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THE (R,S)-SYMMETRIC SOLUTIONS TO THE LEAST-SQUARES PROBLEM OF MATRIX EQUATION $AXB = C$

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ABSTRACT. For real generalized reflexive matrices R, S, i.e., $R^T = R$, $R^2 = I$, $S^T = S$, $S^2 = I$, we say that real matrix X is (R,S)-symmetric, if $RXS = X$. In this paper, an iterative algorithm is proposed to solve the least-squares problem of matrix equation $AXB = C$ with (R,S)-symmetric X. Furthermore, the optimal approximation solution to given matrix X_0 is also derived by this iterative algorithm. Finally, given numerical example and its convergent curve show that this method is feasible and efficient.

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1. Introduction

Throughout this paper, denoted by $R_k^{m\times n}$ the set of all $m \times n$ real matrices with rank k, and $OR^{m\times n}$ the set of all $n \times n$ real orthogonal matrices. Let the superscripts T and I_n be the transpose and identity matrix with order n, respectively. For matrices $A = (a_1, a_2, \ldots, a_n), B \in R^{m\times n}$, $a_i \in R^m$, $R(A)$ and $tr(A)$ represent its range and trace, respectively. The symbol $vec(\cdot)$ stands for the vec operator, i.e., $vec(A) = (a_1^T, a_2^T, \ldots, a_n^T)^T$. Let $A \otimes B$ be the Kronecker product[1] of matrices A and B. Moreover, $\langle A, B\rangle = tr(B^T A)$ is defined as the inner product of the two matrices, which generates the Frobenius norm, i.e., $\|A\| = \sqrt{< A, A >} = \sqrt{tr(A^T A)}$.

We introduce the following conception (see [2,3] for details).

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Definition 1. For given generalized reflexive matrices $R \in R^{m\times m}$, $S \in R^{n\times n}$, i.e., $R^T = R$, $R^2 = I_m$, $S^T = S$, $S^2 = I_n$, we say that matrix $X \in R^{m\times n}$ is (R,S)-symmetric ((R,S)-skew symmetric), if $RXS = X$ $(RXS = -X)$.

The set of all $m \times n$ (R,S)-symmetric ((R,S)-skew symmetric) matrices with respect to R and S is denoted by $GSR^{m\times n}$ $(GSSR^{m\times n})$.

Let $J = (e_n, e_{n-1}, \cdots, e_1)$, here e_i is the i^{th} column of identity matrix I_n. If $JXJ = X$, we say that $X \in R^{n\times n}$ is centro-symmetric matrix, which has practical applications in information theory, linear system theory, linear estimate theory and numerical analysis (see [4,5]). From Definition 1, it is obvious that (R,S)-symmetric matrix is the extension of the centro-symmetric matrix (when $R = S = J$) and reflective matrix [6,7] (when $R = S$), respectively.

Remark 1. In this paper, let R, S be fixed generalized reflexive matrices as in Definition 1.

For above generalized reflexive matrix R, $R^T = R$ implies that the eigenvalues of R belong to the real field, and the absolute of the eigenvalues equal to 1 because of $R^2 = I$. The matrix S has similar properties to R. Hence, we have the following conclusion.

Lemma 1. *For given generalized reflexive matrices R, S, there exist unitary matrices $U_1 \in OR^{m\times m}$, $U_2 \in OR^{n\times n}$ such that*

$$R = U_1 \begin{pmatrix} I_r & 0 \\ 0 & -I_{m-r} \end{pmatrix} U_1^T, \; S = U_2 \begin{pmatrix} I_l & 0 \\ 0 & -I_{n-l} \end{pmatrix} U_2^T.$$

According to Definition 1 and Lemma 1, we can obtain another result on the (R,S)-symmetric matrix.

Lemma 2. *For matrices R, S in Lemma 1, $X \in GSR^{m\times n}$, then*

$$X = U_1 \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} U_2^T,$$

where $X_1 \in R^{r\times l}$, $X_2 \in R^{(m-r)\times(n-l)}$.

From Lemma 2, we can easily get a (R,S)-symmetric matrix by choosing different X_i, but U_i $(i = 1, 2)$ are fixed in Lemma 1.

The problems to be discussed in this paper can be expressed as follows:

Problem I. *Given matrices $A \in R^{p\times m}$, $B \in R^{n\times q}$, $C \in R^{p\times q}$, find $X \in GSR^{m\times n}$ such that*

$$\|AXB - C\| = \min_{Y\in GSR^{m\times n}} \|AYB - C\|.$$

Problem II. *Given matrix $X_0 \in R^{m\times n}$, find $\hat{X}$ such that*

$$\|\hat{X} - X_0\| = \min_{X\in S_E} \|X - X_0\|,$$

where S_E is the solution set of Problem I.

In fact, Problem I is the least-squares problem of the well-known linear matrix equation

$$AXB = C. \tag{1}$$

or the minimum residual problem of (1) under Frobenius norm. This equation has been widely discussed, such as, Dai [8] and Chu [9] have investigated this matrix equation by using the generalized singular value decomposition (GSVD) [10] of matrix pair, and established the solvability conditions. The existence of the reflexive solution of the matrix equation has been studied by Peng et al. in [6]. In addition, Peng [11] presented an iterative algorithm for finding the symmetric solution of matrix equation (1), which can be terminated within finite iteration steps if the roundoff errors were ignored. Deng et al. [12] proposed iterative orthogonal direction methods for Hermitian minimum norm solutions for the complex matrix equation $AXB = C$ motivated by the conjugate gradient method.

However, the matrices A, B and C are experimentally occurring in practice, they may not satisfy the solvability conditions of the matrix equation, therefore, we always need to consider the associated least-squares problem. For instance, Huang [13] and william [3] have discussed the minimum residual problem for the symmetric matrices set and (R,S)-symmetric matrices set by Moore-Penrose inverse, respectively. By constructing different matrix iterative methods, the general solution, symmetric solution to the least-squares problem of matrix equation (1) have been considered in the references [14-16].

Problem II is so-called the optimal approximation problem, which may arise in many areas of science computing and engineering applications (see [16]). Here, the given matrix X_0 can be obtained by experimental observation or statistical distribution information, but it always does not satisfy the experimental requirements or the minimum residual restrictions, and the matrix $\hat{X}$ satisfies the requirements and is closed to X_0 in Frobenius norm or others. For this problem, we refer to [5-7,11-16, 18-20] and references therein.

As the extension of the centro-symmetric matrix, the (R,S)-symmetric matrix has important theoretical value, which has been discussed by matrix decomposition and general inverse in references [2,3], but the methods and results mentioned there are not convenient to be used in practice. Therefore, it is necessary to establish iterative algorithm and study the associated matrix equation problems. In this paper, we solve Problem I and II by constructing iterative algorithm and obtain their solutions. We should point out that if $R = S = I$, our problems to be discussed are the same as those in [16], but the used method is different. More, the iterative algorithm to be proposed here is similar to but different from those in [12,14,15].

This paper is organized as follows. In section 2, an iterative algorithm will be given to solve Problem I. In section 3, we will study Problem II. In section 4, some numerical examples will be given to illustrate the efficiency of this algorithm.

2. The iterative algorithm for solving Problem I

In this section, we will propose an iterative algorithm to find the solution of Problem I. For any (R,S)-symmetric initial iterative matrix $X_1 \in GSR^{m\times n}$, we will show that a solution of which can be obtained within finite iteration steps.

Definition 2. Assume that matrices $M, N \in R^{p\times m}$, where p, m are arbitrary positive integers, then M, N are called to be orthogonal each other, if $tr(N^T M) = 0$.

If $F \in GSR^{m\times n}$, $G \in GSSR^{m\times n}$, it is easy to verify that $tr(G^T F) = 0$.

As we all know, the least-squares solutions of linear equations can be obtained by its normal equation, which is always consistent. Hence, we have the following assertion.

Lemma 3. *Problem I can be transformed as to find the (R,S)-symmetric solutions of the normal equation of matrix equation $AXB = C$ with $X \in GSR^{m\times n}$, that is,*

$$A^T AXBB^T + RA^T AXBB^T S = A^T CB^T + RA^T CB^T S,\ X \in GSR^{m\times n}. \quad (2)$$

Proof. We first prove that the consistency of matrix equation (1) with $X \in GSR^{m\times n}$ is the same as that of the following matrix equation

$$AXB + ARXSB = 2C,\ X \in R^{m\times n}. \quad (3)$$

In fact, if matrix equation (3) is consistent, and $\bar{X}$ is a solution of which, let $\bar{\bar{X}} = \frac{\bar{X}+R\bar{X}S}{2} \in GSR^{m\times n}$, then

$$A\bar{\bar{X}}B = \tfrac{1}{2}A(\bar{X} + R\bar{X}S)B = \tfrac{1}{2}(AXB + ARXSB) = C,$$

which shows that $\bar{\bar{X}}$ is a solution of matrix equation (1). The contrary is obvious.

Hence, matrix equation (1) and (3) have same (R,S)-symmetric solutions. The Problem I can be transformed into finding the (R,S)-symmetric solution of the normal equation of (3). By using the properties of the Kronecker product and *vec* operator, the normal equation of (3) can be expressed as

$$A^T AXBB^T + A^T ARXSBB^T + RA^T AXBB^T S + RA^T ARXSBB^T S = 2A^T CB^T + 2RA^T CB^T S$$

when $X \in GSR^{m\times n}$, we can obtain equality (2) from above equation. □

In the next part, we will obtain the solution to Problem I by constructing the iterative algorithm of matrix equation (2). The algorithm can be stated as follows.

Algorithm 1.

Step 1: Input generalized reflexive matrices $R \in R^{m\times m}$, $S \in R^{n\times n}$, and $A \in R^{p\times m}$, $B \in R^{n\times q}$, $C \in R^{p\times q}$. Choosing arbitrary $X_1 \in GSR^{m\times n}$.

Step 2: Compute

$\mathtt{R}_1 = \mathtt{A}^T\mathtt{CB}^T + \mathtt{RA}^T\mathtt{CB}^T\mathtt{S}\text{-}\mathtt{A}^T\mathtt{AX}_1\mathtt{BB}^T - \mathtt{RA}^T A\mathtt{X}_1\mathtt{BB}^T\mathtt{S},$

$\mathtt{P}_1 = \mathtt{A}^T\mathtt{FAR}_1\mathtt{BB}^T + \mathtt{RA}^T\mathtt{AR}_1\mathtt{BB}^T\mathtt{S},$

$k := 1.$

Step 3: Compute

$\mathtt{X}_{k+1} = \mathtt{X}_k + \frac{\|\mathtt{R}_k\|^2}{\|\mathtt{P}_k\|^2}\mathtt{P}_k.$

Step 4: Compute

$$\begin{aligned}\mathtt{R}_{k+1} &= \mathtt{A}^T\mathtt{CB}^T + \mathtt{RA}^T\mathtt{CB}^T\mathtt{S} - \mathtt{A}^T\mathtt{AX}_{k+1}\mathtt{BB}^T - \mathtt{RA}^T\mathtt{AX}_{k+1}\mathtt{BB}^T\mathtt{S}\\ &= \mathtt{R}_k - \frac{\|\mathtt{R}_k\|^2}{\|\mathtt{P}_k\|^2}(\mathtt{A}^T\mathtt{AP}_k\mathtt{BB}^T + \mathtt{RA}^T\mathtt{AP}_k\mathtt{BB}^T\mathtt{S}),\end{aligned}$$

$\mathtt{P}_{k+1} = \mathtt{A}^T\mathtt{AR}_{k+1}\mathtt{BB}^T + \mathtt{RA}^T\mathtt{AR}_{k+1}\mathtt{BB}^T\mathtt{S} + \frac{\|\mathtt{R}_{k+1}\|^2}{\|\mathtt{R}_k\|^2}\mathtt{P}_k.$

Step 5: If $\mathtt{R}_k = 0$, stop. Otherwise, $\mathtt{k} := \mathtt{k} + 1$, goto step 3.

Algorithm 1 reveals that $X_i,\ P_i \in GSR^{m\times n}$ $(i = 1, 2, \ldots)$. When $R_k = 0$, we obtain a solution of Problem I. The properties of this algorithm will be given out in the form of lemma.

Lemma 4. *The sequences $\{R_i\}$, $\{P_i\}$ generated by Algorithm 1, are orthogonal sequences in matrix inner product space $GSR^{m\times n}$, i.e.,*

$$tr(R_i^T R_j) = 0,\ \ tr(P_i^T P_j) = 0,\ i, j = 1, 2, \ldots, k\,(k \geq 2),\, i \neq j. \tag{3}$$

Proof. We prove the conclusion by the principle of induction. Since $tr(F^T G) = tr(G^T F)$ for arbitrary matrices F, G with suitable size , it is enough to prove that (3) holds for $i > j$.

When $k = 2$, from Algorithm 1, noting that $R_i,\ P_i \in GSR^{m\times n}$, we have

$$\begin{aligned}tr(R_2^T R_1) &= tr(R_1^T R_1) - \tfrac{\|R_1\|^2}{\|P_1\|^2}tr[(A^T AP_1BB^T + RA^T AP_1BB^T S)^T R_1]\\ &= tr(R_1^T R_1) - \tfrac{\|R_1\|^2}{\|P_1\|^2}tr[P_1^T(A^T AR_1BB^T + A^T ARR_1SBB^T)]\\ &= tr(R_1^T R_1) - \tfrac{\|R_1\|^2}{\|P_1\|^2}tr[P_1^T(A^T AR_1BB^T + RA^T AR_1BB^T S)]\\ &- \tfrac{\|R_1\|^2}{\|P_1\|^2}tr[P_1^T(A^T AR_1BB^T - RA^T AR_1BB^T S)]\\ &= tr(R_1^T R_1) - \tfrac{\|R_1\|^2}{\|P_1\|^2}tr(P_1^T P_1)\\ &= 0\end{aligned} \tag{4}$$

and

$$\begin{aligned}tr(P_2^T P_1) &= tr[(A^T AR_2BB^T + RA^T AR_2BB^T S)^T P_1] + \tfrac{\|R_2\|^2}{\|R_1\|^2}tr(P_1^T P_1)\\ &= tr[R_2^T(A^T AP_1BB^T + RA^T AP_1BB^T S)]\\ &+ tr[R_2^T(A^T AP_1BB^T - RA^T AP_1BB^T S)] + \tfrac{\|R_2\|^2}{\|R_1\|^2}\|P_1\|^2\\ &= \tfrac{\|P_1\|^2}{\|R_1\|^2}tr[R_2^T(R_1 - R_2)] + \tfrac{\|R_2\|^2}{\|R_1\|^2}\|P_1\|^2 = 0\end{aligned} \tag{5}$$

Assume that (3) holds for $k = t$, that is, $tr(R_t^T R_j) = 0$, $tr(P_t^T P_j) = 0$, $j = 1, 2, \ldots, t-1$. Being similar to (4) and (5), we can verify that $tr(R_{t+1}^T R_t) = 0$, and $tr(P_{t+1}^T P_t) = 0$.

Furthermore, when $k = t+1$, noting that the assumptions, we can obtain

$$\begin{aligned} tr(R_{t+1}^T R_1) &= tr(R_t^T R_1) - \tfrac{\|R_t\|^2}{\|P_t\|^2} tr[(A^T A P_1 B B^T + R A^T A P_1 B B^T S)^T P_1] \\ &= -\tfrac{\|R_t\|^2}{\|P_t\|^2} tr[P_t^T (A^T A R_1 B B^T + R A^T A R_1 B B^T S)] \\ &\quad - \tfrac{\|R_t\|^2}{\|P_t\|^2} tr[P_t^T (A^T A R_1 B B^T - R A^T A R_1 B B^T S)] \\ &= -\tfrac{\|R_t\|^2}{\|P_t\|^2} tr(P_t^T P_1) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} tr(P_{t+1}^T P_1) &= tr[(A^T A R_{t+1} B B^T + R A^T A R_{t+1} B B^T S)^T P_1] + \tfrac{\|R_{t+1}\|^2}{\|R_t\|^2} tr(P_t^T P_1) \\ &= tr[(A^T A R_{t+1} B B^T + R A^T A R_{t+1} B B^T S)^T P_1] \\ &= tr[R_{t+1}^T (A^T A P_1 B B^T + R A^T A P_1 B B^T S)] \\ &= \tfrac{\|P_1\|^2}{\|R_1\|^2} tr[R_{t+1}^T (R_1 - R_2)] \\ &= 0. \end{aligned}$$

In the same way, we can prove that $tr(R_{t+1}^T R_j) = 0$ and $tr(Q_{t+1}^T Q_j) = 0$ when $2 \le j \le t-1$. We complete the proof. □

Remark 2. From Lemma 4, we can see R_i $(i = 1, 2, \ldots, mn)$ as an orthogonal basis of matrix inner product space $GSR^{m \times n}$. If $R_i \neq 0$, we can compute R_{mn+1} by Algorithm 1, then there must be $tr(R_{mn+1}^T R_i) = 0$, which generates $R_{mn+1} = 0$, that is, X_{mn+1} is a solution of Problem I.

Lemma 5. *Suppose that $\bar{X} \in GSR^{m \times n}$ is a solution of Problem I, then X_k, P_k, R_k in Algorithm 1 satisfy that*

$$tr[(\bar{X} - X_k)^T P_k] = \|R_k\|^2, \; k = 1, 2, \cdots. \tag{6}$$

Proof. When k=1, according to Algorithm 1 and Lemma 4, noting that $R(\bar{X} - X_k)S = \bar{X} - X_k$, we can obtain

$$\begin{aligned} & tr[(\bar{X} - X_1)^T P_1] \\ &= tr[(\bar{X} - X_1)^T (A^T A R_1 B B^T + R A^T A R_1 B B^T S)] \\ &= tr\{R_1^T [A^T A (\bar{X} - X_1) B B^T + A^T A R (\bar{X} - X_1) S B B^T]\} \\ &= tr\{R_1^T [A^T A \bar{X} B B^T + R A^T A \bar{X} B B^T S - A^T A X_1 B B^T - R A^T A X_1 B B^T S]\} \\ &\quad + tr\{R_1^T [(A^T A \bar{X} B B^T - A^T A X_1 B B^T) - R(A^T A \bar{X} B B^T - A^T A X_1 B B^T) S]\} \\ &= tr[R_1^T (A^T C B^T + R A^T C B^T S - A^T A X_1 B B^T - R A^T A X_1 B B^T S)] \end{aligned}$$

$= \|R_1\|^2.$ (7)

Assume that (6) holds for $k = t$, when $k = t + 1$,

$tr[(\bar{X} - X_{t+1})^T P_t]$

$= tr[(\bar{X} - X_t)^T P_t] - \frac{\|R_t\|^2}{\|P_t\|^2} tr(P_t^T P_t)$

$= \|R_t\|^2 - \frac{\|R_t\|^2}{\|P_t\|^2} \|P_t\|^2$

$= 0,$

then, being similar to the proof of (7), we have

$tr[(\bar{X} - X_{t+1})^T P_{t+1}]$

$= tr[(\bar{X} - X_{t+1})^T (A^T A R_{t+1} B B^T + R A^T A R_{t+1} B B^T S)]$

$- \frac{\|R_t\|^2}{\|P_t\|^2} tr[(\bar{X} - X_{t+1})^T P_t]$

$= tr\{R_{t+1}^T [A^T A(\bar{X} - X_{t+1}) B B^T + A^T A R(\bar{X} - X_{t+1}) S B B^T]\}$

$= tr[R_{t+1}^T (A^T C B^T + R A^T C B^T S - A^T A X_{t+1} B B^T - R A^T A X_{t+1} B B^T S)]$

$= \|R_{s+1}\|^2,$

Therefore, (6) holds for all integers k. The proof is completed. □

Remark 3. Lemma 5 implies that, if $R_k \neq 0$, we have $P_k \neq 0$. Therefore, the iteration can not be terminated unless $R_k = 0$.

Based on the previous analysis and the remarks, we have the following main result, its proof is omitted.

Theorem 1. *For arbitrary initial iterative matrix $X_1 \in GSR^{m \times n}$, the solution of Problem I can be obtained within finite iterative steps in the absence of roundoff errors.*

The following lemma is stated from [13].

Lemma 6. *Suppose that $y_0 \in R(M^T)$ is a solution of inconsistent linear equations $My = b$, then y_0 is the unique least-norm solution.*

Theorem 2. *Let initial iterative matrix $X_1 = A^T A H B B^T + R A^T A H B B^T S$, where arbitrary $H \in R^{m \times n}$, or especially, $X_1 = 0 \in R^{m \times n}$, then the solution X^* generated by Algorithm 1 is the least-norm solution of Problem I.*

Proof. Algorithm 1 and Theorem 1 imply that if let initial matrix $X_1 = A^T A H B B^T + R A^T A H B B^T S$, where $H \in R^{m \times n}$ is arbitrary, we can obtain a solution X^* of Problem I, which has form like $X^* = A^T A Y B B^T + R A^T A Y B B^T S$. Hence, it is enough to prove that X^* is the unique least-norm solution of Problem I.

We know that, by making use of the properties of *vec* operator, matrix equation (2) can be transformed as the following linear systems

$$[(BB^T)\otimes(A^TA)+(SBB^T)\otimes(RA^TA)]vec(X) = vec(A^TCB^T+A^TCB^T). \quad (8)$$

In addition, the iterative solution X^* can be rewritten as

$$\begin{aligned} vec(X^*) &= [(BB^T)\otimes(A^TA)+(SBB^T)\otimes(RA^TA)]vec(Y) \\ &\in R[(BB^T)\otimes(A^TA)+(SBB^T)\otimes(RA^TA)] \end{aligned}$$

which implies from Lemma 6 that $vec(X^*)$ is the least-norm solution of the linear systems (8). According to the reversibility of vec operator, X^* is the unique least-norm solution of Problem I. □

3. The solution to problem II

In this section, we discuss the optimal approximation problem for the given matrix $X_0 \in R^{m\times n}$, i.e., Problem II. Connecting with the definition of closed convex set, we can verify that the solution set S_E of Problem I is a closed convex set in the matrix inner product space $GSR^{m\times n}$, so the solution of Problem II is unique. Without loss of generality, we can assume that the given matrix $X_0 \in GSR^{m\times n}$ because of the orthogonality between (R,S)-symmetric and (R,S)-skew symmetric matrices. In fact, suppose that $X \in S_E$, we have

$$\begin{aligned} \|X-X_0\|^2 &= \left\|X - \tfrac{X_0+RX_0S}{2} - \tfrac{X_0-RX_0S}{2}\right\|^2 \\ &= \left\|X - \tfrac{X_0+RX_0S}{2}\right\|^2 + \left\|\tfrac{X_0-RX_0S}{2}\right\|^2. \end{aligned}$$

let $\tilde{X} = X - X_0$, $\tilde{C} = C - AX_0B$, then Problem II means to find the least-norm solution of the new matrix equation

$$A^TA\tilde{X}BB^T + RA^TA\tilde{X}BB^TS = A^T\tilde{C}B^T + RA^T\tilde{C}B^TS. \quad (9)$$

Taking Theorem 2, we know that if let initial matrix

$$\tilde{X}_1 = A^TA\tilde{H}BB^T + RA^TA\tilde{H}BB^TS,$$

where arbitrary $\tilde{H} \in R^{m\times n}$, or especially, $\tilde{X}_1 = 0 \in R^{m\times n}$, then the solution $\tilde{X}^*$ generated by Algorithm 1 is the unique least-norm solution of matrix equation (9). Therefore, the optimal approximation solution $\hat{X}$ can be derived by $\hat{X} = \tilde{X}^* + X_0$.

4. Numerical examples

In this section, we will give some numerical examples tested by MATLAB 6.5 to illustrate our results. Because of the influence of the roundoff errors, we see matrix A as zero matrix if $\|A\| < 1.0e-010$. That is, in the iterative process, if $\|R_k\| < 1.0e-010$, stop the iteration, and X_k is regarded as a required solution.

Example Given matrices A, B, R, S as follows.

$$A = \begin{bmatrix} 5 & -3 & 0 & 3 & 0 & 2 & 8 \\ 0 & -4 & -6 & 4 & -6 & 0 & -4 \\ -6 & 0 & 7 & 0 & 7 & 3 & 1 \\ 0 & 5 & -3 & -5 & -3 & 0 & 3 \\ 4 & -7 & 0 & 7 & 0 & -8 & -3 \\ -1 & 0 & -6 & 0 & -5 & 9 & 0 \\ 0 & -3 & 0 & 3 & -7 & 0 & 1 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} -3 & 5 & -5 & -2 & 5 & -2 \\ 0 & 4 & 9 & 9 & 4 & -6 \\ 6 & -1 & 7 & 0 & -1 & 3 \\ -2 & 4 & 0 & 5 & 4 & 5 \\ -1 & -6 & -2 & 0 & -6 & 2 \\ 0 & -9 & 1 & 1 & -9 & 2 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 28 & -72 & -11 & -56 & -72 & 11 \\ 14 & -62 & 8 & -9 & -62 & 13 \\ -16 & 31 & -30 & -8 & 31 & 7 \\ -11 & 53 & 14 & 29 & 53 & -25 \\ 34 & -80 & 32 & -13 & -80 & 11 \\ -4 & -36 & -33 & -27 & -36 & 20 \\ 15 & -41 & -4 & -32 & -41 & -8 \end{bmatrix}.$$

Let initial iterative matrix $X_1 = 0 \in R^{7\times 6}$, then, by Algorithm 1 and iterating 57 times, we can obtain the least-norm solution of Problem I, i.e.,

$$X_{57} = \begin{bmatrix} -0.2671 & -0.2671 & 0 & -0.2040 & -0.1936 & 0.1936 \\ 0.2101 & 0 & -0.2634 & 0.2046 & -0.2029 & -0.3794 \\ 0 & -0.2101 & -0.2634 & -0.2046 & -0.3794 & -0.2029 \\ -0.0836 & -0.4111 & 0.0833 & -0.3549 & 0.2325 & 0.0855 \\ -0.0769 & -0.0769 & 0 & 0.3907 & -0.2116 & 0.2116 \\ 0.4159 & -0.4159 & 0.0940 & 0 & 0.2039 & 0.2039 \\ -0.4111 & -0.0836 & -0.0833 & -0.3549 & -0.0855 & -0.2325 \end{bmatrix},$$

and the minimum residual $\|R_{57}\| = 7.0490e - 011$.

Suppose that the give matrix in Problem II is

$$X_0 = \begin{bmatrix} -1.4142 & -1.4142 & 0 & -1.0000 & -1.0606 & 1.0606 \\ 1.0000 & 0 & -1.0606 & 1.0606 & -1.0000 & -1.5000 \\ 0 & -1.0000 & -1.0606 & -1.0606 & -1.5000 & -1.0000 \\ -0.5000 & -2.5000 & 0.7071 & -1.7677 & 1.2500 & 0.7500 \\ -0.3535 & -0.3535 & 0 & 2.0000 & -1.0606 & 1.0606 \\ 2.1213 & -2.1213 & 0.5000 & 0 & 1.0606 & 1.0606 \\ -2.5000 & -0.5000 & -0.7071 & -1.7677 & -0.7500 & -1.2500 \end{bmatrix}.$$

Compute $C_0 = AX_0B$, and let initial matrix be 7×6 zero matrix, we can also obtain the least-norm (R,S)-symmetric solution $\tilde{X}^*$ of the new matrix equation (9) by making use of Algorithm 1 with 59 iteration steps. Hence, the solution $\hat{X}$ of Problem II is

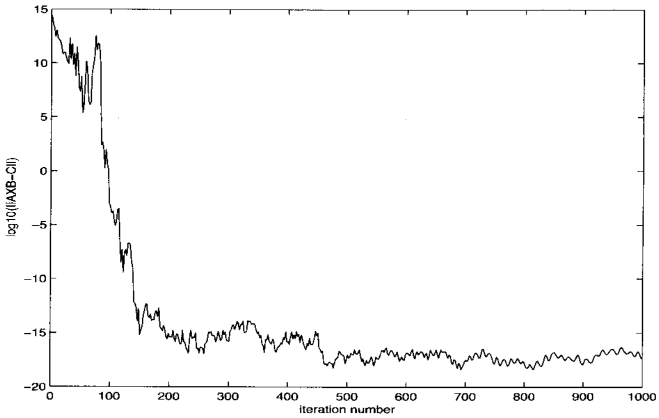

FIGURE 1. Convergence curve for the Frobenius norm of the residual.

$$\hat{X} = \tilde{X}^* + X_0 = \begin{bmatrix} -0.2671 & -0.2671 & 0 & -0.2040 & -0.1936 & 0.1936 \\ 0.1110 & 0 & 0.0112 & 0.2135 & -0.1416 & -0.0782 \\ 0 & -0.1110 & 0.0112 & -0.2135 & -0.0782 & -0.1416 \\ -0.1828 & -0.8656 & 0.3578 & -0.3459 & 0.2938 & 0.3868 \\ -0.0769 & -0.0769 & 0 & 0.3907 & -0.2115 & 0.2115 \\ 0.4159 & -0.4159 & 0.0940 & 0 & 0.2038 & 0.2038 \\ -0.8656 & -0.1828 & -0.3578 & -0.3459 & -0.3868 & -0.2938 \end{bmatrix}$$

Furthermore, in order to show the property of Algorithm 1, we stop the iteration if $\|R_k\| < 1.0e-020$, by Algorithm 1 and iteration 1000 steps, we obtain the convergence curve (i.e., Figure 1) for the Frobenius norm of the residual $(AXB - C)$.

From Figure 1, we can see that the descending of the residual R_k is rapid, which imply that the iterative method in this paper is efficient. Certainly, for the matrices with large orders, the convergence velocity may be slow, this is our urgent work in the future.

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