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SHAPE PRESERVING ADDITIONS OF LR-FUZZY INTERVALS WITH UNBOUNDED SUPPORT

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ABSTRACT. Continuous $t-$norm based shape preserving additions of $LR-$fuzzy intervals with unbounded support is studied. The case for bounded support, which was a conjecture suggested by Mesiar in 1997, was proved by the author in 2002 and 2008. In this paper, we give a necessary and sufficient conditions for a continuous $t-$norm T that induces $LR-$shape preserving addition of $LR-$fuzzy intervals with unbounded support. Some of the results can be deduced from the results given in the paper of Mesiar in 1997. But, we give direct proofs of the results.

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1. Introduction

Fuzzy arithmetic has grown in importance during recent years as a tool of advance in fuzzy optimization and control theory. The usual arithmetic operations of reals can be extended to the arithmetical operations on fuzzy intervals by means of Zadeh's extension principle [32] based on a triangular norm T. Fuzzy arithmetic based on the sup-(t-norm) convolution, with the controllability of the increase of fuzziness, enables us to construct more flexible and adaptable mathematical models in several intelligent technologies based on approximate reasoning and fuzzy logic. Hence a lot of effort is needed to fine exact and good approximative computational formulas for fuzzy arithmetic operations.

Most of results are restricted on the case of bounded supports. The most frequently used fuzzy intervals are those of LR-fuzzy intervals, where L and R are left and right shape functions, respectively. The simplest form of fuzzy quantities

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are triangular fuzzy intervals that has been studied extensively and used in many applications. Some results on fuzzy arithmetic operations and their applications can be found in [4-27]. An important feature of t-norm-based arithmetical operations is that it provides a mean of controlling the growth of uncertainty during calculations. Namely, shape preserving arithmetic operations of LR-fuzzy intervals allow to control the resulting spread. In practical computation, it is natural to require the preserving of the shape of fuzzy intervals during the addition and the multiplication. Recently, Hong [16] showed that T_W, the weakest t-norm, is the only t-norm T that induces a shape preserving multiplications of LR-fuzzy intervals. Mesiar [27] was interested, for a given shapes L and R, in which t-norm T induces a shape preserving addition of LR-fuzzy numbers and suggested this as an open problem in [28]. In [18, 19], Hong answered this question, that is, he gave necessary and sufficient conditions for a continuous $t-$norm T that induces shape preserving additions of $LR-$fuzzy intervals.

Recently, Dombi and Győrbíró [4] and Hong [12] investigated the shape preserving additions of sigmoid and two bell-shaped membership functions which appear in many natural processes and are used in fuzzy neural networks [2, 9] and machine learning applications [32]. These membership functions have unbounded supports. So, it is interesting to study, for a given shapes L and R with with unbounded support, which t-norm T induces a shape preserving addition of LR-fuzzy numbers. In this paper, we give a necessary and sufficient conditions for a continuous $t-$norm T that induces $LR-$shape preserving addition of $LR-$fuzzy intervals with unbounded support. Some of the results can be deduced from the results given in the paper of Mesiar [28]. But, we give direct proofs of the results.

2. Basic definitions

A fuzzy interval A is a fuzzy subset of the real line $\mathbf{R}$ with a continuous, compactly supported, unimodal and normalized membership function. Following Dubois and Prade [6], a fuzzy interval A is a so called LR-fuzzy interval, $A = (a, b, \alpha, \beta)_{LR}$, if the corresponding membership function satisfies for all $x \in \mathbf{R}$

$$A(x) = \begin{cases} 1 & \text{if } a \le x \le b \\ L\left(\dfrac{a-x}{\alpha}\right) & \text{if } a-\alpha \le x \le a, \alpha > 0 \\ R\left(\dfrac{x-b}{\beta}\right) & \text{if } b \le x \le b+\beta, \beta > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $[a, b]$ is the peak of A, $\alpha > 0$ and $\beta > 0$ is the left and the right spread, respectively, and L and R are strictly decreasing continuous function from $[0, 1]$ to $[0, 1]$ such that $L((0, 1)) = R((0, 1)) = (0, 1)$. Recall that L and R is called the left and right shape function, respectively. If $R(x) = L(x) = 1 - x$, we denote $(a, b, \alpha, \beta)_{LR}$ by (a, b, α, β). A function $T\colon [0, 1] \times [0, 1] \to [0, 1]$ is said to be a triangular norm(t-norm) iff T is symmetric, associative, non-decreasing in each argument, and $T(x,1)=x$ for all $x\in[0,1]$ ([3]).

Recall that a continuous t-norm T is Archimedean iff $T(x,x)<x$ for all $x\in(0,1)$. Every continuous Archimedean t-norm T is representable by a continuous and decreasing function $f\colon[0,1]\to[0,\infty]$ with $f(1)=0$ and

$$T(x,y)=f^{[-1]}\Big(f(x)+f(y)\Big).$$

where $f^{[-1]}$ is the pseudo-inverse of f, defined by

$$f^{[-1]}(y):=\begin{cases} f^{-1}(y) & \text{if } y\in[0,f(0)],\\ 0 & \text{if } y\in[f(0),\infty].\end{cases}$$

The function f is the additive generator of T.

If f is bounded, then it can be chosen uniquely so that $f(0)=1$ and the corresponding t-norm T is called a nilpotent t-norm. If f is unbounded, the corresponding t-norm T is called a strict t-norm.

Let T be a given t-norm and let A_1 and A_2 be two fuzzy numbers. Then their T-sum $C=A\oplus B$ is defined by the generalization, proposed in [6], of Zadeh's extension principle [31] as

$$C(z)=\sup_{x+y=z}T\Big(A(x),B(y)\Big),\quad z\in R.$$

If T is a continuous Archimedean t-norm with the additive generator f then

$$C(z)=f^{[-1]}\left(\inf_{x+y=z}(f(A(x))+f(B(y))\right).$$

3. Additions preserving the linearity of fuzzy intervals

Recall that T_M is the strongest t-norm, $T_M(x,y)=\min(x,y)$, and T_W is the weakest t-norm, where $T_W(x,y)=0$ whenever $\max(x,y)<1$ and $T_W(x,y)=\min(x,y)$ otherwise. Let $A_i=(a_i,\beta_1,\alpha_i,\beta_i)_{LR}, i=1,2$. The followings are well known [2, 3].

The addition $\oplus$ based on the weakest t-norm T_W is

$$\begin{aligned}&(a_1,b_1,\alpha_1,\beta_1)_{LR}\oplus(a_2,b_2,\alpha_2,\beta_2)_{LR}\\ &=\Big(a_1+a_2,b_1+b_2,\max(\alpha_1,\alpha_2),\max(\beta_1,\beta_2)\Big)_{LR}.\end{aligned}$$

and the T_M based addition is

$$(a_1,b_1,\alpha_1,\beta_1)_{LR}\oplus(a_2,b_2,\alpha_2,\beta_2)_{LR}=\Big(a_1+a_2,b_1+b_2,\alpha_1+\alpha_2,\beta_1+\beta_2\Big)_{LR}.$$

Some results on the additions preserving the linearity of fuzzy intervals can be found in [6, 18, 19, 21, 22]. The following theorem, part (b), is due to Kolesárová [16, 17].

Theorem 1. *a) Let T be an arbitrary t-norm weaker or equal with the Lukasiewicz t-norm $\mathbf{T}_L$, $T(x,y) \le \mathbf{T}_L(x,y) = \max(0, x+y-1)$, $x, y \in [0,1]$. Then the addition $\oplus$ based on T coincides on linear fuzzy intervals with the addition $\oplus$ based on the weakest t-norm T_W, i.e.,*

$$(a_1, b_1, \alpha_1, \beta_1) \oplus (a_2, b_2, \alpha_2, \beta_2) = \Big(a_1 + a_2, b_1 + b_2, \max(\alpha_1, \alpha_2), \max(\beta_1, \beta_2)\Big).$$

b) Let T be a continuous Archimedean t-norm with twice differentiable, strict convex additive generator f. Then the addition $\oplus$ based on T preserves the linearity of fuzzy intervals if and only if the t-norm T is a member of Yager's family of nilpotent t-norms with parameter $p \in [1, \infty)$, $T = T_p^Y$ and $f(x) = (1-x)^p$. Then $T_1^Y = \mathbf{T}_L$ and for $p \in (1, \infty)$

$$(a_1, b_1, \alpha_1, \beta_1) \oplus (a_2, b_2, \alpha_2, \beta_2) = \Big(a_1 + a_2, b_1 + b_2, (\alpha_1^q + \alpha_2^q)^{1/q}, (\beta_1^q + \beta_2^q)^{1/q}\Big)$$

where $(1/p + 1/q) = 1$, i.e., $q = p/(p-1)$.

The above Theorem 1 gives some sufficient conditions for a t-norm T to serve as a basis for a linearity preserving addition. Mesiar [27, 28] conjectured that the only t-norm based additions preserving the linearity of fuzzy intervals are these described in Theorem 1. Recently, Hong [18, 19] proved the following conjecture due to Mesiar.

Theorem 2. *Let a continuous t-norm T be not weaker or equal with $\mathbf{T}_L$. Then the addition $\oplus$ based on T preserves the linearity of fuzzy intervals if and only if the t-norm T is either T_M or a member of Yager's family of nilpotent t-norms with parameter $p \in (1, \infty)$, $T = T_p^Y$ and $f(x) = (1-x)^p$.*

As corollaries of above results, we have continuous $t-$norm based $LR-$shape preserving addition of fuzzy intervals with bounded support by transformation rule (see [27]).

4. Additions preserving the $LR-$shape of fuzzy intervals with unbounded support

A fuzzy interval A is a called LR-fuzzy interval with unbounded support, $A = (a, b, \alpha, \beta)_{LR}$, if the corresponding membership function satisfies for all $x \in \mathbf{R}$

$$A(x) = \begin{cases} 1, & \text{if } \ a \le x \le b \\ L\left(\dfrac{a-x}{\alpha}\right), & \text{if } \ x \le a, \alpha > 0 \\ R\left(\dfrac{x-b}{\beta}\right), & \text{if } \ b \le x, \beta > 0, \end{cases}$$

where $[a, b]$ is the peak of A, $\alpha > 0$ and $\beta > 0$ is the left and the right spread, respectively, and L and R are strictly decreasing continuous function from $[0, \infty)$

to $[0,1]$ such that $L((0,\infty)) = R((0,\infty)) = (0,1)$. Recall that L and R is called the left and right shape function with unbounded support, respectively.

In this section we consider a necessary and sufficient conditions for a t-norm T to serve as a basis for a LR–shape preserving addition of LR–fuzzy intervals with unbounded support.

Lemma 1. *For positive β_1, β_2,*

$$\inf_{\substack{x+y=z,\\ x\geq 0, y\geq 0,}} \left(\frac{x}{\beta_1}\right)^p + \left(\frac{y}{\beta_2}\right)^p = \begin{cases} \left(\dfrac{z}{(\beta_1^q+\beta_2^q)^{1/q}}\right)^p, & \text{if} \quad p>1,\\ \dfrac{z}{\max\{\beta_1,\beta_2\}}, & \text{if} \quad p=1, \end{cases}$$

where $q = p/(p-1)$ for $p \neq 1$.

Proof. Let us denote $h(x) = \left(\frac{x}{\beta_1}\right)^p + \left(\frac{z-x}{\beta_2}\right)^p$ and find the minimal value of h for $x \in R^+$. We first assume that $p > 1$. The derivative of the function h is:

$$h'(x) = \frac{p}{\beta_1}(\frac{x}{\beta_1})^{p-1} - \frac{p}{\beta_2}(\frac{z-x}{\beta_2})^{p-1}.$$

The only point x for which $h'(x) = 0$ is

$$x_0 = \frac{z}{1+\lambda}, \text{ where } \lambda = (\frac{\beta_2}{\beta_1})^{\frac{p}{(p+1)}}.$$

It is easy to show that h has its minimum at the point x_0 and minimal value of h is

$$h(x_0) = \left[\frac{z}{\beta_1\beta_2(1+\lambda)}\right]^p ({\beta_2}^p + {\beta_1}^p\lambda^p) = \left(\frac{z}{(\beta_1^q+\beta_2^q)^{1/q}}\right)^p.$$

The case for $p = 1$ is easy to check. □

The following two results can be deduced from the results in the paper of Mesiar [28]. Here, we give directly proofs.

Theorem 3. *Let both L and R be shape functions with unbounded support. Let T be a strict t-norm with additive generator $f = (R^{-1})^p$ for some $p \geq 1$ and let $L(x) = R(x^k)$ for some $k > 0$. Then the addition $\oplus_T$ based on T preserves the LR–shape of LR–fuzzy intervals and for $p \in (1,\infty)$*

$$\begin{aligned}&(a_1,b_1,\alpha_1,\beta_1)_{LR} \oplus_T (a_2,b_2,\alpha_2,\beta_2)_{LR}\\ &= \left(a_1+a_2, b_1+b_2, (\alpha_1^s+\alpha_2^s)^{1/s}, (\beta_1^q+\beta_2^q)^{1/q}\right)_{LR}\end{aligned}$$

where $1/p+1/q = 1$, i.e., $q = p/(p-1)$ and $1/(pk)+1/s = 1$, i.e., $s = pk/(pk-1)$.

Proof. Let $A_i = (a_i,\beta_i,\alpha_i,\beta_i)_{LR}, i = 1,2$. We first consider the case for $z \geq b_1 + b_2$. As mentioned in Section 1, the investigated membership function is

$$A_1 \oplus_T A_2(z) = f^{-1}\left(\inf_{x_1+x_2=z} f(A_1(x_1)) + f(A_2(x_2))\right).$$

We put $B_i = (a_i - b_i, 0, \alpha_i, \beta_i)_{LR}$, $i = 1, 2$. Since $A_i(x) = B_i(x - b_i)$ for all x, we have for each z,

$$\begin{aligned} A_1 \oplus_T A_2(z) &= f^{-1}\left(\inf_{x_1+x_2=z} f(A_1(x_1)) + f(A_2(x_2))\right) \\ &= f^{-1}\left(\inf_{y_1+y_2=z-b} (f(B_1(y_1)) + f(B_2(y_2)))\right) = B_1 \oplus_T B_2(z-b) \end{aligned}$$

where $b = b_1 + b_2$. It is well-known by [6] that right-hand side of $B_1 \oplus_T B_2$ depends only on right-hand sides of B_1 and B_2 and hence we have that

$$\begin{aligned} B_1 \oplus_T B_2(z) &= f^{-1}\left(\inf_{\substack{x_1+x_2=z \\ x_i \geq 0, i=1,2}} f(B_1(x_1)) + f(B_2(x_2))\right) \\ &= f^{-1}\left(\inf_{\substack{x_1+x_2=z \\ x_i \geq 0, i=1,2}} f \circ R\left(\frac{x_1}{\beta_1}\right) + f \circ R\left(\frac{x_1}{\beta_1}\right)\right) \\ &= f^{-1}\left(\inf_{\substack{x_1+x_2=z \\ x_i \geq 0, i=1,2}} (\frac{x_1}{\beta_1})^p + (\frac{x_2}{\beta_2})^p\right) \\ &= f^{-1}\left((\frac{z}{(\beta_1^q + \beta_2^q)^{1/q}})^p\right) \\ &= R\left(\frac{z}{(\beta_1^q + \beta_2^q)^{1/q}}\right), \end{aligned}$$

where the fourth equality comes from Lemma 1. Therefore we have for $z \geq b$,

$$A_1 \oplus_T A_2(z) = R\left(\frac{z-b}{(\beta_1^q + \beta_2^q)^{1/q}}\right).$$

The case for $z \leq a_1 + a_2$ is similar. □

Theorem 4. *Let both L and R be shape functions with unbounded support. Let T be an arbitrary t-norm weaker or equal with the t-norm T_R (T_R), where T_R (T_L) is the strict t−norm with the additive generator R^{-1} (L^{-1}), i.e., $T_R(x,y) = R(R^{-1}(x) + R^{-1}(y))$. Then the addition $\oplus_T$ based on T works on the right shape R (the left shape L) of LR−fuzzy intervals exactly as the addition $\oplus_T$ based on the weakest t-norm T_W, i.e. if $T(x,y) \leq T_R(x,y)$ and $T(x,y) \leq T_L(x,y)$, then*

$$\begin{aligned} &(a_1, b_1, \alpha_1, \beta_1)_{LR} \oplus_T (a_2, b_2, \alpha_2, \beta_2)_{LR} \\ &= \Big(a_1 + a_2, b_1 + b_2, \max(\alpha_1, \alpha_2), \max(\beta_1, \beta_2)\Big)_{LR}. \end{aligned}$$

Proof. If we just follow the proof of Theorem 3 in the case of $p = 1$ and use Lemma 1 again, then we immediately have that for $z \geq b = b_1 + b_2$,

$$A_1 \oplus_T A_2(z) = R\left(\frac{z-b}{max\{\beta_1, \beta_2\}}\right).$$

For the case of $z \leq a = a_1 + a_2$, we similarly have that

$$A_1 \oplus_T A_2(z) = L\left(\frac{a-z}{max\{\alpha_1,\alpha_2\}}\right),$$

that is,

$$A_1 \oplus_T A_2(z) = \Big(a_1 + a_2, b_1 + b_2, \max(\alpha_1, \alpha_2), \max(\beta_1, \beta_2)\Big)_{LR}.$$

□

Now, the addition $\oplus$ based on the weakest t-norm T_W coincides on this LR–fuzzy intervals by a known result [6], and hence the result follows.

Theorem 5. *Let R (L) be shape function with unbounded support. Suppose that f is the additive generator of a t-norm T and that the addition based on T preserves the right shape R (the left shape L) of LR–fuzzy intervals. Then either $f \circ R(x) = ax^p$ ($f \circ L(x) = ax^p$) for some $a \in R^+$, $p \in (1, \infty)$ or $T \le T_R$ ($T \le T_L$).*

Here we need some lemmas.

Lemma 2. *If h is a convex function on $[0, \infty)$ such that $h(0) = 0$ and $\inf_{x+y=z}(h(x) + h(y)) = h(z/c)$ for some constant $1 \le c < 2$, then $h(x) = ax^p$, for some $a \in R^+$, $1 \le p < \infty$.*

Proof. Since h is a convex function, we have that $\inf_{x+y=z}(h(x) + h(y)) = 2h(z/2)$, and hence we have that for any $z \in \mathbf{R}$ and for some constant $1 \le c < 2$,

$$2h(z/2) = h(z/c).$$

Let us consider a one to one correspondence between intervals $[1, 2)$ and $[1, \infty)$ given by the mapping $c \leftrightarrow 2^{1-1/p}$. Putting $z/2 = u$ and $2^{1/p} = \lambda$, we get

$$\lambda^p h(u) = h(\lambda u). \qquad (1)$$

For a given $p \in [1, \infty)$ the only continuous increasing solutions of the functional equation (1) in the interval $[1, \infty)$ are functions h given by $h(u) = au^p$ for some $a > 0$ (see [1]). □

We say that a function h defined on $[0, \infty)$ has line support at $x_0 \in [0, \infty)$ if there exists an affine function $H(x) = h(x_0) + m(x - x_0)$ such that $H(x) \le h(x)$ for every $x \in [0, \infty)$. The graph of the support function H is called a supporting line for h at x_0.

Let f be a continuous and decreasing function from $[0, \infty)$ to $[0, \infty)$ with $h(1) = 0$. Considering the convex hull of $\Big\{(x, y) | x \in [0, \infty), h(x) \le y\Big\}$, we can define h^*, the convex function generated by h, to be

$$\begin{aligned} h^*(x) &= \inf\Big\{\lambda_1 h(x_1) + \lambda_2 h(x_2) + \cdots + \lambda_n h(x_n) : \\ &\qquad \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = x\Big\} \\ &= \sup H(x) \end{aligned}$$

where the supremum is taken over all affine functions H such that $H(x) \leq f(x)$ on $[0,\infty)$. (see [29, p21]). It is noted that $f^*(x_0) = f(x_0)$ if and only if there exists supporting line for f at each $x_0 \in (0,\infty)$.

Indeed, we can prove Lemma 2 without the assumption that h is convex on $[0,\infty)$. This is the key idea of this paper.

Lemma 3. *Let g be a continuous strictly increasing function on $[0,\infty)$ such that $g(0)=0$ and $\inf_{x+y=z}(g(x)+g(y)) = g(z/c)$ for some constant $1<c<2$, then g is strictly convex, indeed, we have $g(u)=au^p$, for some $a \in (0,\infty)$, $1<p<\infty$.*

Proof. Let g^* be the convex function generated by g, then $\inf_{x+y=z}(g^*(x)+g^*(y)) = 2g^*(z/2)$. If we show that $g=g^*$, then the result follows immediately from Lemma 2. We first consider that

$$\inf_{x+y=z}(g^*(x)+g^*(y)) \leq \inf_{x+y=z}(g(x)+g(y)) = g(z/c).$$

Since $\inf_{x+y=z}(g^*(x)+g^*(y)) = 2g^*(z/2)$, is a convex function, we have that

$$\inf_{x+y=z}(g^*(x)+g^*(y)) \leq g^*(z/c).$$

Now, let $S = \Big\{z \mid g(z/2) = g^*(z/2)\Big\}$. Then clearly S is nonempty unbounded set. Hence for $z \in S$, we have that

$$\begin{aligned}\inf_{x+y=z}(g^*(x)+g^*(y)) &= 2g^*(z/2) = 2g(z/2)\\ &\geq \inf_{x+y=z}(g(x)+g(y)) = g(z/c)\\ &\geq g^*(z/c),\end{aligned}$$

and then we have that,

$$2g^*(z/2) = \inf_{x+y=z}(g^*(x)+g^*(y)) = g^*(z/c).$$

If $u \in S$, then $g^*(u) = au^p$, for some $a \in (0,\infty)$, $1 \leq p < \infty$ by Lemma 2. We now show that $S=[0,\infty)$. Suppose that $S \neq [0,\infty)$, then there exist $z_l, z_r \in S$ such that $(z_l, z_r) \subset S^c$, since S is a closed set. Let $z^* = (z_l+z_r)/2$, then $z^* \notin S$. Since g^* is convex, $2g^*(z^*) \leq g^*(z_l) + g^*(z_r) = a(z_l)^p + a(z_r)^p$.

On the other hand, considering the supporting line of g^* at $z_l/2$ and $z_r/2$, we consider that

$$\begin{aligned}\inf_{x+y=z^*}(g^*(x)+g^*(y)) &= g^*(z_l/2)+g^*(z_r/2) = g(z_l/2)+g(z_r/2)\\ &\geq \inf_{x+y=z}(g(x)+g(y)) = g(z^*/c)\\ &\geq g^*(z^*/c),\end{aligned}$$

that is, $2g^*(z/2) = \inf_{x+y=z^*}(g^*(x)+g^*(y)) = g^*(z^*/c)$, and hence we have $g^*(z^*) = a((z_l+z_r)/2)^p$, by Lemma 2. Since it is true that $(z_l)^p/2 + (z_r)^p/2 \geq ((z_l+z_r)/2)^p$, only for $p=1$, $g^*(z) = az$, $z \in [0,\infty)$. We now choose a point

$z_0 \neq 0$ such that $g(z_0) = g^*(z_0)$. Noting that $g(z_0) = g^*(z_0) = \inf_{x+y=z_0}(g^*(x) + g^*(y)) \leq \inf_{x+y=z_0}(g(x) + g(y)) = g(z_0/c)$, we have that $g(z_0) \leq g(z_0/c)$, which is a contradiction and complete the proof. □

Proof of Theorem 5. Suppose that the addition based on T preserves the right shape R of $LR-$fuzzy intervals. Let $A_1 = A_2 = (0,0,1,1)_{LR}$. Then as in the proof of Theorem 3, we have

$$R(\frac{z}{c}) = f^{-1}\left(\inf_{\substack{x_1+x_2=z \\ x_i \geq 0, i=1,2}} f \circ R(x_1) + f \circ R(x_1)\right),$$

for some $c \in [1,2)$, since for $c = 2$, we have $T = T_M$. Let $f \circ R(x) = g(x)$, $x \in [0,\infty)$. Then for some constant $1 \leq c < 2$,

$$\inf_{x+y=z}(g(x) + g(y)) = g(z/c).$$

If $1 < c < 2$, then by Lemma 3, $f \circ R(x) = g(x) = ax^p$ for some $a \in \mathbf{R}^+$, $p \in (1,\infty)$. If $c = 1$ then g is idempotent, and hence we have $T \leq T_R$ by Theorem 4 in [17], which completes the proof. □

We finally have the following theorem which completely characterize the continuous $T-$ norm based $LR-$shape preserving addition of fuzzy intervals with unbounded support.

Theorem 6. *Let R (L) be shape function with unbounded support. Let a continuous t-norm T be not weaker or equal with T_R (T_L). Then the addition $\oplus$ based on T preserves the right shape R (the left shape L) of $LR-$fuzzy intervals if and only if the t-norm T is either T_M or $f \circ R(x) = ax^p$ $(f \circ L(x) = ax^p)$ for some $a \in R^+$, $p \in (1,\infty)$*

Proof. The "if" part is immediate. We prove "only if" part. If T is not T_M, then by Lemma 1. 2. 3 [27], T is an Archimedean $t-$norm. Now the result comes from Theorem 5. □

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