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INTERACTION OF SURFACE WATER WAVES WITH SMALL BOTTOM UNDULATION ON A SEA-BED

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ABSTRACT. The problem of interaction of surface water waves by small undulation at the bottom of a laterally unbounded sea is treated on the basis of linear water wave theory for both normal and oblique incidences. Perturbation analysis is employed to obtain the first order corrections to the reflection and transmission coefficients in terms of integrals involving the shape function $c(x)$ representing the bottom undulation. Fourier transform method and residue theorem are applied to obtain these coefficients. As an example, a patch of sinusoidal ripples is considered in both the cases as the shape function. The principal conclusion is that the reflection coefficient is oscillatory in the ratio of twice the surface wave number to the wave number of the ripples. In particular, there is a Bragg resonance between the surface waves and the ripples, which is associated with high reflection of incident wave energy. The theoretical observations are validated computationally.

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1. Introduction

The problems of interaction between surface waves and a pre-existing or a fixed pattern of undulation on an otherwise flat bed are important for their possible applications in the areas of coastal and marine engineering, and as such these are being studied by scientists and engineers with immense interest. These problems have received an increasing amount of attention because such pattern may comprise shore parallel bars or tidally generated features such as sand waves, lying transverse to the direction of the wave propagation. These problems are, in general, somewhat difficult to solve analytically although there exist various approximate mathematical techniques by which quantities of physical

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interests, namely the reflection and transmission coefficients, can be estimated numerically. For example, the problem of free surface flow over an undulating bed, called the mild-slope equation, was initially devised by Berkhoff [1] and by Smith and Sprinks [15] independently. That was further extended by Kirby [8] (extended mild-slope equation) and by Chamberlain and Porter [2] (modified mild slope equation) which introduced approximate analytical techniques essentially involving depth-averaging under the assumption of the small variation of the bed.

However, the existence of a class of mostly naturally occurring bottom standing obstacles such as sand ripples, which can be assumed to be small in some sense, allows for some sort of perturbation technique to be employed to obtain the first order corrections to the reflection and transmission coefficients. Miles [14] considered the diffraction of surface wave obliquely incident on a small cylindrical deformation of the bottom of a sea. Davies and Heathershaw [5] also considered the problem of water wave scattering by a sinusoidally varying topography on the sea-bed for normal incidence, with the introduction of a linear friction term in the free surface condition.

Mei [13] presented a theory that strong reflection could be induced by sandbars themselves when the Bragg resonance condition was met. Hara and Mei [6] extended the linearized theory on Bragg scattering of surface waves by periodic sandbars to include second-order effects of the free surface and also of the bars. They also described new experiments which demonstrated the physical features of the problem. Mandal and Basu [9] generalized the problem in [14] by including the effect of surface tension at the free surface. Using Green's integral theorem, the reflection and transmission coefficients were obtained up to the first order in terms of integrals involving the shape function describing the bottom undulation.

Martha and Bora [10]-[12] have dealt with both normal and oblique scattering of surface wave propagation over a small undulation on the bottom of a sea. By employing perturbation analysis the velocity potential, reflection coefficient and transmission coefficients up to the first order were obtained by using Green's function technique and finite cosine transformation. The results were demonstrated with a number of practical examples. Recently Warke *et al* [16] obtained closed form solutions for scattering of surface waves by wavy or exponential bed topography. Numerical computations indicated that when solitary or sinusoidal wave conditions were applied at the boundary, water surface elevation attained almost a Gaussian profile.

Here we formulate the scattering problem for both normal and oblique incidences of a train of surface waves propagating from negative infinity over a sea-bed having small undulation. In both the cases, the governing boundary value problem is reduced to a simpler one for the first order correction of the potential using perturbation analysis involving a small parameter ε. Applying Fourier transform method to the reduced problem, it is found that the integrand contains certain singularities and hence the residue theorem is used while employing contour integration to evaluate the first order correction of the potential. The reflection and transmission coefficients are evaluated approximately up to the first order of ε in terms of integrals involving the shape function. For a patch of sinusoidal ripples, a special form of the undulation, the integrals for the reflection and transmission coefficients are evaluated explicitly. When the ripple wave number is equal to twice the surface wave number, the first order reflection coefficient is found to be increasing with the number of ripples. This is consistent with the result obtained earlier in [5] while studying the surface wave propagation over sinusoidal topography.

2. Case-I: Normal incidence

2.1. Statement and formulation

A right-handed rectangular Cartesian co-ordinate system is considered in which x-axis is the position of the undisturbed free surface of the sea and y-axis is measured positive vertically downwards from the undisturbed free surface. The bottom of the sea with small undulation is described by $y = h + \varepsilon c(x)$ where $c(x)$ is a function with compact support and describes the bottom undulation, h denotes the uniform finite depth of the sea far to either side of the undulation of the bottom so that $c(x) \to 0$ as $|x| \to \infty$ and the non-dimensional number $\varepsilon(\ll 1)$ a measure of smallness of the undulation. It is also assumed that the fluid is incompressible and inviscid, and the motion is irrotational. Assuming linear theory, we propose to solve for the complex-valued potential function $\phi(x, y)$ describing small motion in water, and satisfying the following equations:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \quad \text{in } -\infty < x < \infty, \quad 0 \leq y \leq h + \varepsilon c(x), \tag{1}$$

$$\frac{\partial\phi}{\partial y} + K\phi = 0 \quad \text{on} \quad y = 0, \tag{2}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on} \quad y = h + \varepsilon c(x), \tag{3}$$

where $K = \sigma^2/g, \sigma$ is the angular frequency of the incoming water wave train with time dependence $e^{-i\sigma t}$, g the acceleration due to gravity, and $\partial/\partial n$ the normal derivative at a point (x, y) on the bottom. The time dependent term is dropped throughout the analysis.

It can be assumed that a progressive wave train represented by the velocity potential

$$\phi_0(x, y) = \cosh k_0(h - y)e^{ik_0 x} \tag{4}$$

is incident upon the bottom undulation from negative infinity where k_0 is the wave number of the incident wave and is the unique positive root of the equation

$$K = k \tanh kh. \tag{5}$$

It is, then, partially reflected by and partially transmitted over the undulation so that ϕ has an asymptotic behaviour given by

$$\phi(x,y) \sim \begin{cases} \phi_0(x,y) + R\phi_0(-x,y), & x \to -\infty, \\ T\phi_0(x,y), & x \to +\infty, \end{cases} \tag{6}$$

where the constants R and T, respectively, are the usual reflection and transmission coefficients defined to be the ratio of amplitudes of the reflected and transmitted waves, respectively, to that of the incident wave, and are to be determined. Hence,

$$\phi(x,y) \sim \begin{cases} (e^{ik_0x} + Re^{-ik_0x}) \cosh k_0(h-y), & x \to -\infty, \\ Te^{ik_0x} \cosh k_0(h-y), & x \to +\infty. \end{cases} \tag{7}$$

2.2. Method of solution

The bottom condition $\partial\phi/\partial n = 0$ on $y = h + \varepsilon c(x)$ can be approximated up to the first order of the small parameter ε as

$$\frac{\partial \phi}{\partial y} - \varepsilon \frac{\partial}{\partial x}\left\{c(x)\frac{\partial \phi}{\partial x}\right\} = 0 \text{ on } y = h. \tag{8}$$

The boundary condition (8) and the fact that a wave train propagating in a sea of uniform finite depth experiences no reflection, together suggest that ϕ, R and T introduced above can be expressed in terms of ε as

$$\left.\begin{aligned} \phi &= \phi_0 + \varepsilon\phi_1 + O(\varepsilon^2) \\ R &= \varepsilon R_1 + O(\varepsilon^2) \\ T &= 1 + \varepsilon T_1 + O(\varepsilon^2) \end{aligned}\right\}. \tag{9}$$

Using (9) in (1), (2), (8) and (7) we find that $\phi_1(x,y)$ satisfies the BVP described by

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 0 \quad \text{in} \quad -\infty < x < \infty,\ 0 \le y \le h, \tag{10}$$

$$\frac{\partial \phi_1}{\partial y} + K\phi_1 = 0 \quad \text{on} \quad y = 0, \tag{11}$$

$$\frac{\partial \phi_1}{\partial y} = ik_0 \frac{d}{dx}\{c(x)e^{ik_0x}\} \equiv p(x) \quad \text{on} \quad y = h, \tag{12}$$

$$\text{and} \quad \phi_1(x,y) \sim \begin{cases} R_1 e^{-ik_0x} \cosh k_0(h-y), & \text{as} \quad x \to -\infty, \\ T_1 e^{ik_0x} \cosh k_0(h-y), & \text{as} \quad x \to +\infty. \end{cases} \tag{13}$$

To solve this boundary value problem we now assume that the first order potential $\phi_1(x,y)$ is such that Fourier transform of ϕ_1 with respect to x, denoted by $\overline{\phi}_1$, exists and is given by

$$\overline{\phi}_1(\xi,y) = \int_{-\infty}^{\infty} \phi_1(x,y)e^{i\xi x}dx, \tag{14}$$

together with the inverse

$$\phi_1(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\phi}_1(\xi,y)e^{-i\xi x}d\xi. \tag{15}$$

We observe that such Fourier transform exists if we make an artificial assumption that K possesses a small imaginary part, given by $i\mu'\sigma/g$, where $\mu' > 0$ is very small and will be taken to be zero (in eliminating sense) at the end of the analysis.

Now, taking Fourier transform of the governing equation (10) and boundary conditions (11) and (12), we obtain

$$\frac{\partial^2\overline{\phi}_1}{\partial y^2} - \xi^2\overline{\phi}_1 = 0 \quad \text{in } -\infty < \xi < \infty,\ 0 \le y \le h, \tag{16}$$

$$\frac{\partial\overline{\phi}_1}{\partial y} + K\overline{\phi}_1 = 0 \qquad \text{on } y = 0, \tag{17}$$

$$\frac{\partial\overline{\phi}_1}{\partial y} = \Lambda(\xi) \qquad \text{on } y = h, \tag{18}$$

where $$\Lambda(\xi) = \int_{-\infty}^{\infty} p(x)e^{i\xi x}dx. \tag{19}$$

The solution of (16) subject to the boundary conditions (17) and (18) is

$$\overline{\phi}_1(\xi,y) = \frac{\xi\cosh\xi y - K\sinh\xi y}{\xi[\xi\sinh\xi h - K\cosh\xi h]}\Lambda(\xi). \tag{20}$$

Taking inverse Fourier transform, the solution for the first-order velocity potential $\phi_1(x,y)$ can be written in the form

$$\phi_1(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\xi\cosh\xi y - K\sinh\xi y}{\xi[\xi\sinh\xi h - K\cosh\xi h]}\Lambda(\xi)e^{-i\xi x}d\xi. \tag{21}$$

We now obtain the final result from (21) by contour integration using the residue theorem.

We observe that equation (21) also has certain singularities (lying on the ξ-axis) other than $\xi = 0$. Replacing K by $\widehat{K} = (\sigma^2 + i\mu'\sigma)/g$ in equation (21), the singularities of (21) are displaced off the ξ-axis to the upper and the lower half planes. Hence, we write

$$\phi_1(x,y) = \lim_{\mu'\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(\xi,y)}{G_{\mu'}(\xi,h)}e^{-i\xi x}d\xi, \tag{22}$$

where

$$F(\xi,y) = [\xi\cosh\xi y - \widehat{K}\sinh\xi y]\Lambda(\xi), \tag{23}$$

$$G_{\mu'}(\xi,h) = \xi[\xi\sinh\xi h - \widehat{K}\cosh\xi h]. \tag{24}$$

If $\widehat{K} = \widehat{K}_1 + i\widehat{K}_2$, then $\widehat{K}_2 = \mu' K/\sigma$ which is very small and if $\zeta = \alpha + i\beta$ is a zero of the expression (24), then ζ can be determined as

$$\zeta = \pm\alpha_n \pm i\beta_n \quad \text{and} \quad \zeta = \pm(k_0 + \gamma) \pm i\beta_n' \tag{25}$$

where β_n's are roots of $\beta\tan\beta h + K = 0$ and

$$\left.\begin{aligned}\alpha_n = \widehat{K}_2\alpha_n^{(1)}, \quad \alpha_n^{(1)} &= \frac{-\cos\beta_n h}{(\widehat{K}_1 h - 1)\sin(\pm\beta_n h) - (\pm\beta_n h)\cos\beta_n h},\\ \gamma = \widehat{K}_2\alpha_n^{'(1)}, \quad \alpha_n^{'(1)} &= \frac{k_0^2 h - \widehat{K}_1[1+(k_0h)^2/2]}{\pm k_0\widehat{K}_1\widehat{K}_2 h - (\pm 2k_0\widehat{K}_2 h)},\\ \beta_n' = \widehat{K}_2\beta_n^{'(1)}, \quad \beta_n^{'(1)} &= \frac{2+(k_0h)^2}{\pm 4k_0 h \pm (k_0h)^3 - (\pm 2k_0h^2\widehat{K}_1)},\end{aligned}\right\} \tag{26}$$

where $\beta_n > 0$. Here the contour consists of the portion $-R$ to R on the real ξ-axis and a semicircle centered at the origin and having a large radius R. The semicircle must be taken in the upper half ζ plane ($\zeta = \xi + i\eta$) in anticlockwise direction or in the lower half plane in clockwise direction according as $x < 0$ or $x > 0$. In the limit as $R \to \infty$, the required range of integration is recovered, since the integration along the semicircle makes a zero contribution. Hence, by using the residue theorem,

$$\begin{aligned}\phi_1(x,y) &= \lim_{\mu'\to 0}(-i)\Bigg[\sum_{n=1}^{\infty}\text{Res}\left\{\frac{F(\zeta,y)\,e^{-i\zeta x}}{G_{\mu'}(\zeta,h)}\right\}\Bigg|_{\zeta=\alpha_n+i\beta_n}\\ &\quad +\text{Res}\left\{\frac{F(\zeta,y)\,e^{-i\zeta x}}{G_{\mu'}(\zeta,h)}\right\}\Bigg|_{\zeta=(k_0+\gamma)+i\beta_n'}\Bigg] \quad \text{for } x<0,\end{aligned} \tag{27}$$

and

$$\begin{aligned}\phi_1(x,y) &= \lim_{\mu'\to 0} i\Bigg[\sum_{n=1}^{\infty}\text{Res}\left\{\frac{F(\zeta,y)\,e^{-i\zeta x}}{G_{\mu'}(\zeta,h)}\right\}\Bigg|_{\zeta=-\alpha_n-i\beta_n}\\ &\quad +\text{Res}\left\{\frac{F(\zeta,y)\,e^{-i\zeta x}}{G_{\mu'}(\zeta,h)}\right\}\Bigg|_{\zeta=-(k_0+\gamma)-i\beta_n'}\Bigg] \quad \text{for } x>0,\end{aligned} \tag{28}$$

which imply that

$$\begin{aligned}\phi_1(x,y) &= \lim_{\mu'\to 0}(-i)\sum_{n=1}^{\infty}\text{Res}\left\{\frac{F(\zeta,y)\,e^{-i\zeta x}}{G_{\mu'}(\zeta,h)}\right\}\Bigg|_{\zeta=\alpha_n+i\beta_n}\\ &\quad + \frac{-2ik_0^2}{2k_0h+\sinh 2k_0h}\left\{\int_{-\infty}^{\infty} c(x)e^{2ik_0x}dx\right\}\cosh k_0(h-y)e^{-ik_0x}\\ &\quad \text{for } x<0,\end{aligned} \tag{29}$$

and

$$\begin{aligned}\phi_1(x,y) &= \lim_{\mu'\to 0} i\sum_{n=1}^{\infty}\text{Res}\left\{\frac{F(\zeta,y)\,e^{-i\zeta x}}{G_{\mu'}(\zeta,h)}\right\}\Bigg|_{\zeta=-\alpha_n-i\beta_n}\\ &\quad + \frac{2ik_0^2}{2k_0h+\sinh 2k_0h}\left\{\int_{-\infty}^{\infty} c(x)dx\right\}\cosh k_0(h-y)e^{ik_0x}\\ &\quad \text{for } x>0.\end{aligned} \tag{30}$$

The first term on right hand side of each of equations (29) and (30) represents the non-propagating modes which decay rapidly away from the undulation and the second term represents a propagating mode from the region of the bed disturbance. Comparing equations (29) and (30) with equation (13), the reflection and transmission coefficients can, respectively, be written as

$$R_1 = \frac{-2ik_0^2}{2k_0h + \sinh 2k_0h} \int_{-\infty}^{\infty} c(x) e^{2ik_0x} dx, \tag{31}$$

and

$$T_1 = \frac{2ik_0^2}{2k_0h + \sinh 2k_0h} \int_{-\infty}^{\infty} c(x) dx. \tag{32}$$

The results (31) and (32) may be interpreted as the results obtained in [14] for normal incidence and in [9] for normal incidence when surface tension is negligible.

R_1 and T_1 in equations (31) and (32) can be evaluated once the shape function $c(x)$ is known. Next we consider a special form of the function $c(x)$.

2.3. Example of a special bed surface

We now consider the interaction of progressive surface waves with a patch of sinusoidal ripples on the bed which do not imply any restrictions on the bed wave number. The bed surface is given by

$$c(x) = \begin{cases} a\sin(lx+\delta), & L_1 \le x \le L_2, \\ 0 & \text{otherwise}, \end{cases} \tag{33}$$

where

$$L_1 = \frac{-n\pi - \delta}{l}, \quad L_2 = \frac{m\pi - \delta}{l},$$

with a and l as the amplitude and the wave number, respectively, of the sinusoidal ripples and δ an arbitrary phase angle; and m and n as positive integers. This represents a patch of sinusoidal ripples on an otherwise flat bottom, the patch consisting of $(n+m)/2$ ripples having the same wave number l. For this specific bed profile we obtain the reflection and transmission coefficients, respectively, as

$$\begin{aligned} R_1 &= \frac{-2ik_0^2 a}{2k_0h + \sinh 2k_0h} \frac{l}{l^2 - (2k_0)^2} \\ &\times \left[(-1)^n e^{2ik_0L_1} - (-1)^m e^{2ik_0L_2}\right] \end{aligned} \tag{34}$$

and

$$T_1 = \frac{2ik_0^2}{2k_0h + \sinh 2k_0h} \frac{(-a)}{l} \left[(-1)^m - (-1)^n\right]. \tag{35}$$

For the case in which there is an integer number of ripple wavelengths in the patch $L_1 \le x \le L_2$ such that $m = n$ and $\delta = 0$, as considered in [5] also, we find R_1 and T_1, respectively, as

$$R_1 = \frac{2k_0a}{2k_0h + \sinh 2k_0h} \frac{(-1)^m(2k_0/l)}{(2k_0/l)^2 - 1} \sin\left(\frac{2k_0m\pi}{l}\right), \tag{36}$$

and

$$T_1 = 0. \tag{37}$$

These results match exactly with those obtained in [5]. Equation (36) illustrates that for a given number of m ripples, the first order wave reflection coefficient is an oscillatory function in the quotient of twice the surface wave number and the ripple wave number. Furthermore, if the bed wave number is equal to twice the surface wave number, i.e., $2k_0 = l$, equation (36) reveals that there is a resonant Bragg-type interaction between the surface waves and bed forms. This resonant interaction is reported in [3], [4] and [13]. The resonant interaction over sandbars is described in [7] with experiment demonstration. Hence, at resonance, we find from equation (36) that

$$R_1 = \frac{k_0 a m \pi}{2k_0 h + \sinh 2k_0 h}, \tag{38}$$

from which we observe that the reflection coefficient increases linearly in the number of ripples m in the patch. It indicates that relatively few bottom undulations, with wave number equal to approximately twice the surface wave number, may give rise to a very substantial reflected wave. A possible consequence of this is a coupling between ripple growth and wave reflection, which may be important for problems of coastal protection.

3. Case II: Oblique incidence

3.1. Statement and formulation

We consider a right-handed rectangular Cartesian co-ordinate system in which xz-plane is the position of the undisturbed free surface of the sea and the y-axis is measured positive vertically downwards from the undisturbed free surface. Considering the same small undulation and other assumptions as in the formulation for the normal incidence case as described in Section 2.1, we proceed to handle the case of oblique incidence. Assuming linear theory, we want to solve for the complex-valued potential function $\psi(x, y, z)$ describing the small motion in water and satisfying the following equations:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0 \qquad \text{in } 0 \le y \le h + \varepsilon c(x), \tag{39}$$

$$\frac{\partial\psi}{\partial y} + K\psi = 0 \qquad \text{on } y = 0, \tag{40}$$

$$\frac{\partial\psi}{\partial n} = 0 \qquad \text{on } \quad y = h + \varepsilon c(x), \tag{41}$$

where $\partial/\partial n$ denotes the normal derivative at a point (x, y, z) on the bottom.

It can be assumed that a progressive wave train represented by the velocity potential

$$\psi_0(x, y, z) = \cosh k_0(h - y)e^{i(\mu x + \nu z)} \tag{42}$$

is obliquely incident upon the bottom undulation from negative infinity, where k_0, the wave number of the incident wave, is the unique positive real root of equation (5) and

$$\mu = k_0 \cos\theta, \quad \nu = k_0 \sin\theta \quad (0 \le \theta < \pi/2), \tag{43}$$

where θ is the angle of oblique incidence of the wave train ($\theta = 0$ corresponds to normal incidence), μ and ν are, respectively, the x and z components of k_0.

It is, then, partially reflected by and partially transmitted over the undulation so that ψ has the asymptotic behaviour given by

$$\psi(x,y,z) \sim \begin{cases} \psi_0(x,y,z) + R\psi_0(-x,y,z), & \text{as } x \to -\infty, \\ T\psi_0(x,y,z), & \text{as } x \to +\infty. \end{cases}$$

which imply

$$\psi(x,y,z) \sim \begin{cases} (e^{i\mu x} + Re^{-i\mu x})e^{i\nu z} \cosh k_0(h-y), & \text{as } x \to -\infty, \\ Te^{i(\mu x + \nu z)} \cosh k_0(h-y), & \text{as } x \to +\infty. \end{cases} \tag{44}$$

3.2. Method of solution

The bottom condition (41) can be approximated up to the first order of the small parameter ε as

$$\frac{\partial \psi}{\partial y} - \varepsilon \left[\frac{\partial}{\partial x} \left\{ c(x) \frac{\partial \psi}{\partial x} \right\} + c(x) \frac{\partial^2 \psi}{\partial z^2} \right] = 0 \text{ on } y = h. \tag{45}$$

Now, in view of the geometry of the problem, i.e., because of the uniformity in the z-direction, $\psi(x,y,z)$ can be written as

$$\psi(x,y,z) = \phi(x,y)e^{i\nu z}. \tag{46}$$

Then $\phi(x,y)$ satisfies the equations

$$(\nabla^2 - \nu^2)\phi = 0 \quad \text{in } -\infty < x < \infty,\ 0 \le y \le h, \tag{47}$$

$$\frac{\partial \phi}{\partial y} + K\phi = 0 \qquad \text{on} \quad y = 0, \tag{48}$$

$$\frac{\partial \phi}{\partial y} - \varepsilon \left[\frac{\partial}{\partial x} \left\{ c(x) \frac{\partial \phi}{\partial x} \right\} - \nu^2 c(x)\phi(x,y) \right] = 0 \quad \text{on } y = h, \tag{49}$$

$$\phi(x,y) \sim \begin{cases} (e^{i\mu x} + Re^{-i\mu x}) \cosh k_0(h-y), & \text{as } x \to -\infty, \\ Te^{i\mu x} \cosh k_0(h-y), & \text{as } x \to +\infty. \end{cases} \tag{50}$$

where ∇^2 is the two-dimensional Laplacian operator.

Thus the bottom condition, in effect, reduces approximately to a condition on $y = h$. This suggests that we may consider the governing partial differential equation for ϕ to hold in the strip $0 \le y \le h, -\infty < x < \infty$ along with the boundary conditions (48), (49) and the far field requirements (50). It may be noted here that both R and T depend upon ε.

In view of the boundary condition (49) and with the similar consideration as in the normal incidence case, we can express ϕ, R and T in terms of the perturbation parameter ε as in equation (9) with

$$\phi_0(x,y) = \cosh k_0(h-y)e^{i\mu x}. \tag{51}$$

Using $\phi_0(x,y)$ in (9) and applying it to (47)-(50) we find that $\phi_1(x,y)$ satisfies equation (47) and the condition (48) together with the following additional conditions:

$$\frac{\partial \phi_1}{\partial y} = i\mu \frac{d}{dx}\left\{c(x)e^{i\mu x}\right\} - \nu^2 c(x)e^{i\mu x} \equiv V(x) \quad \text{on } y = h, \tag{52}$$

$$\text{and} \quad \phi_1(x,y) \sim \begin{cases} R_1 e^{-i\mu x}\cosh k_0(h-y), & \text{as } x \to -\infty, \\ T_1 e^{i\mu x}\cosh k_0(h-y), & \text{as } x \to +\infty. \end{cases} \tag{53}$$

Applying Fourier transform, as given by equation (14), to the boundary value problem, we get

$$\overline{\phi}_1(\widehat{\xi}, y) = \frac{\widehat{\xi}\cosh\widehat{\xi}y - K\sinh\widehat{\xi}y}{\widehat{\xi}[\widehat{\xi}\sinh\widehat{\xi}h - K\cosh\widehat{\xi}h]}\Lambda(\xi). \tag{54}$$

where $\widehat{\xi}^2 = \xi^2 + \nu^2$ and

$$\Lambda(\xi) = \int_{-\infty}^{\infty} V(x)e^{i\xi x}dx. \tag{55}$$

Taking inverse Fourier transform, the solution $\phi_1(x,y)$ for the first-order velocity potential can be found as

$$\phi_1(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\widehat{\xi}\cosh\widehat{\xi}y - K\sinh\widehat{\xi}y}{\widehat{\xi}[\widehat{\xi}\sinh\widehat{\xi}h - K\cosh\widehat{\xi}h]}\Lambda(\xi)e^{-i\xi x}d\xi. \tag{56}$$

We now obtain the final result from (56) by contour integration using the residue theorem as done previously.

Here also we observe that the equation (56) has certain singularities (lying on the ξ-axis) other than $\widehat{\xi} = 0$. Replacing K by $\widehat{K}$ (as defined in Section 2.2) in equation (56), the singularities of (56) are displaced off the ξ-axis to the upper and the lower half planes. Hence, we write

$$\phi_1(x,y) = \lim_{\mu' \to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(\widehat{\xi},y)}{G_{\mu'}(\widehat{\xi},h)}\, e^{-i\xi x}d\xi, \tag{57}$$

where

$$F(\widehat{\xi},y) = [\widehat{\xi}\cosh\widehat{\xi}y - \widehat{K}\sinh\widehat{\xi}y]\Lambda(\xi), \tag{58}$$

$$G_{\mu'}(\widehat{\xi},h) = \widehat{\xi}[\widehat{\xi}\sinh\widehat{\xi}h - \widehat{K}\cosh\widehat{\xi}h]. \tag{59}$$

If $\widehat{K} = \widehat{K}_1 + i\widehat{K}_2$, and if $\hat{\zeta} = \alpha + i\beta$ is a zero of the expression (59), then $\hat{\zeta}$ can be determined as

$$\hat{\zeta} = \pm\alpha_n \pm i\beta_n \quad \text{and} \quad \hat{\zeta} = \pm(k_0 + \gamma) \pm i\beta_n'. \tag{60}$$

where $\alpha_n, \beta_n, \gamma$ and β_n' are given by equation (26).

Substituting $\hat{\zeta} = \pm\sqrt{\zeta^2 + \nu^2}$, the roots $\hat{\zeta} = \pm\alpha_n \pm i\beta_n$ give

$$\zeta \approx \pm\sqrt{-(\beta_n^2 + \nu^2) \pm i\widehat{K}_2(2\alpha_n'^{(1)}\beta_n)}$$

up to the first order of $\widehat{K}_2$, which gives

$$\zeta \approx \pm i s_n \text{ as } \mu' \to 0, \text{ where } s_n = \sqrt{\beta_n^2 + \nu^2}, \tag{61}$$

and the roots $\hat{\zeta} = \pm(k_0 + \gamma) \pm i\beta_n'$ give

$$\zeta \approx \pm k_0 \cos\theta = \pm\mu \tag{62}$$

up to the first order of $\widehat{K}_2$ and when $\mu' \to 0$. Again we consider the same contour as in Section 2.2 and perform the integration along it. Then by using the residue theorem,

$$\begin{aligned} \phi_1(x,y) &= \lim_{\mu' \to 0}(-i)\Bigg[\sum_{n=1}^{\infty} \operatorname{Res}\left\{\frac{F(\hat{\zeta},y)\, e^{-i\zeta x}}{G_{\mu'}(\hat{\zeta},h)}\right\}\Bigg|_{\zeta = i s_n} \\ &+ \operatorname{Res}\left\{\frac{F(\hat{\zeta},y)\, e^{-i\zeta x}}{G_{\mu'}(\hat{\zeta},h)}\right\}\Bigg|_{\zeta=\mu}\Bigg] \text{ for } x < 0, \end{aligned} \tag{63}$$

and

$$\begin{aligned} \phi_1(x,y) &= \lim_{\mu' \to 0}(i)\Bigg[\sum_{n=1}^{\infty} \operatorname{Res}\left\{\frac{F(\hat{\zeta},y)\, e^{-i\zeta x}}{G_{\mu'}(\hat{\zeta},h)}\right\}\Bigg|_{\zeta = -i s_n} \\ &+ \operatorname{Res}\left\{\frac{F(\hat{\zeta},y)\, e^{-i\zeta x}}{G_{\mu'}(\hat{\zeta},h)}\right\}\Bigg|_{\zeta=-\mu}\Bigg] \text{ for } x > 0. \end{aligned} \tag{64}$$

After simplifying and then comparing equations (63) and (64) with equation (53), the reflection and transmission coefficients can, respectively, be written as

$$R_1 = \frac{-2ik_0^2 \sec\theta \cos 2\theta}{2k_0 h + \sinh 2k_0 h} \int_{-\infty}^{\infty} c(x) e^{2i\mu x} dx, \tag{65}$$

and

$$T_1 = \frac{2ik_0^2 \sec\theta}{2k_0 h + \sinh 2k_0 h} \int_{-\infty}^{\infty} c(x) dx. \tag{66}$$

The results (65) and (66) may be interpreted as the results obtained in [14] and in [9] when surface tension is negligible. It is observed, as the expression of R_1 contains a $\cos 2\theta$ term, that for oblique incidence at $\theta = \frac{\pi}{4}$ of a wave train, the reflection coefficient R_1 up to the first order vanishes independently of the shape of bottom undulation, as also mentioned in [14] and in [9]. Also the results for normal incidence of Section 2 can be obtained by putting $\theta = 0$.

Equations (65) and (66) can be evaluated once the shape function $c(x)$ is known. Next we consider a special form for the function $c(x)$.

3.3. A special bed surface

Now consider the interaction of progressive surface waves with a patch of sinusoidal ripples on the bed where the patch is given by equation (33).

For this case we obtain the reflection and transmission coefficients, respectively, as

$$R_1 = \frac{-2ik_0^2 a \sec\theta \cos 2\theta}{2k_0 h + \sinh 2k_0 h} \frac{l}{l^2 - (2\mu)^2} \Big[(-1)^n e^{2i\mu L_1} - (-1)^m e^{2i\mu L_2} \Big], \tag{67}$$

and

$$T_1 = \frac{2ik_0^2 \sec\theta}{2k_0 h + \sinh 2k_0 h} \frac{(-a)}{l} \Big[(-1)^m - (-1)^n \Big]. \tag{68}$$

For the case in which there is an integer number of ripple wavelengths in the patch $L_1 \leq x \leq L_2$ such that $m = n$ and $\delta = 0$, we find R_1 and T_1, respectively, as

$$R_1 = \frac{2k_0 a \sec\theta \cos 2\theta}{2k_0 h + \sinh 2k_0 h} \frac{(-1)^m (2k_0/l)}{(2\mu/l)^2 - 1} \sin\left(\frac{2\mu m\pi}{l}\right), \tag{69}$$

and

$$T_1 = 0. \tag{70}$$

These results exactly match with those obtained in [5] when $\theta = 0$. Equation (69) illustrates that for a given number of m ripples, the first order wave reflection coefficient is an oscillatory function in the ratio of twice the x-component of the wave number and the ripple wave number. Furthermore, if the bed wave number is twice the x-component of the wave number $(2\mu/l = 1)$, then equation (69) reveals that there is a resonant Bragg-type interaction between the surface waves and bed forms as described in Section 2.3. Hence, at resonance, we find from equation (69) that

$$R_1 = \frac{k_0 a \sec^2\theta \cos 2\theta}{2k_0 h + \sinh 2k_0 h} m\pi, \tag{71}$$

from which we observe that R_1 becomes a constant multiple of m, the number of ripples in the patch. It indicates that relatively few bottom undulations with its wave number equal to approximately twice the x-component of the surface wave number, may give rise to a very substantial reflected wave as described in Section 2.3.

4. Numerical results

The numerical computation is shown here for the first order reflection coefficient given by equation (69). In figure 1, $|R_1|$ is plotted against the wave number $k_0 h$ for one, three and five ripples with $a/h = 0.1, lh = 1$. From the graph, for the single ripple, it is clear that its peak value is attained when the wave number of the bottom undulation lh becomes approximately twice as large as surface wave number $k_0 h$. This is most evident in the curve which has its maximum value 0.0732 at $k_0 h = 0.5442$.

Again when the number of ripples is increased to three, the same general feature of $|R_1|$ is now observed with the modification that the overall values of R_1 is now increased to 0.2167 at $k_0 h = 0.5442$ in comparison to the case of $m = 1$; the oscillating nature of $|R_1|$ against $k_0 h$ is more pronounced and the number of zeros of $|R_1|$ also increases. Again this phenomenon becomes clearly

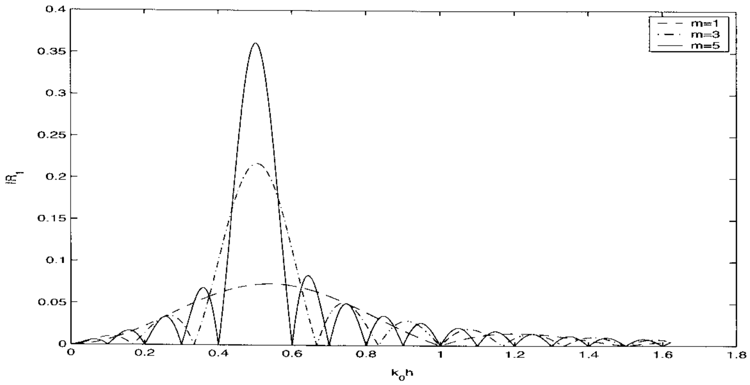

FIGURE 1. Reflection coefficient against the wave number k_0h for $\theta = 0; a/h = 0.1; lh = 1; m = 1, 3$ and 5.

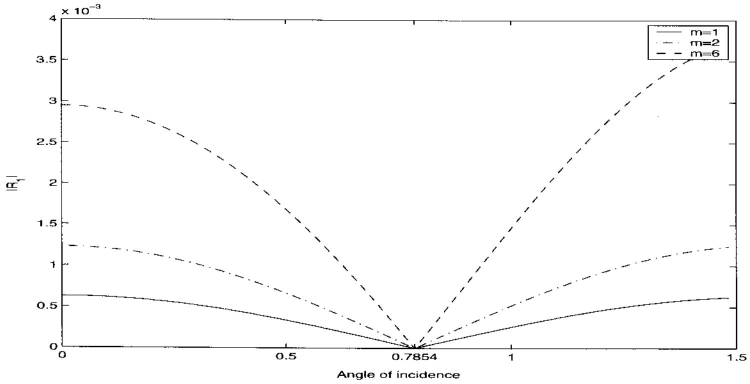

FIGURE 2. Reflection coefficient against the angle of incidence θ for $k_0h = 0.03163; a/h = 0.1; lh = 1; m = 1, 2$ and 6

evident when m is increased to 5 and the maximum value of $|R_1|$ is 0.3607 at $k_0h = 0.5$. A general observation that follows is that as the number of ripples increases, the peak value of $|R_1|$ increases and it becomes more oscillatory.

In figure 2, the reflection coefficient $|R_1|$ is depicted against the angle of incidence θ for $k_0h = 0.03163, a/h = 0.1, lh = 1$ and for ripples $m = 1, m = 2$ and $m = 6$, respectively. From the graph it is clear that for $\theta = \frac{\pi}{4}$, $|R_1|$ vanishes independently of the shape of the function which validates equation (65).

5. Conclusion

Fourier transform method is used to solve the problem of surface water wave interaction with small undulation on an otherwise flat bottom of a sea for both the normal and oblique incidences. Initially, by using perturbation technique, the formulated boundary value problem is reduced to one in the first order potentials whereby obtaining the reflection and transmission coefficients. Due to the presence of singularities in the integrals, residue theorem is employed to evaluate the integrals appearing in the first order correction of the potential. After deriving the velocity potential, the reflection and transmission coefficients up to first order are obtained. Application of these results for a sinusoidal bottom undulations yields results which coincide exactly with the results for the same given in the literature. From the numerical results it is observed that the reflection coefficient increases when the number of ripples increases. It indicates that relatively few bottom undulations, at resonance, may give rise to a very substantial reflected wave field. A possible consequence of this is a coupling between ripple growth and wave reflection, which may be important for problems of coastal protection. The present method clearly has a more general approach than the one employed in [5] and it has definite advantages over some other methods which employ Green's function technique. The procedure followed here is simple and very easy to apply.

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