} = Z_T$, $Z_t = E_Q[Z_T/\mathcal{F}^S_t]$, So we only prove $Z_T \in L^1((\mathcal{F}^S_0, Q)$. Let $Q \in M^2_{loc}(P, \mathcal{G})$, but S is a R-local martingale, thus S^iZ is a $Q-$local martingale, By Itô formula, $[S^i, Z] = S^iZ - \int S^i_- dZ - \int Z_- dS^i$. Note that S^iZ is a $Q-$ local martingale, we have get $[S^i, Z] = 0$, thus Z is orthogonal to price processes S. Since the market is complete for the restricted information investor, there exist a $\mathcal{G}_0-$random variable a and a $\mathcal{G}_0-$predictable portfolio $\varphi \in L^2(\Omega \times [0, T], dR \otimes [S])$ such that $Z_T = a + \sum_{i=i}^{d} \int_0^T \varphi^i dS^i$.
Because S^i is a $(\mathcal{F}^S, Q)-$ martingale, $\varphi \cdot S$ is a $(\mathcal{F}^S, Q)-$ martingale. Thus $Z_t - Z_0 = \sum_{i=i}^{d} \int_0^t \varphi^i dS^i$ is strongly orthogonal to the stable space generalized by prices, then it is orthogonal to itself; therefore $Z_t - Z_0 = 0$ and Z_T is a $\mathcal{F}^S_0-$ measurable random variable. □

Remark 5. By martingale pricing method, the price of a contingent claim is its expected payoff under a special equivalent martingale measure. Because there are many equivalent martingale measures in an incomplete market, there are many non-arbitrage price for a contingent claim. Under an explicit market, we can deduce the link among many arbitrage-less prices using theorem 8.

4. A jump-diffusion model for restricted information Market

In the section we presents an example of the restricted information market. Consider the market with only two tradeable assets. Assume the risk asset price process S is composed of two components. one is simulated by a Brownian motion which fluctuates continuously, another is modelled by a stochastic point process which only jumps in discrete times $T_1 < T_2 <, \cdots, < T_n < T$. T_n denotes the instant when $n-$th jump on risk asset price occurs for great events. More explicitly, assume S is the unique solution of the stochastic equation below.

$$\begin{cases} dS_t = S_{t-}(u(t, S_{t-})dW_t + dR_t), \\ R_t = \sum_{n=1}^{N_t} Z_n, \end{cases} \tag{2}$$

where N is a doubly stochastic Poisson process, $u(t, S_t)$ is a bounded function, (Z_n) is a iid-sequence of random variables, Z_n denotes the return ratio of investors for the $n-$th great event. Besides this we shall work under a technical assumption as below:

Consider a adapted Markovian process X and a continuous function λ : $[0, T] \times R \to R$, taking its values in the interval $[\lambda_1, \lambda_2]$, $0 < \lambda_1 < \lambda_2 < \infty$, λ_1, λ_2 are constants. Besides $\{\mathcal{F}_t\}, \{\mathcal{F}_t^S\}$,we also consider natural filtration $\{\mathcal{F}_t^N\}$ generated by N and $\{\mathcal{H}_t\}$: $\mathcal{H}_t = \mathcal{F}_t \vee \sigma(X_s, 0 \le s \le T)$. Obviously $\mathcal{H}_0$ contains all the future information of X. Moreover make the following assumptions :

Assumption A) N is a doubly stochastic Poisson with driving process X and intensity function $\lambda(t, X)$,i.e. N is a point process which admits the $(P, \mathcal{H})-$ intensity $\lambda(t, X_t)$, specially by point process theory, $N_t - \int_0^t \lambda(s, X_s)ds$ is $(P, \mathcal{H})-$martingale. Obviously, assumption **A)** demonstrates that given information about all the future of the state variable, N is a Poisson process with conditionally deterministic intensity $\lambda(t, X_t)$. Additional, assumption **A)** has rational economical meanings: $\lambda(t, X_t)$ corresponds to the rate at which mew economic information is absorbed by the agents active on the market for the stock; it makes perfect sense to assume that this rate is influenced by some exogenous stochastic factor. Generally speaking, market absorptivity is affected by season so we take the time-dependency in $\lambda(t, X_t)$.

Assumption B) Let W^2 is a Brownian motion orthogonal to W^1 , α and β are continuous functions, X is the unique solution of the stochastic differential equation

$$dX_t = \alpha(X_t)dt + \beta(X_t)dW_t^2.$$

Assumption C) (Z_n) is a sequence of independent, identically $\nu-$distribute random variables, (Z_n) is independent of N, X and satisfy:

$$\nu(-1,\infty)=1;\ \int_{-\infty}^{\infty} z\nu(dz)=0; \sigma^2=\int_{-\infty}^{\infty} z^2\nu(dz)<\infty.$$

Obviously we can refer to $\{\Omega, P, \mathcal{F}^S\}$(that is, investors only get market information $\mathcal{F}^S$)as a restricted information market.

5. Risk-minimizing hedging strategies under complete information

We are now in position to discuss the problem of hedging claims under the above model. To the end, we start with considering the character of price process S.

Theorem 9. *S is a square integrable $P-$martingale.*

Proof. By stochastic exponential formula:

$$\begin{aligned} S_t &= S_0 e^{(\sum_{n=1}^{N_t} Z_n)} \Pi_{n=1}^{N_t}(1+Z_n)e^{(-Z_n)} exp(\int_0^t u(s,S_s)dW_s^1 - \tfrac{1}{2}\int_0^t u(s,S_s)^2 ds) \\ &= S_0 \Pi_{n=1}^{N_t}(1+Z_n) exp(\int_0^t u(s,S_s)dW_s^1 - \tfrac{1}{2}\int_0^t u(s,S_s)^2 ds) \\ &= S_0 exp\left(\sum_{n=1}^{N_t} \log(1+Z_n)\right) exp(\int_0^t u(s,S_s)dW_s^1 - \tfrac{1}{2}\int_0^t u(s,S_s)^2 ds). \end{aligned}$$

Denote By P^R the stochastic measure in $[0,T]\times R$ generated by pure jump process R satisfying: for every function $W:\Omega\times[0,T]\times R\to R$, then

$$\int_0^t \int_{-\infty}^{+\infty} W(\omega;s,z)P^R(ds\times dz)=\sum_{n=1}^{\infty} W(\omega;T_n(\omega),z_n(\omega))1_{\{T_n(\omega)\le t\}}. \tag{3}$$

By definition: R has the $(P,\mathcal{F}_t)-$ local characteristics $(\lambda(t,X_t),\nu(dz))$, i.e., defining the signed random measure q^R:

$$q^R(dt\times dz)=P^R(dt\times dz)-\nu(dz)\lambda(t,X_t)dt. \tag{4}$$

So for every predictable W: $\Omega\times[0,T]\times R\to R$ with finite integral

$$\int_0^T \int_{-\infty}^{+\infty} |W(\omega;t,z)|\nu(dz)\lambda(t,X_t)dt,$$

we have that

$$\int_0^t \int_{-\infty}^{+\infty} W(\omega;s,z)P^R(ds\times dz) \text{ is a } P- \text{ local martingale.}$$

Then

$$\begin{aligned} R_t &= \int_0^t \int_{-\infty}^{+\infty} zP^R(ds\times dz)=\int_0^t \int_{-\infty}^{+\infty} z\left(q^R(dt\times dz)-\nu(dz)\lambda(t,X_t)dt\right) \\ &= \int_0^t \int_{-\infty}^{+\infty} zq^R(dt\times dz), \end{aligned}$$

the final equality deriving from assumption **C)**. So R is a $P-$local martingale. For proving the fact that S is a square integrable martingale, we next calculate

the quadratic variation of R and S .
Since R is a pure jump process, by *Itô* formula we have:

$$R_t^2 = 2\int_0^t R_{s\_}dR_s + \sum_{n=1}^{N_t}(Z_n)^2.$$

But $\int_0^\cdot R_{s\_}dR_s$ is a local martingale and moreover by definition of P^R, we have:

$$\begin{aligned}\sum_{n=1}^{N_t} {Z_n}^2 &= \int_0^t\int_{-\infty}^{+\infty} z^2 P^R(ds\times dz)\\ &= \int_0^t\int_{-\infty}^{+\infty} z^2 q^R(ds\times dz) + \int_0^t\int_{-\infty}^{+\infty} z^2\nu(dz)\lambda(t,X_t)dt\\ &= \int_0^t\int_{-\infty}^{+\infty} z^2 q^R(ds\times dz) + \int_0^t \sigma^2\lambda(s,X_s)ds.\end{aligned}$$

Then $R^2 - \int_0^\cdot \sigma^2\lambda(s,X_s)ds = 2\int_0^\cdot R_{s\_}dR_s + \int_0^\cdot\int_{-\infty}^{+\infty} z^2q^R(ds\times dz)$ is a $P-$ local martingale. Thus $[R,R]-\int_0^\cdot \sigma^2\lambda(s,X_s)ds$ is a local martingale. By the definition of conditional variation

$$\langle R,R\rangle_t = \int_0^t \sigma^2\lambda(s,X_s)ds;\ \langle S,S\rangle_t = \int_0^t \sigma^2\lambda(s,X_s)S_{s-}^2 + S_{s-}^2u^2(s,S_{s-})ds. \tag{5}$$

Finally we prove S is a square integrable martingale. By the definition of $\{\mathcal{G}_t\}$

$$\begin{aligned} E[S_t^2] &= E\left[E[S_t^2/\mathcal{H}_0]\right] = \bar{L}_t E[K_t],\\ \bar{L}_t &\hat{=} S_0^2\sum_{n=0}^{\infty} E\left[\exp\left(2\sum_{i=1}^{n}\log(1+Z_n)\right)\right]P[N_t=n/\mathcal{H}_0],\\ K_t &\hat{=} \exp(2\textstyle\int_0^t u(s,S_s)dW_s^1 - \int_0^t u^2(s,S_s)ds).\end{aligned}$$

Because $(Z_n)_n$ is a iid-sequence, we get

$$E\left[\exp\left(2\sum_{i=1}^{n}\log(1+Z_n)\right)\right] = \prod_{i=1}^{n}E[(1+Z)^2] = (1+\sigma^2)^n.$$

For every $0\le t<\bar{T}\le T$, define $\Lambda(t,\bar{T})$: $\Lambda(t,\bar{T})\hat{=}\int_t^{\bar{T}}\lambda(s,X_s)ds$, then

$$\bar{L}_t = S_0^2\sum_{n=0}^{\infty}\frac{(1+\sigma^2)^n\Lambda(0,t)^n}{n!\exp(\Lambda(0,t))} = S_0^2\exp(\sigma^2\Lambda(0,t)).$$

But $\lambda\in[\lambda_1,\lambda_2]$, $\bar{L}_t \le {S_0}^2\exp(\sigma^2\lambda_2 T) < \infty$. Obviously $E[K_t]<\infty$, thus $E{S_t}^2<\infty$, the conditional quadric variation of S satisfies

$$E[\langle S,S\rangle_t] = E\left[\int_0^t \sigma^2\lambda(s,X_s)S_{s-}^2 + S_{s-}^2u^2(s,S_{s-})ds\right] \le C\int_0^t E[S_s^2]ds < \infty.$$

Thus S is a P-square integrable martingale. $\square$

Next We will discuss the risk-minimizing strategies of $T-$ claims $H(S_T)$ under complete information $\mathcal{F}$.
Because of the Markovian-property of S and X, we can define:

$$h(t,S_t,X_t)\hat{=}E[H(S_T)/\mathcal{F}_t].$$

For simplifying mathematical disposal, as usual, we stipulate that h is $\mathcal{C}^{1,2,2}$ continuous function. Assume the integro-differential equation below has unique solution.

$$\left\{\begin{array}{l} h(T,S_T,X_T)=H(S_T), \\ h_t+\alpha h_x+\frac{1}{2}\beta^2 h_{xx}+\frac{1}{2}S_t^2u^2(t,S_t)h_{ss} \\ \quad +\lambda(t,X)\int_{-\infty}^{+\infty}[h(t,S_t(1+z),X_t)-h(t,S_t,X_t)]\nu(dz)=0. \end{array}\right. \tag{6}$$

For argument in depth, we recall a result derived from[4].

Lemma 10. *If the K-W decomposition of $H\in L^2(P,\mathcal{F})$ is: $H=E[H]+\int_0^T\vartheta_s^H dS_s+L_T^H$, then its $\mathcal{F}-$ risk-minimizing hedging strategy is given by $\vartheta_t=\vartheta_t^H$; $V_t=E[H/\mathcal{F}_t]$.*
If we let $H_t=E[H/\mathcal{F}_t]$, then

$$\vartheta_t=\frac{d\langle H,S\rangle_t}{d\langle S,S\rangle_t}. \tag{7}$$

Theorem 11. *Let h is the unique solution of the integro-differential equation (6), then for every contingent claim $H(S_T)\in L^2(P)$, there exists unique $\mathcal{F}-$risk-minimizing hedging strategy $\varphi=(\widetilde{\vartheta},\widetilde{V})$ satisfying*

$$\widetilde{\vartheta}_t=\frac{d\langle H,S\rangle_t}{d\langle S,S\rangle_t}=\frac{f(t,S_{t\_},X_t)+S_{t\_}u^2(t,S_{t\_})h_t}{\sigma^2\lambda(t,X_t)S_{t-}+S_{t-}u^2(t,S_{t-})}, \tag{8}$$

$$\widetilde{V}_t=E[H(S_T)/\mathcal{F}_t]=h(t,S_t,X_t), \tag{9}$$

where

$$f(t,S_{t\_},X_t)=\int_0^t\int_{-\infty}^{+\infty}[h(s,S_{s-}(1+z),X_s)-h(s,S_{s-},X_s)]z\nu(dz)\lambda(s,X_s)ds.$$

Proof. Since $E[H(S_T)/\mathcal{F}_t]\hat{=}h(t,S_t,X_t)$, h is a $\mathcal{C}^{1,2,2}$ continuous function, by *Itô* formula we have

$$\begin{aligned} h(t,S_t,X_t) &= h(0,S_0,X_0))+\int_0^t h_t ds+\int_0^t h_x dX_s+\int_0^t h_s dS_s \\ &\quad +\frac{1}{2}\int_0^t h_{ss}d[S,S]_s^c+\frac{1}{2}\int_0^t h_{xx}d[X,X]_s+\int_0^t h_{sx}d[S,X]_s^c \\ &\quad +\sum_{0\le s\le t}\{h(s,S_s,X_s)-h(s,S_{s\_},X_s)-h_s(S,S_{s\_},X_s)\Delta S_s\} \\ &= h(0,S_0X_0)+\int_0^t h_s S_{s-}u(s,S_{s\_})dW_s^1+\int_0^t\beta dW_s^2 \\ &\quad +\int_0^t[h_t+\alpha h_x+\frac{1}{2}\beta^2h_{xx}+\frac{1}{2}S^2u^2(s,S_{s-})h_{ss}ds \\ &\quad +\int_0^t\int_{-\infty}^{+\infty}[h(s,S_{s-}(1+z),X_s)-h(s,S_{s-},X_s)]P^R(ds\times dz). \end{aligned}$$

Using the equality (4), then

$$\begin{aligned}&\int_0^t\int_{-\infty}^{+\infty}[h(s,S_{s-}(1+z),X_s)-h(s,S_{s-},X_s)]P^R(ds\times dz)\\&=\int_0^t\int_{-\infty}^{+\infty}[h(s,S_{s-}(1+z),X_s)-h(s,S_{s-},X_s)]q^R(ds\times dz)\\&\quad+\int_0^t\int_{-\infty}^{+\infty}[h(s,S_{s-}(1+z),X_s)-h(s,S_{s-},X_s)]\nu(dz)\lambda(s,X_s)ds. \qquad (10)\end{aligned}$$

By definition $h(t,S_t,X_t)$ is a martingale, so the predictable finite variance component is 0. Thus $h(t,S_t,X_t)$ satisfies integro-differential equation (6) and

$$\begin{aligned}h(t,S_t,X_t) &= h(0,S_0,X_0)) + \int_0^t h_s S_{s-}u(s,S_{s-})dW_s^1 + \int_0^t \beta dW_s^2\\&\quad+\int_0^t\int_{-\infty}^{+\infty}[h(s,S_{s-}(1+z),X_s)-h(s,S_{s-},X_s)]q^R(ds\times dz),\end{aligned}$$

$$\begin{aligned}\langle H,W^1\rangle_t &= \langle h(0,S_0,X_0)) + \int_0^t h_s S_{s-}u(s,S_{s-})dW_s^1 + \int_0^t \beta dW_s^2, W^1\rangle\\&\quad+\langle\int_0^t\int_{-\infty}^{+\infty}[h(s,S_{s-}(1+z),X_s)-h(s,S_{s-},X_s)]q^R(ds\times dz),W^1\rangle\\&=\int_0^t h_s S_{s-}u(s,S_{s-})ds.\end{aligned}$$

Below we calculate $\langle H,R\rangle_t$. It is well known that if $g_(t,\omega,z), g_2(t,\omega,z)$ is $\{\mathcal{F}_t\}$−predictable, then

$$\begin{aligned}&\langle\int_0^\cdot\int_{-\infty}^{+\infty} g_1(s,\omega,z)q^R(ds\times dz), \int_0^\cdot\int_{-\infty}^{+\infty} g_2(s,\omega,z)q^R(ds\times dz)\rangle\\&=\int_0^\cdot\int_{-\infty}^{+\infty} g_1(s,\omega,z)g_2(s,\omega,z)\nu(dz)\lambda(s,X_s)ds. \qquad (11)\end{aligned}$$

Thus

$$\begin{aligned}\langle H,R\rangle_t &= \langle h(0,S_0,X_0)) + \int_0^t h_s S_{s-}u(s,S_{s-})dW_s^1 + \int_0^t \beta dW_s^2,\\&\quad\int_0^t\int_{-\infty}^{+\infty} zq^R(ds\times dz)\rangle + \langle\int_0^t\int_{-\infty}^{+\infty}[h(s,S_{s-}(1+z),X_s)\\&\quad-h(s,S_{s-},X_s)]q^R(ds\times dz), \int_0^t\int_{-\infty}^{+\infty} zq^R(ds\times dz)\rangle\\&=\int_0^t\int_{-\infty}^{+\infty}[h(s,S_{s-}(1+z),X_s)-h(s,S_{s-},X_s)]z\nu(dz)\lambda(s,X_s)ds.\end{aligned}$$

Recall $\langle S,S\rangle_t = \int_0^t \sigma^2\lambda(s,X_s)S_{s-}^2 + S_{s-}^2u^2(s,S_{s-})ds$, so by Lemma 2, Theorem 3 is proved. □

6. Risk-minimizing hedging strategies under restricted information

In the section we assume that the investor only know the past price information $\mathcal{F}^S$. What is his risk-minimizing hedging strategy?
For every stochastic process U on $(\Omega,\mathcal{F},\mathcal{P})$, denote by ${}^{(o,\mathcal{F}^S)}U$ (${}^{(p,\mathcal{F}^S)}U$; $U^{p,\mathcal{F}^S}$) its optional (predictable; predictable dual) projection on filtration $\{\mathcal{F}^S\}$. Next we introduce a result derived from [13].

Lemma 12. *If $\mathcal{F}$−the risk-minimizing strategy for $\mathcal{F}^S$−measurable contingent claim H is $\varphi=(\vartheta,V)$, then its $\mathcal{F}^S$-risk-minimizing strategy $\hat\varphi=(\hat\vartheta,\hat V)$ is given by*

$$\hat\vartheta_t = \frac{d(\int\vartheta d\langle S,S\rangle)_t^{p,\mathcal{F}^S}}{d\langle S,S\rangle_t^{p,\mathcal{F}^S}} \quad;\quad \hat V_t = E[V_t/\mathcal{F}_t^S]. \qquad (12)$$

Moreover, if $\langle S,S\rangle$ is $\mathcal{F}^S$-predictable, and $\langle S,S\rangle$ has density v with respect to Lebesgue measure. then $\hat\vartheta_t = \frac{{}^{(p,\mathcal{F}^S)}(v_t\vartheta_t)}{{}^{(p,\mathcal{F}^S)}(v_t)}$.

Theorem 13. *Let $H(S_T) \in L^2(P)$, and its $\mathcal{F}$-risk-minimizing strategy is $\varphi = (\vartheta, V)$, then its $\mathcal{F}^S$ risk-minimizing strategy $\hat{\varphi} = (\hat{\vartheta}, \hat{V})$ is given by*

$$\hat{\vartheta} = S_{t_-}^{-1} \frac{\left(E[f(t, S_{t_-}, X_t) + S_{t_-} u^2(t, S_{t_-}) h_t / \mathcal{F}_{t_-}^S]\right)^-}{\left(E[\sigma^2 \lambda(t, X_t) + u^2(t, S_{t_-}) / \mathcal{F}_{t_-}^S]\right)^-}; \quad \hat{V}_t = E[h(t, S_t, X_t)/\mathcal{F}_t^S].$$

where process $(\cdot)^-$ denotes the left continuous version of process $(\cdot)$

Proof. By [3], wealth process $\hat{V}_t = E[H(S_T)/\mathcal{F}_t^S]$. Using property of conditional expectation, we have $\hat{V}_t = E[E[H(S_T)/\mathcal{F}_t]/\mathcal{F}_t^S] = E[h(t, S_t, X_t)/\mathcal{F}_t^S]$
Next we will find $\hat{\vartheta}$. Denote by v the density of $d\langle S, S\rangle$ with respect to Lebesgue measure, i.e. $v_t = \sigma^2 \lambda(t, X_t) S_t^2 + S_t^2 u(s, S_t)$. From Lemma 12,

$$\hat{\vartheta} = \frac{d(\int \vartheta d\langle S, S\rangle)_t^{p, \mathcal{F}^S}}{d\langle S, S\rangle_t^{p, \mathcal{F}^S}} = \frac{{}^{(p, \mathcal{F}^S)}(v_t \vartheta_t)}{{}^{(p, \mathcal{F}^S)}(v_t)}. \tag{13}$$

It remains to determine the predictable projections in (12). Define for a continuous function $g : [0, T] \times R^+ \times R \to R$ the process $U_t^g \hat{=} g(t, S_{t_-}, X_t)$. First calculate ${}^{(p, \mathcal{F}^S)} U_t^g$, $0 \le t \le T$. Consider conditional expectation $E[U_t^g / \mathcal{F}_t^S]$. It is well known that there is a right-continuous Y which is a version of this conditional expectation and which is continuous except at the jump-times T_n of the process N. By [8] Y is indistinguishable from ${}^{(o, \mathcal{F}^S)} U_t^g$. Denote by $\left({}^{(o, \mathcal{F}^S)} U_t^g\right)^-$ the left-continuous version of this process. Since N is quasi left-continuous, for every predictable stopping time τ, $\Delta N_\tau = 0$, then for every predictable stopping time τ, we have ${}^{(o, \mathcal{F}^S)} U_\tau^g = \left({}^{(o, \mathcal{F}^S)} U^g\right)_\tau^-$.
By definition of optional prediction, for every $\{\mathcal{F}_t^S\}$ stopping time τ, we have $E\left[U_\tau^g 1_{\{\tau < \infty\}}\right] = E\left[{}^{(o, \mathcal{F}^S)} U_\tau^g 1_{\{\tau < \infty\}}\right]$.
Together two equalities above, then for every predictable stopping time τ

$$E\left[U_\tau^g 1_{\{\tau < \infty\}}\right] = E\left[\left({}^{(o, \mathcal{F}^S)} U_\tau^g\right)^- 1_{\{\tau < \infty\}}\right].$$

Since $\left({}^{(o, \mathcal{F}^S)} U_\tau^g\right)^-$ is $\{\mathcal{F}_t^S\}$ adapted and left-continuous, then it is $\{\mathcal{F}_t^S\}$ predictable. Hence $\left({}^{(o, \mathcal{F}^S)} U_\tau^g\right)^-$ is indistinguishable from the predictable projection ${}^{(p, \mathcal{F}^S)} U_\tau^g$. So from(12), we have

$$\hat{\vartheta} = \frac{\left(E[S_{t_-} f(t, S_{t_-}, X_t) + S_{t_-}^2 u^2(t, S_{t_-}) h_t / \mathcal{F}_{t_-}^S]\right)^-}{\left(E[\sigma^2 \lambda(t, X_t) S_{t_-}^2 + S_{t_-}^2 u^2(t, S_{t-}) / \mathcal{F}_{t-}^S\right)^-}$$

$$= S_{t_-}^{-1} \frac{\left(E[f(t, S_{t_-}, X_t) + S_{t_-} u^2(t, S_{t_-}) h_t / \mathcal{F}_{t_-}^S]\right)^-}{\left(E[\sigma^2 \lambda(t, X_t) + u^2(t, S_{t_-}) / \mathcal{F}_{t_-}^S]\right)^-}. \qquad \square$$

Remark 6. Using non-linear filtration theory, we can by computing conditional expectation $E[g(t, X, S)/\mathcal{F}_t^S]$ get the optimal strategy(see for detail in [20]).

7. Conclusions

The paper discusses an imperfect market with restricted information. Based on constructing restricted markets and martingale theory, we strictly prove the result that non-arbitrage market for general information investors is also non-arbitrage for restricted information investors. Also the explicit relation of minimal martingale measure between two different information markets is given, and a link among all equivalent martingale measures in the class of restricted local martingale measures, which is important in pricing contingent claims, is derived. As an example of restricted information markets, we construct a jump-diffusion model for risky assets and give a risk-minimizing optimal portfolio strategy to a investor. Although our method is standard stochastic analysis approach, the model is new and suitable for describing real asset price processes. the results are also new and the explicit optimal strategies , which will contribute to financial investors in their trade, are given.

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