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CLASSIFICATION OF REFLEXIBLE EDGE-TRANSITIVE EMBEDDINGS OF $K_{m,n}$ FOR ODD m, n

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Abstract. In this paper, we classify reflexible edge-transitive embeddings of complete bipartite graphs $K_{m,n}$ for any odd positive integers m and n. As a result, for any odd m, n, it will be shown that there exists only one reflexible edge-transitive embedding of $K_{m,n}$ up to isomorphism.

1. Preliminaries

In this paper, we consider a 2-cell embedding of a simple connected graph into a closed orientable surface, simply called a (orientable) *map*. The embedded graph is called the *underlying graph* of the map.

For a simple connected graph G, an *arc* is an ordered pair (u, v) of adjacent vertices in G. In a combinatorial way, a map $\mathcal{M}$ can be described by a pair $\mathcal{M} = (G; R)$, where R is a permutation of the arc set D whose orbits coincide with the sets of arcs based at the same vertex. The permutation R is called the *rotation* of the map $\mathcal{M}$. In the cycle decomposition of the permutation R, the cycle permuting the arcs based at a vertex v is said to be the *local rotation* R_v at v.

Given two maps $\mathcal{M}_1 = (G_1; R_1)$ and $\mathcal{M}_2 = (G_2; R_2)$, a *map isomorphism* $\phi : \mathcal{M}_1 \to \mathcal{M}_2$ is a graph isomorphism $\phi : G_1 \to G_2$ such that $\phi R_1(u, v) = R_2\phi(u, v)$ for any arc (u, v) in G_1. Furthermore, if $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M} = (G; R)$, ϕ is called a *map automorphism* of $\mathcal{M}$. The set of all automorphisms of $\mathcal{M}$ is a group under composition, called the *automorphism group* of $\mathcal{M}$ and denoted by $\mathrm{Aut}\,(\mathcal{M})$. Since $\mathrm{Aut}\,(\mathcal{M})$ is a subgroup of $\mathrm{Aut}\,(G)$, we consider it as an acting group on the vertex set $V(G)$, the edge set $E(G)$ or the arc set D according to the context. The group $\mathrm{Aut}\,(\mathcal{M})$ of $\mathcal{M}$ acts semi-regularly on the arc set D. If it acts regularly, the map is called *regular*. The map $\mathcal{M}$ is said to be *vertex-transitive* and *edge-transitive* if $\mathrm{Aut}\,(\mathcal{M})$ acts transitively on $V(G)$ and $E(G)$, respectively. When G is bipartite, if the set of partite set preserving map automorphisms acts transitively on $E(G)$ then $\mathcal{M}$ is called *edge-transitive*

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by partite set preserving automorphisms. For a map $\mathcal{M} = (G; R)$, if $\mathcal{M}$ and $\mathcal{M}^{-1} = (G; R^{-1})$ are isomorphic, $\mathcal{M}$ is called *reflexible* and if not, *chiral.*

Classifying highly symmetric embeddings of a given class of graphs is an important problem in topological graph theory. In recent years, there has been particular interest in the regular embeddings of complete bipartite graphs $K_{n,n}$ by several authors [1]-[9]. The reflexible regular embeddings and self-Petrie dual regular embeddings of $K_{n,n}$ into orientable surfaces were classified in [7]. Recently, the classification of regular embeddings of $K_{n,n}$ is completed by G. Jones [3] and J. H. Kwak and the author classify the nonorientable regular embeddings of $K_{n,n}$ [8]. In this paper, we classify the reflexible edge-transitive embeddings of $K_{m,n}$ for any odd positive integers m, n. Note that for different m and n, edge-transitive embeddings of $K_{m,n}$ are the most symmetric ones.

This paper is organized as follows. In the next section, a relation between edge-transitive embeddings of $K_{m,n}$ and products of two cyclic groups with generators is considered. In Section 3, we consider products of two cyclic groups with generators in the automorphism group $\mathrm{Aut}\,(K_{m,n})$ of complete bipartite graph $K_{m,n}$. In the final section, the classification of reflexible edge-transitive embeddings of $K_{m,n}$ is given for odd m, n.

2. Edge-transitive embeddings and bicyclic triples

Regular embeddings of complete bipartite graph $K_{n,n}$ are related to groups Γ with two generators satisfying some conditions [4]. Using this relation, G. Jones classify regular embeddings of $K_{n,n}$ [3]. In this section, we consider similar kind of relations between edge-transitive embeddings of $K_{m,n}$ by partite set preserving automorphisms and groups with two generators satisfying some conditions.

In [4], G. Jones et al. showed that any finite group Γ is isomorphic to $\mathrm{Aut}\,(\mathcal{M})$ for some regular embedding of $K_{n,n}$ if and only if Γ has cyclic subgroups $X = \langle x \rangle$ and $Y = \langle y \rangle$ of order n such that (i) $\Gamma = XY$ (ii) $X \cap Y = 1$ (iii) there is an automorphism α of Γ transposing x and y. They call the triple (Γ, x, y) satisfying the above conditions the *n-isobicyclic* triple. For any n-isobicyclic triple (Γ, x, y), the corresponding embedding and its underlying graph can be defined as follows: the vertices of the underlying graph are the cosets gX and gY of X and Y in Γ, the edges are the elements of Γ and the incidence is given by containment. By conditions $(i), (ii)$, the underlying graph is a complete bipartite graph $K_{n,n}$. The elements x and y define a cyclic order $g, gx, gx^2, \ldots$ or $g, gy, gy^2, \ldots$ of edges around each vertex gX or gY. These local orientations determine local rotations, that is, an embedding $\mathcal{M}$ of the complete bipartite graph $K_{n,n}$. For any $h \in \Gamma$, a left multiplication L_h sending an edge g to $h^{-1}g$ and vertices gX and gY to $h^{-1}gX$ and $h^{-1}gY$, respectively, is a map automorphism. Hence, the map $\mathcal{M}$ is edge-transitive by partite set preserving automorphisms. Furthermore, the map $\mathcal{M}$ is regular by the third condition of n-isobicyclic. For two n-isobicyclic triples (Γ_1, x_1, y_1)

and (Γ_2, x_2, y_2), two induced regular embeddings $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic if and only if there exists a group isomorphism from Γ_1 to Γ_2 given by $x_1 \mapsto x_2$ and $y_1 \mapsto y_2$. (For more information, the reader is referred to [4].) Note that for any n-isobicyclic triple (Γ, x, y) and its induced regular embedding $\mathcal{M}$, the reflection $\mathcal{M}^{-1}$ is also regular and $\mathcal{M}^{-1}$ is isomorphic to the regular embedding induced by n-isobicyclic triple (Γ, x^{-1}, y^{-1}). Therefore, the map $\mathcal{M}$ is reflexible if and only if there exists an automorphism β of Γ which sends x and y to x^{-1} and y^{-1}, respectively.

For different positive integers m and n, the most symmetric embedding of the complete bipartite graph $K_{m,n}$ is edge-transitive one. As one see, only using the first and second conditions of n-isobicyclic triple, one can define an embedding of $K_{n,n}$ and show that the induced map is edge-transitive. Hence, edge-transitive embedding of $K_{m,n}$ can be also represented by a group with two generators. For a group Γ having cyclic subgroups $X = \langle x \rangle$ of order m and $Y = \langle y \rangle$ of order n, we call the triple (Γ, x, y) (m, n)-*bicyclic* if it satisfies that (i) $\Gamma = XY$ (ii) $X \cap Y = 1$. Then, for any (m, n)-bicyclic triple (Γ, x, y), one can define an embedding of $K_{m,n}$ by similar way to define an embedding of $K_{n,n}$ with n-isobicyclic triple. We denote this embedding by $\mathcal{M}(\Gamma, x, y)$. One can see that $\mathcal{M}(\Gamma, x, y)$ is edge-transitive. Furthermore, every edge-transitive embedding of $K_{m,n}$ by partite set preserving automorphisms is isomorphic to $\mathcal{M}(\Gamma, x, y)$ for some (m, n)-bicyclic triple (Γ, x, y). Note that the edge set in $\mathcal{M}(\Gamma, x, y)$ is Γ itself. For two (m, n)-bicyclic triples (Γ_1, x_1, y_1) and (Γ_2, x_2, y_2), if there exists a group isomorphism ϕ from Γ_1 to Γ_2 given by $x_1 \mapsto x_2$ and $y_1 \mapsto y_2$ then two induced edge-transitive embeddings $\mathcal{M}(\Gamma_1, x_1, y_1)$ and $\mathcal{M}(\Gamma_2, x_2, y_2)$ are isomorphic by the map isomorphism induced by ϕ. Conversely, if two edge-transitive embeddings $\mathcal{M}(\Gamma_1, x_1, y_1)$ and $\mathcal{M}(\Gamma_2, x_2, y_2)$ are isomorphic, then there exists an isomorphism ψ which sends the edge labelled the identity in Γ_1 to the edge labelled the identity in Γ_2. Then, if we consider ψ as mapping from Γ_1 to Γ_2, ψ preserves group operations and hence it is an isomorphism from Γ_1 to Γ_2 given by $x_1 \mapsto x_2$ and $y_1 \mapsto y_2$. In summary, we have the following lemma.

Lemma 2.1. *Let m, n be two positive integers(not necessarily distinct).*

(1) *For any edge-transitive embedding $\mathcal{M}$ of $K_{m,n}$ by partite set preserving automorphisms, $\mathcal{M}$ is isomorphic to $\mathcal{M}(\Gamma, x, y)$ for some (m, n)-bicyclic triple (Γ, x, y).*
(2) *For two (m, n)-bicyclic triples (Γ_1, x_1, y_1) and (Γ_2, x_2, y_2), two edge-transitive embeddings $\mathcal{M}(\Gamma_1, x_1, y_1)$ and $\mathcal{M}(\Gamma_2, x_2, y_2)$ are isomorphic if and only if there exists a group isomorphism from Γ_1 to Γ_2 given by $x_1 \mapsto x_2$ and $y_1 \mapsto y_2$.*

For any (m, n)-bicyclic triple (Γ, x, y), there exists a subgroup of the automorphism group $\mathrm{Aut}\,(K_{m,n})$ which is isomorphic to Γ. Hence, one can consider x and y as permutations of vertices of $K_{m,n}$. In the next section, we deal with (m, n)-bicyclic triple in $\mathrm{Aut}\,(K_{m,n})$.

3. (m,n)-bicyclic triples in Aut $(K_{m,n})$

For any positive integer n, let $[n]$ denote the set $\{0,1,\ldots,n-1\}$. Let $V=\{0,1,\ldots,(n-1)\}\cup\{0',1',\ldots,(m-1)'\}=[n]\cup[m]'$ be the vertex set of $K_{m,n}$ as partite sets, and $D=\{(i,j'),(j',i)\mid 0\le i\le n-1 \text{ and } 0\le j\le m-1\}$ the arc set, where (i,j') is the arc emanating from i to j' and (j',i) denotes its inverse. We denote the symmetric group on $[n]$ and $[m]'$ by S and S', respectively. Let S_0 and S'_0 be their stabilizer of 0 and $0'$. Note that Aut $(K_{m,n})$ is isomorphic to $S\times S'$ when $m\ne n$; $S\wr\mathbb{Z}_2$ when $m=n$. We identify the integers $0,1,2,\ldots$ with their residue classes modulo m or n according to the context.

For any (m,n)-bicyclic triple (Γ,x,y), since Γ is isomorphic to a subgroup of Aut $(K_{m,n})$, one can consider x and y as permutations of vertices of $K_{m,n}$. Furthermore, one can restrict $x\in$ Aut $(K_{m,n})_0$, the stabilizer of 0 in Aut $(K_{m,n})$ and $y\in$ Aut $(K_{m,n})_{0'}$, the stabilizer of $0'$ in Aut $(K_{m,n})$. Since Aut $(K_{m,n})$ contains all permutations of vertices of each partite set, there exists a $\phi\in$ Aut $(K_{m,n})$ such that

$$x^{\phi}=\phi^{-1}x\phi=\alpha(0'\ 1'\ \cdots\ (m-1)')\quad\text{and}\quad y^{\phi}=\phi^{-1}y\phi=\beta(0\ 1\ \cdots\ n-1)$$

for some $\alpha\in S_0$ and $\beta\in S'_0$. From now on, for any $\alpha\in S_0$ and $\beta\in S'_0$, let

$$x_\alpha=\alpha(0'\ 1'\ \cdots\ (m-1)')\quad\text{and}\quad y_\beta=\beta(0\ 1\ \cdots\ n-1)$$

for our convenience. Then, it suffices to consider x_α and y_β as candidate for the second and third coordinates of (m,n)-bicyclic triple to classify edge-transitive embeddings of $K_{m,n}$ by partite set preserving automorphisms.

Lemma 3.1. *For any $\alpha\in S_0$ and $\beta\in S'_0$, the group $\langle x_\alpha,y_\beta\rangle$ acts transitively on the edge set of $K_{m,n}$.*

Proof. For any $i\in[n]$ and $j'\in[m]'$, we have

$$y_\beta^i x_\alpha^{\beta^{-i}(j)}(0)=y_\beta^i(0)=i\quad\text{and}\quad y_\beta^i x_\alpha^{\beta^{-i}(j)}(0')=y_\beta^i(\beta^{-i}(j)')=j'.$$

Hence, for any $i\in[n]$ and $j'\in[m]'$, the edge incident to 0 and $0'$ can be sent to the edge incident to i and j' by an element in $\langle x_\alpha,y_\beta\rangle$. □

For any $\alpha\in S_0$ and $\beta\in S'_0$, let $\Gamma=\langle x_\alpha,y_\beta\rangle$. Then, $|\Gamma|=mn$ is a necessary condition for the triple $(\Gamma,x_\alpha,y_\beta)$ to be (m,n)-bicyclic. The next lemma shows that it is also a sufficient condition.

Lemma 3.2. *For any $\alpha\in S_0$ and $\beta\in S'_0$, the triple $(\langle x_\alpha,y_\beta\rangle,x_\alpha,y_\beta)$ is (m,n)-bicyclic if and only if $|\langle x_\alpha,y_\beta\rangle|=mn$.*

Proof. 'Only if' is trivial. Hence, it suffices to show 'if' part. Let $\Gamma=|\langle x_\alpha,y_\beta\rangle|$ and assume that $|\Gamma|=mn$. By Lemma 3.1, Γ acts regularly on the edge set of $K_{m,n}$. Since $x_\alpha^m(0)=0$ and $x_\alpha^m(0')=0'$, namely, x_α^m fixes the arc $(0,0')$, x_α^m is the identity element. It means that $\langle x_\alpha\rangle$ is a cyclic group of order m. Similarly, one can show that $\langle y_\beta\rangle$ is a cyclic group of order n. If $g\in\langle x_\alpha\rangle\cap\langle y_\beta\rangle$

then $g(0) = 0$ and $g(0') = 0'$ which implies that g is the identity. Hence, $\langle x_\alpha \rangle \cap \langle y_\beta \rangle = \{1\}$. Since $|\langle x_\alpha \rangle \cdot \langle y_\beta \rangle| = \frac{|\langle x_\alpha \rangle| \cdot |\langle y_\beta \rangle|}{|\langle x_\alpha \rangle \cap \langle y_\beta \rangle|} = mn$, $\Gamma = \langle x_\alpha \rangle \cdot \langle y_\beta \rangle$. Therefore, $(\Gamma, x_\alpha, y_\beta)$ is an (m, n)-bicyclic triple. □

By Lemma 3.2, we need to characterize $\alpha \in S_0$ and $\beta \in S'_0$ satisfying $|\langle x_\alpha, y_\beta \rangle| = mn$ to classify the edge-transitive embeddings of $K_{m,n}$ by partite set preserving automorphisms. To do this, we denote

$$\mathrm{ET}_{m,n} = \{(\alpha, \beta) : \alpha \in S_0,\ \beta \in S'_0 \ \text{ and } \ |\langle x_\alpha, y_\beta \rangle| = mn\}.$$

Note that for any $(\alpha, \beta) \in \mathrm{ET}_{m,n}$, $(\langle x_\alpha, y_\beta \rangle, x_\alpha, y_\beta)$ is an (m, n)-bicyclic and hence $\mathcal{M}(\langle x_\alpha, y_\beta \rangle, x_\alpha, y_\beta)$ is an edge-transitive embedding of $K_{m,n}$. Furthermore, for any edge-transitive embedding $\mathcal{M}$ of $K_{m,n}$ by partite set preserving automorphism, there exists $(\alpha, \beta) \in \mathrm{ET}_{m,n}$ such that $\mathcal{M}$ is isomorphic to $\mathcal{M}(\langle x_\alpha, y_\beta \rangle, x_\alpha, y_\beta)$. If both $\alpha \in S_0$ and $\beta \in S'_0$ are the identity permutations then one can easily check that $(\alpha, \beta) \in \mathrm{ET}_{m,n}$. In this case, the group $\langle x_\alpha, y_\beta \rangle$ is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$. We call the induced edge-transitive embedding $\mathcal{M}(\langle x_\alpha, y_\beta \rangle, x_\alpha, y_\beta)$ the *standard embedding* of $K_{m,n}$. Hence, for any positive integer m, n, there exists at least one edge-transitive embedding $\mathcal{M}$ of $K_{m,n}$ by partite set preserving automorphisms.

Remark 1. (1) For any $(\alpha, \beta) \in \mathrm{ET}_{m,n}$, $\langle x_\alpha, y_\beta \rangle = \{x_\alpha^i y_\beta^j \ \mid \ i \in [m], \ \ j \in [n] \ \} = \{y_\beta^j x_\alpha^i \ \mid \ i \in [m], \ \ j \in [n] \ \}$. Hence, if α satisfies some properties then β also satisfies these properties and vice versa.
(2) Let $\mathcal{M}$ be an embedding of $K_{m,n}$ for different m and n. Then, any automorphism of $\mathcal{M}$ is partite set preserving. Let $m = n$ be odd and let $\mathcal{M}$ be an edge-transitive embedding of $K_{n,n}$. If a subgroup Γ of $\mathrm{Aut}\,(\mathcal{M})$ acts regularly on the edge set then $|\Gamma| = n^2$ is odd and hence there exists no partite set reversing element in Γ. Hence, for odd n, every edge-transitive embedding of $K_{n,n}$ is edge-transitive embeddings of $K_{n,n}$ by partite set preserving automorphisms. On the other hand, for even n, we do not know whether the above statement is also true or not.

The next lemma shows that for different $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathrm{ET}_{m,n}$, two induced edge-transitive embeddings are non-isomorphic.

Lemma 3.3. *For any $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathrm{ET}_{m,n}$, the induced edge-transitive embeddings $\mathcal{M}(\langle x_{\alpha_1}, y_{\beta_1} \rangle,\ x_{\alpha_1}, y_{\beta_1})$ and $\mathcal{M}(\langle x_{\alpha_2}, y_{\beta_2} \rangle, x_{\alpha_2}, y_{\beta_2})$ are isomorphic if and only if $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$.*

Proof. It suffices to show 'only if' part. Assume that $\mathcal{M}(\langle x_{\alpha_1}, y_{\beta_1} \rangle, x_{\alpha_1}, y_{\beta_1})$ and $\mathcal{M}(\langle x_{\alpha_2}, y_{\beta_2} \rangle, x_{\alpha_2}, y_{\beta_2})$ are isomorphic. By Lemma 2.1, there exists an isomorphism $\phi : \langle x_{\alpha_1}, y_{\beta_1} \rangle \to \langle x_{\alpha_2}, y_{\beta_2} \rangle$ given by $x_{\alpha_1} \mapsto x_{\alpha_2}$ and $y_{\beta_1} \mapsto y_{\beta_2}$. For any $i \in [m]$, there exists $a(i) \in [m], b(i) \in [n]$ such that $y_{\beta_1} x_{\alpha_1}^i = x_{\alpha_1}^{a(i)} y_{\beta_1}^{b(i)}$. It implies that

$$\beta_1(i') = y_{\beta_1} x_{\alpha_1}^i (0') = x_{\alpha_1}^{a(i)} y_{\beta_1}^{b(i)} (0') = a(i)'.$$

Because $\phi : \langle x_{\alpha_1}, y_{\beta_1}\rangle \to \langle x_{\alpha_2}, y_{\beta_2}\rangle$ given by $x_{\alpha_1} \mapsto x_{\alpha_2}$ and $y_{\beta_1} \mapsto y_{\beta_2}$ is an isomorphism, it also holds that $y_{\beta_2} x^i_{\alpha_2} = x^{a(i)}_{\alpha_2} y^{b(i)}_{\beta_2}$. Hence, $\beta_2(i') = a(i)'$, which means $\beta_1 = \beta_2$. Similarly, one can show that $\alpha_1 = \alpha_2$. □

By Lemma 3.3, the number of edge-transitive embeddings of $K_{m,n}$ by partite set preserving automorphisms up to isomorphism is equal to the cardinality $|\mathrm{ET}_{m,n}|$. Note that for any $(\alpha, \beta) \in \mathrm{ET}_{m,n}$, the stabilizers $\langle x_\alpha, y_\beta\rangle_0$ and $\langle x_\alpha, y_\beta\rangle_{0'}$ are cyclic groups $\langle x_\alpha\rangle$ of order m and $\langle y_\beta\rangle$ of order n, respectively.

Lemma 3.4. *For an $(\alpha, \beta) \in \mathrm{ET}_{m,n}$, $\langle\alpha\rangle$ and $\langle\beta\rangle$ are cyclic groups of order $|\alpha^i(1) \mid i \in [n]\}|$ and $|\beta^i(1') \mid i \in [m]\}|$, the cardinality of the orbit containing 1 and 1′, respectively. Furthermore, they are divisors of m and n, respectively.*

Proof. Let $|\{\alpha^i(1) \mid i \in [n]\}| = d_1$ and $|\{\beta^i(1') \mid i \in [m]\}| = d_2$. Then, d_1 and d_2 are divisors of the orders $|\langle x_\alpha\rangle| = m$ and $|\langle y_\beta\rangle| = n$, respectively. Note that

$$\alpha^{d_1}(1) = 1 \text{ and } y_\beta^{-1} x_\alpha^{d_1} y_\beta(0) = 0,$$

which implies that, as a conjugate of $x^{d_1}_\alpha$, $y^{-1}_\beta x^{d_1}_\alpha y_\beta$ belongs to the vertex stabilizer $\langle x_\alpha, y_\beta\rangle_0 = \langle x_\alpha\rangle$. Since d_1 is a divisor of m, $y^{-1}_\beta x^{d_1}_\alpha y_\beta = x^{rd_1}_\alpha$ for some $r \in [m]$ such that $(r, \frac{m}{d_1}) = 1$. Now, suppose to the contrary that $|\langle\alpha\rangle| \neq d_1$. Then, there exists $k \in [n]$ such that $\alpha^{d_1}(k) \neq k$. Let q be the largest element in $[n]$ such that $\alpha^{d_1}(q) \neq q$. On the other hand,

$$\alpha^{rd_1}(q) = x^{rd_1}_\alpha(q) = y^{-1}_\beta x^{d_1}_\alpha y_\beta(q) = y^{-1}_\beta x^{d_1}_\alpha(q+1) = y^{-1}_\beta(q+1) = q,$$

contradictory to $\alpha^{rd_1}(q) \neq q$. Therefore, $|\langle\alpha\rangle| = d_1$. Similarly, one can show that $|\langle\beta\rangle| = d_2$. □

Corollary 3.5. *For any prime numbers p, q, there exists only one edge-transitive embedding of $K_{p,q}$ up to isomorphism.*

Proof. For any $(\alpha, \beta) \in \mathrm{ET}_{p,q}$, it follows from Lemma 3.4 that $|\langle\alpha\rangle|$ and $|\langle\beta\rangle|$ are divisors of p and q, respectively. Since $|\langle\alpha\rangle| < p$ and $|\langle\beta\rangle| < q$, α and β are the identity and hence $|\mathrm{ET}_{p,q}| = 1$. It implies that there exists only one edge-transitive embedding of $K_{p,q}$, namely, the standard embedding of $K_{p,q}$ up to isomorphism. □

From now on, we denote $\beta(i')$ by $\beta(i)$ for any $i \in [m]$ for our convenience. The following lemma is related to a characterization of the set $\mathrm{ET}_{m,n}$.

Lemma 3.6. *Let $\alpha \in S_0$ and $\beta \in S'_0$. Then, $(\alpha, \beta) \in \mathrm{ET}_{m,n}$ if and only if for each $i \in [m]$, there exist $a(i) \in [m]$ and $b(i) \in [n]$ such that $\alpha^i(k) = \alpha^{a(i)}(k + b(i)) - 1$ for all $k \in [n]$ and $\beta(t + i) = \beta^{b(i)}(t) + a(i)$ for all $t \in [m]$. In this case, we have $a(i) = \beta(i)$ and $b(i) = -\alpha^{-i}(-1)$.*

Proof. (⇒) If $|\langle x_\alpha, y_\beta\rangle| = mn$, then $\langle x_\alpha, y_\beta\rangle = \{x^i_\alpha y^j_\beta \mid i \in [m], \ j \in [n]\}$. Therefore, for each $i \in [m]$, there exist $a(i) \in [m]$ and $b(i) \in [n]$ such that $y_\beta x^i_\alpha = x^{a(i)}_\alpha y^{b(i)}_\beta$. By taking their values of $k \in [n]$ and $t' \in [m]'$, we have

$$\alpha^i(k) + 1 = \alpha^{a(i)}(k + b(i)) \text{ and } \beta(t + i) = \beta^{b(i)}(t) + a(i).$$

($\Leftarrow$) The equalities $\alpha^i(k) = \alpha^{a(i)}(k + b(i)) - 1$ for all $k \in [n]$ and $\beta(t+i) = \beta^{b(i)}(t) + a(i)$ for all $t \in [m]$ are nothing but the conditions to have the equality $y_\beta x_\alpha^i = x_\alpha^{a(i)} y_\beta^{b(i)}$. This equality implies $\langle x_\alpha, y_\beta \rangle = \{x_\alpha^i y_\beta^j \mid i \in [m], \ j \in [n]\}$, namely, $|\langle x_\alpha, y_\beta \rangle| = mn$.

Inserting $k = -b(i)$ and $t = 0$ to the equations $\alpha^i(k) = \alpha^{a(i)}(k + b(i)) - 1$ and $\beta(t+i) = \beta^{b(i)}(t) + a(i)$, we have $b(i) = -\alpha^{-i}(-1)$ and $a(i) = \beta(i)$. □

4. A classification of reflexible embeddings for odd m, n

In this section, we classify reflexible edge-transitive embeddings of $K_{m,n}$ for odd m, n.

First, we characterize $(\alpha, \beta) \in \mathrm{ET}_{m,n}$ whose induced edge-transitive embedding is reflexible.

Lemma 4.1. *For any $(\alpha, \beta) \in \mathrm{ET}_{m,n}$, $\mathcal{M}(\langle x_\alpha, y_\beta \rangle, x_\alpha, y_\beta)$ is reflexible if and only if $-\alpha^{-1}(-k) = \alpha(k)$ for any $k \in [n]$ and $-\beta^{-1}(-t) = \beta(t)$ for any $t \in [m]$.*

Proof. Let $(\alpha, \beta) \in \mathrm{ET}_{m,n}$. Then, the reflection of edge-transitive embedding $\mathcal{M}(\langle x_\alpha, y_\beta \rangle, x_\alpha, y_\beta)$ is $\mathcal{M}(\langle x_\alpha^{-1}, y_\beta^{-1} \rangle, x_\alpha^{-1}, y_\beta^{-1})$. Note that

$$x_\alpha^{-1} = \alpha^{-1}(0' \ (m-1)' \ (m-2)' \cdots 2' \ 1') \quad \text{and} \quad y_\beta^{-1} = \beta^{-1}(0 \ n-1 \ n-2 \cdots 2 \ 1).$$

Let $\sigma = (0)(1 \ n-1)(2 \ n-2) \cdots$ and $\delta = (0')(1' \ (m-1)')(2' \ (m-2)') \cdots$ be involutions in S and S', respectively. Let $\phi = \sigma\delta$ as an automorphism of $K_{m,n}$. Then,

$$\phi^{-1} x_\alpha^{-1} \phi = \sigma\alpha^{-1}\sigma(0' \ 1' \ 2' \ \cdots (m-1)') \quad \text{and} \quad \phi^{-1} y_\beta^{-1} \phi = \delta\beta^{-1}\delta(0 \ 1 \ 2 \ \cdots n-1).$$

Since $(\langle x_\alpha^{-1}, y_\beta^{-1} \rangle, x_\alpha^{-1}, y_\beta^{-1})$ is also (m,n)-bicyclic, $(\sigma\alpha^{-1}\sigma, \delta\beta^{-1}\delta) \in \mathrm{ET}_{m,n}$ and furthermore $\mathcal{M}(\langle x_\alpha^{-1}, y_\beta^{-1} \rangle, x_\alpha^{-1}, y_\beta^{-1})$ is isomorphic to $\mathcal{M}(\langle x_{\sigma\alpha^{-1}\sigma}, y_{\delta\beta^{-1}\delta} \rangle, x_{\sigma\alpha^{-1}\sigma}, y_{\delta\beta^{-1}\delta})$ by Lemma 2.1(2). By Lemma 3.3, $\mathcal{M}(\langle x_\alpha, y_\beta \rangle, x_\alpha, y_\beta)$ and its reflection $\mathcal{M}(\langle x_\alpha^{-1}, y_\beta^{-1} \rangle, \ x_\alpha^{-1}, y_\beta^{-1})$ are isomorphic if and only if $(\alpha, \beta) = (\sigma\alpha^{-1}\sigma, \delta\beta^{-1}\delta)$, namely, $-\alpha^{-1}(-k) = \alpha(k)$ for any $k \in [n]$ and $-\beta^{-1}(-t) = \beta(t)$ for any $t \in [m]$. □

For our convenience, we denote

$$\begin{aligned} \mathrm{RET}_{m,n} = \{(\alpha, \beta) \in \mathrm{ET}_{m,n} : -\alpha^{-1}(-k) &= \alpha(k) \ \text{ for any } k \in [n] \text{ and} \\ -\beta^{-1}(-t) &= \beta(t) \ \text{ for any } t \in [m] \ \}. \end{aligned}$$

By Lemmas 3.3 and 4.1, the number of reflexible edge transitive embeddings of $K_{m,n}$ by partite set preserving automorphisms up to isomorphism is equal to the cardinality $|\mathrm{RET}_{m,n}|$. Note that if $\alpha \in S$ and $\beta \in S'$ are the identity permutations, then (α, β) belongs to $\mathrm{RET}_{m,n}$ by Lemma 4.1. So, for any two positive integers m and n, there exists at least one reflexible edge-transitive embedding of $K_{m,n}$.

Lemma 4.2. *Let $(\alpha, \beta) \in \mathrm{RET}_{m,n}$ and let $d_1 = |\langle \alpha \rangle|$ and $d_2 = |\langle \beta \rangle|$. Then,*

(1) $\alpha^{-i}(-k) = -\alpha^i(k)$ *for any* $k \in [n]$ *and* $i \in [m]$, *and* $\beta^{-j}(-t) = -\beta^j(t)$ *for any* $t \in [m]$ *and* $j \in [n]$*;*
(2) *it holds that* $\alpha(k) \equiv -k(\text{mod } d_2)$ *for any* $k \in [n]$ *and* $\beta(t) \equiv -t(\text{mod } d_1)$ *for any* $t \in [m]$.

Proof. (1) By using $\alpha^{-1}(-k) = -\alpha(k)$ repeatedly, one can see that for any $k \in [n]$ and $i \in [m]$,

$$\alpha^{-i}(-k) = \alpha^{-i+1}(-\alpha(k)) = \alpha^{-i+2}(-\alpha^2(k)) = \cdots = \alpha^{-1}(-\alpha^{i-1}(k)) = -\alpha^i(k).$$

Similarly, one can show that $\beta^{-j}(-t) = -\beta^j(t)$ for any $t \in [m]$ and $j \in [n]$.
(2) By Lemma 3.6, for each $i \in [m]$, there exist $a(i) \in [m]$ and $b(i) \in [n]$ such that $\alpha^i(k) = \alpha^{a(i)}(k + b(i)) - 1$ for all $k \in [n]$ and $\beta(t + i) = \beta^{b(i)}(t) + a(i)$ for all $t \in [m]$. Furthermore, $a(i) = \beta(i)$ and $b(i) = -\alpha^{-i}(-1) = \alpha^i(1)$. Inserting $k = 0$ to the equation $\alpha^i(k) = \alpha^{a(i)}(k + b(i)) - 1$, we have $b(i) = \alpha^{-a(i)}(1) = \alpha^{-\beta(i)}(1)$. Hence, $\alpha^i(1) = \alpha^{-\beta(i)}(1)$ for any $i \in [m]$. Since the order of α is equal to the cardinality of the orbit containing 1 by Lemma 3.4, $\beta(i) \equiv -i(\text{mod } d_1)$. By symmetry between α and β, it also holds that $\alpha(k) \equiv -k(\text{mod } d_2)$ for any $k \in [n]$. □

By Lemma 4.2, $b(i) = \alpha^i(1) \equiv (-1)^i(\text{mod } d_2)$. Hence, for any $(\alpha, \beta) \in \text{RET}_{m,n}$ with $d_1 = |\langle\alpha\rangle|$ and $d_2 = |\langle\beta\rangle|$, we have

$$\beta(t + i) = \beta^{b(i)}(t) + a(i) = \beta^{\alpha^i(1)}(t) + \beta(i) = \beta^{(-1)^i}(t) + \beta(i)$$

for all $t, i \in [m]$. By symmetry, it also holds $\alpha(k + j) = \alpha^{(-1)^j}(k) + \alpha(j)$ for all $k, j \in [n]$. The following theorem is the main theorem of this paper.

Theorem 4.3. *If both* m *and* n *are odd then there exists only one reflexible edge-transitive embedding of* $K_{m,n}$ *up to isomorphism, namely, the standard embedding of* $K_{m,n}$ *is the only reflexible edge-transitive embedding of* $K_{m,n}$.

Proof. Let $(\alpha, \beta) \in \text{RET}_{m,n}$ and let $d_1 = |\langle\alpha\rangle|$ and $d_2 = |\langle\beta\rangle|$. Since d_1 and d_2 are divisors of m and n, respectively, both d_1 and d_2 are odd.

As the first consideration, suppose that one of d_1 and d_2 is 1. For our convenience, let $d_1 = 1$, namely, let α be the identity permutation. Then, by Lemma 4.2, $\alpha(1) = 1 \equiv -1(\text{mod } d_2)$. It implies that $d_2 = 1$ or 2. Since d_2 is odd, d_2 is also 1, namely, β is also the identity permutation. Hence if one of d_1 and d_2 is 1 then the other is also 1.

Suppose that $d_1 \geq 3$ or $d_2 \geq 3$. Let $d_1 \geq 3$. By lemma 4.2, $\beta(t) \equiv -t(\text{mod } d_1)$ for all $t \in [m]$, which implies that the order d_2 of β is even. It is a contradiction. By the same reason, if $d_2 \geq 3$ then a contradiction occurs.

Therefore, α and β should be the identity permutations. Because we chose $(\alpha, \beta) \in \text{RET}_{m,n}$ arbitrarily, $|\text{RET}_{m,n}| = 1$ and the standard embedding of $K_{m,n}$ is the only reflexible edge-transitive embedding of $K_{m,n}$ up to isomorphism. □

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