East Asian Mathematical Journal
Vol. 25 (2009), No. 4, pp. 527–532

ON THE EXISTENCE OF SOLUTIONS OF EXTENDED GENERALIZED VARIATIONAL INEQUALITIES IN BANACH SPACES

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Abstract. In this paper, we study the following extended generalized variational inequality problem, introduced by Noor (for short, EGVI) : Given a closed convex subset K in q-uniformly smooth Banach space B, three nonlinear mappings $T : K \to B^*$, $g : K \to K$, $h : K \to K$ and a vector $\xi \in B^*$, find $x \in K, h(x) \in K$ such that $\langle Tx-\xi, g(y)-h(x)\rangle \geq 0$, for all $y \in K, g(y) \in K$.[see [2]: M. Aslam Noor, Extended general variational inequalities, Appl. Math. Lett. 22 (2009) 182-186.] By using sunny nonexpansive retraction Q_K and the well-known Banach's fixed point principle, we prove existence results for solutions of (EGVI). Our results extend some recent results from the literature.

1. Introduction and preliminaries

In what follows, we always let B be a real Banach space with dual space B^*, $\langle \cdot, \cdot \rangle$ be the dual pair between B and B^*, and 2^B denote the family of all the nonempty subsets of X. The generalized duality mapping $J_q(x) : X \to 2^X$ is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^*\rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\},$$

where $q > 1$ is a constant. In particular, $J_2 = J$ is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2}J_2(x)$, for all $x \in B$, and $J_q(x)$ is single-valued if B^* is strictly convex. In the sequel, unless otherwise specified, we always suppose that B is a real Banach space such that $J_q(x)$ is single-valued and H is a Hilbert space. If $B = H$, then J_2 becomes the identity mapping of H.

Received August 13, 2009; Accepted October 24, 2009.
2000 *Mathematics Subject Classification.* 47H05, 47H10, 47J05.
Key words and phrases. sunny nonexpansive retraction, extended generalized variational inequalities, q-uniformly smooth Banach space, Banach's fixed point principle.
This work was financially supported by the Key Project of Chinese Ministry of Education (NO207104) and The National Natural Science Foundation of China (Grant No. 60873203).

The modulus of smoothness of B is the function $\rho_B : [0, +\infty) \to [0, +\infty)$ defined by

$$\rho_B(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\| - 1) : \|x\| \leq 1, \|y\| \leq t\right\}.$$

A Banach space B called uniformly smooth if

$$\lim_{t\to\infty} \frac{\rho_B(t)}{t} = 0.$$

B is called $q-$uniformly smooth if there exists a constant $c > 0$, such that

$$\rho_B(t) \leq ct^q, \qquad q > 1.$$

Note that J_q is single-valued if B is uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [8] proved the following Lemma.

Lemma 1.1. ([8]) *Let B be a real uniformly smooth Banach space. Then, B is $q-$uniformly smooth if and only if there exists a constant $c_q > 0$, such that for all$x, y \in B$*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

Let K be a nonempty, closed and convex subset of B and $T : K \to B$, $g : K \to K$ and $h : K \to K$ three nonlinear mappings. (Satisfying some suitable conditions) We consider a extended generalized variational inequalities (EGVI)(see [6]) as follows: to find $x \in K, h(x) \in K$ such that

$$\langle Tx - \xi, J(g(y) - h(x))\rangle \geq 0, \quad \text{for every } \ y \in K, g(y) \in K \tag{1.1}$$

where $\xi \in B$.

If $B = H$, Hilbert space, $g = h : K \to K$ is bijective on K with $R(g) = K$, then the above (EGVI) can write to

$$\langle Tx - \xi, g(y) - g(x)\rangle \geq 0, \quad \text{for every } \ y \in K, \tag{1.2}$$

which is generalized variational inequality introduced and studied by Noor [7] in 1988.

For $g = h = I, \xi = 0$, where I is the identity operator, problem (1.1) is equivalent to finding $u \in K$ such that

$$\langle Tx, y - x\rangle \geq 0, \qquad \forall y \in K,$$

which is known as the classical variational inequality.

We would like to emphasize that problem (1.1) is equivalent to that of finding $x \in K, h(x) \in K$, such that

$$\langle \rho(Tx - \xi) + h(x) - g(x), J(g(y) - h(x))\rangle \geq 0, \ \text{ for all } \ y \in K, g(y) \in K, \rho > 0. \tag{1.3}$$

This equivalent formulation easy seen from newly setting T.

Let K be a nonempty, closed and convex subset of B. A mapping $Q : B \to K$ is said to be sunny if

$$Q(Qx + t(x - Qx)) = Qx, \forall x \in B, \forall t \geq 0.$$

A mapping $Q : B \to K$ is said to be a retraction or a projection if $Qx = x, \forall x \in K$. If B is smooth then the sunny nonexpansive retraction of B onto K is uniquely decided (see [8]).

Proposition. (See Bruck [2], Goebel and Reich [4].) Let B be a smooth Banach space and let K be a nonempty subset of B. Let $Q_K : B \to K$ be a retraction and let J be the normalized duality mapping on B. Then the following are equivalent:
(a) Q_K is sunny and nonexpansive.
(b) $\|Q_K x - Q_K y\|^2 \leq \langle x - y, J(Q_K x - Q_K y)\rangle \forall x, y \in B$.
(c) $\langle x - Q_K x, J(Q_K x - y)\rangle \geq 0, \forall y \in K$.

Theorem 1.1. Let B be a reflexive and smooth Banach space, $T : K \to B$, $g : K \to K$ and $h : K \to K$ three nonlinear mappings, where $g : K \to K$ be bijective on K with $R(g) = K$. Then the $x \in K$ is a solution of the general variational inequality (1.3) if and only if $x \in K$ is a solution of the operator equation

$$h(x) = Q_K[g(x) - \rho(Tx - \xi)], \quad , \tag{1.4}$$

Proof. Variational inequality (1.3) can write to

$$\langle [g(x) - \rho(Tx - \xi)] - h(x), J(h(x) - g(y))\rangle \geq 0.$$

By Proposition (c), above formula is equivalent to

$$h(x) = Q_K[g(x) - \rho(Tx - \xi)].$$

This completes the proof of Theorem 1.1. □

2. Main results

The following theorem is one of the main results of this work which is motivated by the results in [6], [5].

Definition 2.1. Let $T, q : K \to B$ be two single-valued operators. The operator T is said to be
(i) accretive if

$$\langle Tx - Ty, J_q(x - y)\rangle \geq 0, \ \forall x, y \in B;$$

(ii) strongly accretive if there exists a constant $r > 0$, such that

$$\langle Tx - Ty, J_q(x - y)\rangle \geq r\|x - y\|^q, \ \forall x, y \in B;$$

We rewrite the relation (1.6) in the following form:

$$F(x) = x - h(x) + Q_K[g(x) - \rho(Tx - \xi)], \tag{2.1}$$

which is used to study the existence of a solution of the extended general variational inequality (1.3). We now study those conditions under which the extended general variational inequality (1.3) has a solution.

Theorem 2.1. *Let B be a $q-$uniformly smooth Banach space, K be a closed convex set in B and $\xi \in B$. Suppose that g, h and T satisfy the following conditions*

(i) $g : K \to K$ is bijective on K with $R(g) = K$, strongly accretive with constants $\alpha > 0$ and Lipschitz continuous with constants $s > 0$.

(ii) $h : K \to K$ is strongly accretive with constants $\sigma > 0$ and Lipschitz continuous with constants $\delta > 0$.

(iii) $T : K \to B$ is Lipschitz continuous and strongly accretive with constants $b > 0$ and $t > 0$, respectively.

If there exists some constant $\rho > 0$ such that

$$\sqrt[q]{1 - q\rho t + c_q \rho^q b^q} + \sqrt[q]{1 - q\sigma + c_q \delta^q} + \sqrt[q]{1 - q\alpha + c_q s^q} < 1, \tag{2.2}$$

where

$$1 - q\rho t + c_q \rho^q b^q > 0,\ 1 - q\sigma + c_q \delta^q > 0,\ 1 - q\alpha + c_q s^q > 0, \tag{2.3}$$

then there exists a unique solution $x \in K, h(x) \in K$ of the extended general variational inequality (1.3).

Proof. From Theorem 1.1, it follows that problems (2.1) and (1.6) are equivalent. Thus it is enough to show that the map $F(x)$ defined by (2.1), has a fixed point. By Lemma 1.1, for all $x \neq y \in K$ we have

$$\begin{aligned}
&\|F(x) - F(y)\| \\
&= \|x - y - (h(x) - h(y)) + Q_K[g(x) - \rho(Tx - \xi)] - Q_K[g(y) - \rho(Ty - \xi)]\| \\
&\leq \|x - y - (h(x) - h(y))\| + \|Q_K[g(x) - \rho(Tx - \xi)] - Q_K[g(y) - \rho(Ty - \xi)]\| \\
&\leq \|x - y - (h(x) - h(y))\| + \|[g(x) - \rho(Tx - \xi)] - [g(y) - \rho(Ty - \xi)]\| \\
&= \|x - y - (h(x) - h(y))\| + \|g(x) - g(y) - \rho(Tx - Ty)\|. \\
&\leq \|x - y - (h(x) - h(y))\| + \|x - y - [g(x) - g(y)]\| + \|x - y - \rho[Tx - Ty]\|
\end{aligned} \tag{2.4}$$

since $T : K \to B$ is strongly accretive and B is $q-$uniformly smooth Banach space, by Lemma 1.1, we have the following estimate:

$$\begin{aligned}
&\|x - y - \rho(Tx - Ty)\|^q \\
&\leq \|x - y\|^q - q\rho\langle (Tx - Ty), J_q(x - y)\rangle + c_q \rho^q \|Tx - Ty\|^q \\
&\leq \|x - y\|^q - q\rho t\|x - y\|^q + c_q \rho^q b^q \|x - y\|^q \\
&= [1 - q\rho t + c_q \rho^q b^q]\|x - y\|^q.
\end{aligned} \tag{2.5}$$

Since $g : K \to K$ is s-strongly accretive and B is $q-$uniformly smooth Banach space, by Lemma 1.1, we have

$$\begin{aligned}
&\|x - y - (g(x) - g(y))\|^q \\
&\leq \|x - y\|^q - q\langle g(x) - g(y), J_q(x - y)\rangle + c_q \|g(x) - g(y)\|^q \\
&\leq (1 - q\alpha + c_q s^q)\|x - y\|^q.
\end{aligned} \tag{2.6}$$

Since $h : K \to K$ is σ-strongly accretive and B is $q-$uniformly smooth Banach space, by Lemma 1.1, we have

$$\begin{aligned}&\|x-y-(h(x)-h(y))\|^q\\ &\le \|x-y\|^q - q\langle h(x)-h(y), J_q(x-y)\rangle + c_q\|h(x)-h(y)\|^q\\ &\le (1-q\sigma+c_q\delta^q)\|x-y\|^q.\end{aligned} \tag{2.7}$$

Combining (2.5),(2.6) with (2.7), one has

$$\|F(x)-F(y)\| \le \mu\|x-y\|, \tag{2.8}$$

where

$$\mu = \sqrt[q]{1-q\rho t+c_q\rho^q b^q} + \sqrt[q]{1-q\sigma+c_q\delta^q} + \sqrt[q]{1-q\alpha+c_q s^q}.$$

From (2.2),(2.3), it follows that $0 < \mu < 1$, which implies that the map $F(u)$ defined by (2.1), has a fixed point, which is the unique solution of (1.3). $\square$

Remark 2.1. When $q = 2$, the condition (2.2),(2.3) become the following

$$\left|\rho - \frac{t}{b^2c_2}\right| < \frac{\sqrt{t^2-b^2c_2k(2-k)}}{b^2c_2}, \quad t > b\sqrt{k(2-k)}, \quad k < 1. \tag{2.9}$$

$$k = \sqrt{1-2\sigma+c_2\delta^2} + \sqrt{1-2\alpha+c_2s^2}. \tag{2.10}$$

Remark 2.2. This paper extends and improves all related papers appeared, such as [6] and references therein in the following senses: form Hilbert space to slightly more general q-uniformly smooth Banach space.

References

[1] Ya. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, in: A. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Monotonic and Accretive Type, Marcel Dekker, New York, 1996, 15–50.

[2] R. E. Bruck, *Nonexpansive projections on subsets of Banach spaces*, Pacific J. Math. **47** (1973), 341–355.

[3] J. Fan, *A Mann type iterative scheme for variational inequalities in noncompact subsets of Banach spaces*, J. Math. Anal. Appl. **337** (2008), 1041–1047.

[4] K. Goebel and S. Reich, *Uniform convexity, in: Hyperbolic Geometry and Nonexpansive Mappings*, Marcel Dekker, New York, 1984.

[5] J. Li, *On the existence of solutions of variational inequalities in Banach spaces*, J. Math. Anal. Appl. **295** (2004) 115–126.

[6] M. Aslam Noor, *Extended general variational inequalities*, Appl. Math. Lett. **22** (2009) 182–186.

[7] M. Aslam Noor, *General variational inequalities*, Appl. Math. Lett. **1** (1988), 119–121.

[8] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonl. Anal. **16 (12)** (1991), 1127–1138 .

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