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MIXED VECTOR EQUILIBRIUM-LIKE PROBLEMS IN BANACH SPACES

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Abstract. In this paper, we consider a new class of generalized mixed vector equilibrium-like problems in Banach spaces. By using Fan-KKM Theorem and Nadler's Theorem, we prove the existence theorem of solution for this class of generalized mixed vector equilibrium-like problems.

1. Introduction and preliminaries

(Vector) equilibrium problem is a unified model of several problems, including optimization problems, fixed point problems, variational inequality problems, Nash equilibria problems, saddle point problems, complementarity problems as special cases and provides a natural unified framework for studying many problems in finance, economics, network analysis, transportation and elasticity [1-3, 5, 8, 9, 14]. We consider a new class of generalized mixed vector equilibrium-like problem and by using Fan-KKM Theorem [4] and Nadler's Theorem [12] we prove some existence theorems of solutions for the class of generalized mixed vector equilibrium-like problems. Furthermore these existence theorems can be applied to derive some existence results of solutions for generalized mixed vector variational-like problem. It is worth pointing out that there are no assumptions of pseudo monotonicity in our existence results.

Definition 1.1. Let D be a nonempty subset of a vector space X. Then a multifunction $T : D \to 2^X$ is called a KKM map, where 2^X denotes the collection of all nonempty subsets of X if for each nonempty finite subset $\{u_1, u_2, \cdots, u_n\}$ of D, $\mathrm{co}\{u_1, u_2, \cdots, u_n\} \subset \bigcup_{i=1}^{n} Tu_i$, where $\mathrm{co}\{u_1, u_2, \cdots, u_n\}$ denotes the convex hull of $\{u_1, u_2, \cdots, u_n\}$.

Fan-KKM Theorem. [4] *Let D be an arbitrary set in a Hausdorff topological vector space X. Let $T : D \to 2^X$ be a KKM map such that Tu is closed for all*

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$u \in D$ and is compact for at least one $u \in D$. Then

$$\bigcap_{u \in D} Tu \neq \emptyset.$$

Nadler's Theorem. [12] *Let $(X, \|.\|)$ be a normed vector space and H the Hausdorff metric on the collection $CB(X)$ of all closed and bounded subsets of X, induced by a metric d in terms of $d(x,y) = \|x - y\|$, defined by,*

$$H(A,B) = \max(\sup_{u \in A} \inf_{v \in B} \|u - v\|, \sup_{v \in B} \inf_{u \in A} \|u - v\|),$$

for A and B in $CB(X)$. If A and B are any two members in $CB(X)$, then for each $\epsilon > 0$ and each $u \in A$, there exists $v \in B$ such that

$$\|u - v\| \leq (1 + \epsilon) H(A,B).$$

Lemma 1.1. [11] *Let Y be a topological vector space with a pointed closed and convex cone C such that $intC \neq \emptyset$, then for all $x, y, z \in Y$ we have*

(i) $x - y \in -intC$ and $x \notin -intC \Rightarrow y \notin -intC$;
(ii) $x + y \in -C$ and $x + z \notin -intC \Rightarrow z - y \notin -intC$;
(iii) $x + z - y \notin -intC$ and $-y \in -C \Rightarrow x + z \notin -intC$;
(iv) $x + y \notin -intC$ and $y - z \in -C \Rightarrow x + z \notin -intC$.

Definition 1.2. [11] Let $f : D \times D \to Y$ be a vector valued bifunction, then $f(x,y)$ is said to be hemicontinuous with respect to y, if for any given $x \in D$

$$\lim_{\lambda \to 0^+} f(x, \lambda y_1 + (1 - \lambda) y_2) = f(x, y_2) \text{ for all } y_1, y_2 \in D.$$

Definition 1.3. [15] Let X, Y be two real Banach spaces and $L(X,Y)$ be the space of all linear and continuous operators of X into Y. A bifunction $N(\cdot,\cdot) : L(X,Y) \to L(X,Y)$ is called continuous in the first argument if for any $u, v \in L(X,Y)$,

$$\|N(u,\cdot) - N(v,\cdot)\| \to 0 \text{ as } \|u - v\| \to 0.$$

In a similar way we can define the continuity of N in the second argument.

Definition 1.4. [15] A single valued mapping $h : D \to Y$ is said to be P-convex if

$$h(tu_1 + (1-t)u_2) \in th(u_1) + (1-t)h(u_2) - P, \text{ for } u_1, u_2 \in D \text{ and } t \in [0,1].$$

Remark 1.1. It is easy to prove that $h(u,v)$ is P-convex with respect to u if and only if for any $v \in D$, $h(\sum_{i=1}^{n} t_i u_i, v) \in \sum_{i=1}^{n} t_i h(u_i, v) - P$, for all $u_i \in D$ and $t_i \in [0,1]$, $(i = 1, 2, \cdots, n)$ with $\sum_{i=1}^{n} t_i = 1$.

Throughout the rest of this paper, by "$\rightarrow$ " and "$\rightharpoonup$ " we denote the strong convergence and weak convergence respectively.

2. Main result

Throughout this paper, X and Y are real Banach spaces and D is a nonempty convex subset of X. A map $C : D \rightarrow 2^Y$ has convex cone-values $C(u)$ whose interior $\operatorname{int} C(u)$ is nonempty for each $u \in D$. With the maps A, $T : D \rightarrow 2^{L(X,Y)}$, $N : L(X,Y) \times L(X,Y) \rightarrow L(X,Y)$, $f : L(X,Y) \times D \times D \rightarrow Y$ and $h : D \times D \rightarrow Y$, we consider the following mixed vector equilibrium-like problems (MVELP):

(MVELP 1) Find $u_0 \in D$ such that

$$f(N(p,q), u_0, v) + h(u_0, v) - h(u_0, u_0) \notin -\operatorname{int} C(u_0)$$

for all $v \in D$, $p \in A(v)$ and $q \in T(v)$.

(MVELP 2) Find $u_0 \in D$ such that for some $s_0 \in A(u_0)$ and $t_0 \in T(u_0)$

$$f(N(s_0,t_0), u_0, v) + h(u_0, v) - h(u_0, u_0) \notin -\operatorname{int} C(u_0)$$

for all $v \in D$.

Considering (MVELP 1) and (MVELP 2) in the following Theorem 2.1 and Theorem 2.2, we assume the following condition (A) primarily;

(A) (A-1) $C(u)$ is a pointed closed and convex cone with $C(u) \neq Y$ for all $u \in D$,
(A-2) A and T have nonempty compact set-values,
(A-3) A map $W : D \rightarrow 2^Y$ defined by $W(u) = Y \setminus (-\operatorname{int} C(u))$ for $u \in D$, has a weakly closed graph $G_r(W)$ in $X \times Y$.
(A-4) h is weakly continuous in the first and second arguments.
(A-5) for each $(z, v) \in L(X,Y) \times D$, $f(z, \cdot, v)$ and $h(v, \cdot)$ are weakly continuous from D to Y.

Theorem 2.1. *If we add the following condition (B) to (A);*

(B) *there exists a map $g : D \times D \rightarrow Y$ such that*
(B-1) *$\forall u$, $v \in D$, $g(u,v) \notin -\operatorname{int} C(u)$ implies $f(N(p,q), u, v) + h(u,v) - h(u,u) \notin -\operatorname{int} C(u)$ for $p \in A(u)$, $q \in T(u)$,*
(B-2) *for each finite subset E of D, and for each $u \in coE$, a map $\ell : D \rightarrow Y$ defined by $\ell(v) = g(u,v)$ is P-convex,*
(B-3) *for each $v \in D$, $g(v,v) \notin -\operatorname{int} C(v)$,*
(B-4) *there exists a weakly compact convex subset K of D and $v_0 \in K$ such that*

$$g(u, v_0) \in -\operatorname{int} C(u) \quad \textit{for} \quad u \in D \setminus K.$$

Then

(I) *(MVELP 1) is solvable.*

Moreover, we add the following condition (C) to (A) and (B);

(C) (C-1) *N is continuous in the first and second arguments,*

(C-2) *for each u, $v \in D$, $f(N(p_\lambda, q_\lambda), v_\lambda, v_\lambda) \in C(u)$ for all $p_\lambda \in A(u_\lambda)$, $q_\lambda \in T(u_\lambda)$ and $f(N(p_\lambda, q_\lambda), u, v_\lambda) + f(N(p_\lambda, q_\lambda), v_\lambda, u) = 0$, where $v_\lambda = u + \lambda(v - u)$, $\lambda \in (0, 1)$,*

(C-3) *for each $(z, u) \in L(X, Y) \times D$, a map $f(z, u, \cdot) + h(u, \cdot) : D \to Y$ is P-convex, where $P = \bigcap_{u \in D} C(u)$,*

(C-4) *for any u, $v \in D$,*

$$H(A(u + \lambda(v - u)), A(u)) \to 0$$

and

$$H(T(u + \lambda(v - u)), T(u)) \to 0$$

as $\lambda \to 0^+$ for the Hausdorff metric H in $CB(L(X, Y))$, the collection of all closed and bounded subsets of $L(X, Y)$,

(C-5) *for each net $\{\lambda\} \subset (0, 1)$ converging to 0^+,*

$$\begin{aligned} p_\lambda &\to s_0, \quad p_\lambda \in A(v_\lambda) \\ q_\lambda &\to t_0, \quad q_\lambda \in T(v_\lambda) \end{aligned}$$

implies

$$f(N(p_\lambda, q_\lambda), v_\lambda, v) - f(N(s_0, t_0), u, v) \to 0,$$

where $v_\lambda = u + \lambda(v - u)$ for u, $v \in D \times D$,

(C-6) *there exists a weakly compact convex subset $K \subseteq D$ such that for each $u \in D \backslash K$ there exists $v_0 \in D$ satisfying*

$$f(N(p, q), u, v) + h(u, v) - h(u, u) \in -int C(u)$$

for all $p \in A(v)$, $q \in T(v)$.

Then there exists a solution $u_0 \in D$ such that

$$f(N(p, q), v, u_0) + h(v, u_0) - h(v, v) \notin -int C(u_0)$$

for all $v \in D$ and $p \in A(v)$, $q \in T(v)$.

Moreover additionally, the following conditions are stisfied:

(C-7) *$L(X, Y)$ is reflexive,*

(C-8) *A and T have bounded closed convex set-values,*

(C-9) *for each net $\{\lambda\} \subset (0, 1)$ such that $\lambda \to 0^+$*

$$\left.\begin{aligned} p_\lambda &\to s_0, \quad \text{for all } p_\lambda \in A(v_\lambda) \\ q_\lambda &\to t_0, \quad \text{for all } q_\lambda \in T(v_\lambda) \end{aligned}\right\}$$

$$\Rightarrow \; f(N(p_\lambda, q_\lambda), v_\lambda, v) - f(N(s_0, t_0), v_\lambda, v) \rightharpoonup 0$$

where $v_\lambda = u + \lambda(v - u)$ for $(u, v) \in D \times D$.

(II) Then (MVELP 2) is solvable.

Proof. (I) Define a map $G : D \to 2^K$ by for $v \in D$,

$$G(v) = \{u \in K : f(N(p,q), u, v) + h(u,v) - h(u,u) \notin -\mathrm{int}\, C(u) \\ \text{for } p \in A(v),\ q \in T(v)\},$$

then
(i) $G(v)$ is weakly closed.
Indeed, for any sequence $\{u_n\}$ in $G(v)$ converging to $u_0 \in K$,

$$f(N(p,q), u_n, v) + h(u_n, v) - h(u_n, u_n) \notin -\mathrm{int}\, C(u_n)$$

for $p \in A(v)$ and $q \in T(v)$. Then by conditions (A-4) and (A-5),

$$f(N(p,q), u_n, v) + h(u_n, v) - h(u_n, u_n) \rightharpoonup f(N(p,q), u_0, v) + h(u_0, v) - h(u_0, u_0).$$

Hence by condition (A-3),

$$f(N(p,q), u_0, v) + h(u_0, v) - h(u_0, u_0) \notin -\mathrm{int}\, C(u_0),$$

which shows that $G(v)$ is weakly closed.

(ii) $\bigcap_{v \in D} G(v)$ is nonempty.
Indeed, since K is weakly compact, it is sufficient to show that the family $\{G(v)\}_{v \in D}$ has the fip(finite intersection property). Let $M = \{v_j : j = 1, 2, \cdots, m\}$ be any finite subset of D.
Since $V := coM$ is a compact convex subset of D, it is a weakly compact convex subset of D. Define a map $F : V \to 2^V$ by, for $v \in V$

$$F(v) = \{u \in V : g(u,v) \notin -\mathrm{int}\, C(u)\},$$

then by condition (B-3), $F(v)$ is nonempty. And F is a KKM map. If F is not a KKM map, then there exists a finite subset $\{y_i : i = 1, 2, \cdots, n\}$ of V and scalars $\alpha_i \geq 0$ with $\sum_{i=1}^{n} \alpha_i = 1$ such that

$$\sum_{i=1}^{n} \alpha_i y_i \notin \bigcup_{i=1}^{n} F(y_i).$$

Thus

$$g\left(\sum_{i=1}^{n} \alpha_i y_i, y_i\right) \in -\mathrm{int}\, C\left(\sum_{i=1}^{n} \alpha_i y_i\right).$$

By condition (B-2),

$$\begin{aligned} g\left(\sum_{i=1}^{n}\alpha_i y_i, \sum_{i=1}^{n}\alpha_i y_i\right) &\in \sum_{i=1}^{n}\alpha_i \; g\left(\sum_{i=1}^{n}\alpha_i y_i, y_j\right) - P \\ &\subset \sum_{i=1}^{n}\alpha_i \left(-\operatorname{int} C\left(\sum_{i=1}^{n}\alpha_i y_i\right)\right) - C\left(\sum_{i=1}^{n}\alpha_i y_i\right) \\ &\subset -\operatorname{int} C\left(\sum_{i=1}^{n}\alpha_i y_i\right) - C\left(\sum_{i=1}^{n}\alpha_i y_i\right) \\ &= -\operatorname{int} C\left(\sum_{i=1}^{n}\alpha_i y_i\right), \end{aligned}$$

which contradicts condition (B-3).

Since $F(v) \subset G(v)$ for all $v \in V$ by condition (B-1), $G(v)$ is also nonempty for all $v \in V$.

Since the closure $cl_V(F(v))$ is closed in V for all $v \in V$, it is also compact.

Hence by F-KKM Theorem, $\bigcap_{v\in V} cl_V(F(v))$ is nonempty. Choose $\overline{u} \in \bigcap_{v\in V} cl_V(F(v))$, since $v_0 \in K$ and $F(v_0) \subset K$ by condition (B-4),

$$\begin{aligned} \overline{u} \in cl_V(F(v_0)) &\leq cl_D(F(v_0)) \\ &= cl_K(F(v_0)) \subseteq K. \end{aligned}$$

Moreover it is easy to see that for each $v \in V$,

$$\begin{aligned} M_v := \{u \in V : f(N(p,q),u,v) + h(u,v) - h(u,u) \notin -\operatorname{int} C(u) \\ \text{for } p \in A(v),\ q \in T(v)\}, \end{aligned}$$

is weakly closed.

Since $\overline{u} \in \bigcap_{j=1}^{m} cl_V(F(v_j))$ and

$$\begin{aligned} cl_V(F(v_j)) &= cl_V\left(\{u \in V : g(u,v_j) \notin -\operatorname{int} C(u)\}\right) \\ &\subseteq cl_V(M_{v_j}) \\ &= M_{v_j} \quad \text{for} \quad j = 1,2,\cdots,m, \end{aligned}$$

$$f(N(p,q),\overline{u},v_j) + h(\overline{u},v_j) - h(\overline{u},\overline{u}) \notin -\operatorname{int} C(\overline{u})$$

for $p \in A(v_j)$, $q \in T(v_j)$ for $j = 1,2,\cdots,m$. Hence $\overline{u} \in \bigcap_{j=1}^{m} G(v_j)$. Thus $\{G(v) : v \in D\}$ has the fip, which implies that $\bigcap_{v\in D} G(v)$ is nonempty. Hence (MVELP 1) is solvable.

That is, there exists $u_0 \in K \subseteq D$ such that

$$\begin{aligned} &f(N(p,q),u_0,v) + h(u_0,v) - h(u_0,u_0) \notin -\operatorname{int} C(u_0) \\ &\text{for all } v \in D,\ p \in A(v),\ q \in T(v). \end{aligned}$$

(II) For $v \in D$, letting $v_\lambda = \lambda v + (1-\lambda)u_0$, $0 < \lambda < 1$, we have $v_\lambda \in D$. Since $u_0 \in G(v_\lambda)$,

$$f(N(p_\lambda, q_\lambda), u_0, v_\lambda) + h(u_0, v_\lambda) - h(u_0, u_0) \notin -\operatorname{int} C(u_0) \tag{2.1}$$

for $p_\lambda \in A(v_\lambda)$, $q_\lambda \in T(v_\lambda)$.

Since a map $v \mapsto f(z,u,v) + h(u,v)$ is P-convex for $(z,u) \in L(X,Y) \times D$, by conditions (C-1) and (C-3) we have

$$\begin{aligned}
&f(N(p_\lambda, q_\lambda), v_\lambda, v_\lambda) + h(v_\lambda, v_\lambda) - h(u_0, u_0)\\
&= f(N(p_\lambda, q_\lambda), v_\lambda, \lambda v + (1-\lambda)u_0) + h(v_\lambda, \lambda v + (1-\lambda)u_0) - h(u_0, u_0)\\
&\in \lambda[f(N(p_\lambda, q_\lambda), v_\lambda, v) + h(v_\lambda, v) - h(u_0, u_0)]\\
&\quad + (1-\lambda)[f(N(p_\lambda, q_\lambda), v_\lambda, u_0) + h(v_\lambda, u_0) - h(u_0, u_0)] - P\\
&\subseteq \lambda[f(N(p_\lambda, q_\lambda), v_\lambda, v) + h(v_\lambda, v) - h(u_0, u_0)]\\
&\quad + (1-\lambda)[f(N(p_\lambda, q_\lambda), v_\lambda, u_0) + h(v_\lambda, u_0) - h(u_0, u_0)] - C(u_0)\\
&\subseteq \lambda[f(N(p_\lambda, q_\lambda), v_\lambda, v) + h(v_\lambda, v) - h(u_0, u_0)]\\
&\quad - (1-\lambda)[f(N(p_\lambda, q_\lambda), u_0, v_\lambda) + h(u_0, v_\lambda) - h(u_0, u_0)] - C(u_0).
\end{aligned} \tag{2.2}$$

Hence

$$f(N(p_\lambda, q_\lambda), v_\lambda, v) + h(v_\lambda, v) - h(u_0, u_0) \notin -\operatorname{int} C(u_0).$$

Indeed, suppose to the contrary that

$$f(N(p_\lambda, q_\lambda), v_\lambda, v) + h(v_\lambda, v) - h(u_0, u_0) \in -\operatorname{int} C(u_0).$$

Since $-\operatorname{int} C(u_0)$ is a convex cone

$$\lambda[f(N(p_\lambda, q_\lambda), v_\lambda, v) + h(v_\lambda, v) - h(u_0, u_0)] \in -\operatorname{int} C(u_0).$$

Since condition (C-2) implies that

$$f(N(p_\lambda, q_\lambda), v_\lambda, v_\lambda) \in C(u_0),$$

so from (2.2) we derive

$$\begin{aligned}
&(1-\lambda)[f(N(p_\lambda, q_\lambda), u_0, v_\lambda) + h(u_0, v_\lambda) - h(u_0, u_0)]\\
&\in \lambda[f(N(p_\lambda, q_\lambda), v_\lambda, v) + h(v_\lambda, v) - h(u_0, u_0)] - f(N(p_\lambda, q_\lambda), v_\lambda, v_\lambda) - C(u_0)\\
&\subseteq -\operatorname{int} C(u_0) - C(u_0) - C(u_0)\\
&\subseteq -\operatorname{int} C(u_0) - C(u_0)\\
&= -\operatorname{int} C(u_0).
\end{aligned}$$

Thus

$$f(N(p_\lambda, q_\lambda), u_0, v_\lambda) + h(u_0, v_\lambda) - h(u_0, u_0) \in -\operatorname{int} C(u_0),$$

which contradicts (2.1).

On the other hand, since $A(v_\lambda)$ and $A(u_0)$ (respectively, $T(v_\lambda)$ and $T(u_0)$) are bounded closed subsets of $L(X,Y)$, by Nadler's Theorem for $p_\lambda \in A(v_\lambda)$ (resp., $q_\lambda \in T(v_\lambda)$), there exists an $s_\lambda \in A(u_0)$ (resp., $t_\lambda \in T(u_0)$) such that

$$\|p_\lambda - s_\lambda\| \le (1+\lambda)H(A(v_\lambda), A(u_0))$$
$$(\text{resp., } \|q_\lambda - t_\lambda\| \le (1+\lambda)H(T(v_\lambda), T(u_0))).$$

Since $L(X,Y)$ is reflexive and $A(u_0)$ (resp., $T(u_0)$) is a bounded closed and convex subsets in $L(X,Y)$, $A(u_0)$ (resp., $T(u_0)$) is also weakly compact in $L(X,Y)$. Hence we may assume $s_\lambda \rightharpoonup s_0 \in A(u_0)$ (resp., $t_\lambda \rightharpoonup t_0 \in T(u_0)$) as $\lambda \to 0^+$. Moreover for each $\varphi \in (L(X,Y))^*$, the dual space of $L(X,Y)$,

$$\begin{aligned}
&\|\varphi(N(p_\lambda, q_\lambda) - N(s_0, t_0))\| \\
&\le |\varphi(N(p_\lambda, q_\lambda) - N(s_\lambda, q_\lambda))| + |\varphi(N(s_\lambda, q_\lambda) - N(s_0, q_\lambda))| \\
&\quad + |\varphi(N(s_0, q_\lambda) - N(s_0, t_\lambda)| + |\varphi(N(s_0, t_\lambda) - N(s_0, t_0)| \\
&\le \|\varphi\|\,\|p_\lambda - s_\lambda\| + \|\varphi\|\,\|s_\lambda - s_0\| + \|\varphi\|\,\|q_\lambda - t_\lambda\| + \|\varphi\|\,\|t_\lambda - t_0\| \\
&\le \|\varphi\|(1+\lambda)H(A(v_\lambda), A(u_0)) + \|\varphi\|\,\|s_\lambda - s_0\| \\
&\quad + \|\varphi\|(1+\lambda)H(T(v_\lambda), T(u_0)) + \|\varphi\|\,\|t_\lambda - t_0\|.
\end{aligned}$$

Hence by condition (C-4),

$$p_\lambda \rightharpoonup s_0 \text{ and } q_\lambda \rightharpoonup t_0 \text{ as } \lambda \to 0^+.$$

Thus according to condition (C-5),

$$\|f(N(p_\lambda, q_\lambda), v_\lambda, v) - f(N(s_0, t_0), u_0, v)\| \to 0 \text{ as } \lambda \to 0^+.$$

Since $h : D \times D \to Y$ is continuous with respect to the first and second arguments, by condition (C-5),

$$f(N(p_\lambda, q_\lambda), v_\lambda, v) + h(v_\lambda, v) - h(v_\lambda, v_\lambda) - f(N(s_0, t_0), u_0, v) + h(u_0, v) - h(u_0, u_0)$$
$$\rightharpoonup 0 \text{ as } \lambda \to 0^+.$$

Since $W(u_0) = Y \setminus (-\operatorname{int} C(u_0))$ is weakly closed, (MVELP 2) is solvable. □

Theorem 2.2. *If we change (A-1) with (A′-1) and (B) with (B′), then (MVELP 1) is solvable. Moreover we add (C) to them, then (MVELP 2) is also sovable.*

(A′-1) *Assume that $C(u)$ is a closed proper convex solid cone of Y with $C(u) \ne Y$ for all $u \in D$.*

(B′) *there exists a bifunction $g : D \times D \to Y$ such that*

(B′-1) *$g(u,v) \notin -int C(u)$ for u, $v \in D$,*

(B′-2) *$g(u,v) - f(N(p,q), u, v) \in -C(u)$ for u, $v \in D$, $p \in A(v)$, $q \in T(v)$,*

(B′-3) *$\{v \in D : g(u,v) + h(v,u) - h(u,u) \in int C(u)\}$ is convex for each $u \in D$;*

(B′-4) *there exists a weakly compact convex subset* $K \subseteq D$ *such that for each* $u \in D \backslash K$ *there exists* $v_0 \in D$ *satisfying*

$$f(N(p,q),u,v_0) + h(u,v_0) - h(u,u) \in -int\, C(u) \text{ for } p \in A(u),\ q \in T(u).$$

Proof. Define a map $G : D \to 2^K$ by for $v \in D$,

$$G(v) = \{u \in K : \ f(N(p,q),u,v) + h(u,v) - h(u,u) \notin -\mathrm{int}\, C(u) \text{ for all } p \in A(v),\ q \in T(v)\}.$$

Following the proof of Theorem 2.1, we can prove that $G(v)$ is weakly closed for each $v \in D$. We now claim that $\bigcap_{v \in D} G(v) \neq \emptyset$. Indeed, since K is weakly compact, it is sufficient to show that the family $\{G(v)\}_{v\in D}$ has the fip. Let $\{v_1, v_2, \cdots, v_n\}$ be a finite subset of D and set $B = \mathrm{co}\{K \cup \{v_1, v_2, \cdots, v_n\}\}$. Then B is a weakly compact and convex subset of D.

We define maps $F_1, F_2 : B \to 2^B$ as follows:

$$F_1(v) \ = \ \{u \in B : \ f(N(p,q),u,v) + h(u,v) - h(u,u) \notin -\mathrm{int}\, C(u),$$

for all $p \in A(v),\ q \in T(v)\}$, for all $v \in B$,
and

$$F_2(v) \ = \ \{u \in B : \ g(u,v) + h(v,u) - h(u,u) \notin -\mathrm{int}\, C(u)\} \text{ for all } v \in B.$$

By conditions (B′-1) and (B′-2), we have

$$g(v,v) + h(v,v) - h(v,v) \notin -\mathrm{int}\, C(v), \text{ for all } v \in B \text{ and}$$

$$g(v,v) - f(N(p,q),v,v) \in -C(v) \text{ for all } p \in A(v),\ q \in T(v).$$

Now Lemma 1.1 (ii) guarantees that

$$f(N(p,q),v,v) + h(v,v) - h(v,v) \notin -\mathrm{int}\, C(v) \text{ for all } p \in A(v),\ q \in T(v)$$

and so $F_1(v)$ is nonempty. Now since $F_1(v)$ is a weakly closed subset of the weakly compact subset B, we know that $F_1(v)$ is weakly compact.

Next, we claim that F_2 is a KKM-map. Indeed suppose that there exists a finite subset $\{u_1, u_2, \cdots, u_m\}$ of B and $\alpha_i \geq 0$, $i = 1, 2, \cdots, m$ with $\sum_{i=1}^{m} \alpha_i = 1$ such that

$$\hat{u} \ = \ \sum_{i=1}^{m} \alpha_i u_i \in \bigcup_{j=1}^{m} F_2(u_j).$$

Then

$$g(\hat{u}, u_j) + h(\hat{u}, u_j) - h(\hat{u}, \hat{u}) \in -\mathrm{int}\, C(\hat{u}),\ j = 1, 2, \cdots, m.$$

From condition (B′-3), we derive

$$g(\hat{u}, \hat{u}) \ = \ g(\hat{u}, \hat{u}) + h(\hat{u}, \hat{u}) - h(\hat{u}, \hat{u}) \in -\mathrm{int}\, C(\hat{u}),$$

which contradicts condition (B′-1). Thus F_2 is a KKM-map. From condition (B′-2) and Lemma 1.1(ii), we have

$$F_2(v) \subseteq F_1(v) \text{ for all } v \in B.$$

Indeed if $u \in F_2(v)$, then

$$g(u,v) + h(u,v) - h(u,u) \notin -\operatorname{int} C(u).$$

By condition (B′-2) we have

$$g(u,v) - f(N(p,q),u,v) \in -C(u), \text{ for all } p \in A(v),\ q \in T(v).$$

Consequently it follows from Lemma 1.1 (ii) that

$$f(N(p,q),u,v) + h(u,v) - h(u,u) \notin -\operatorname{int} C(u) \text{ for all } p \in A(v),\ q \in T(v)$$

that is $u \in F_1(v)$. This shows that F_1 is also a KKM map.

According to Fan-KKM Theorem, there exists $\bar{u} \in B$ such that $\bar{u} \in F_1(v)$ for all $v \in B$, that is, there exists $\bar{u} \in B$ such that

$$f(N(p,q),\bar{u},v) + h(\bar{u},v) - h(\bar{u},\bar{u}) \notin -\operatorname{int} C(\bar{u}),$$

for all $v \in B$, $p \in A(v)$, $q \in T(v)$.

By condition (C-6), we get $\bar{u} \in K$ and $\bar{u} \in G(v_i)$, $i = 1, 2, \cdots, n$. Hence $\{G(v)\}_{v\in D}$ has the fip and moreover

$$\bigcap_{v\in D} G(v) \neq \emptyset,$$

that is, there exists $u_0 \in K \subseteq D$ such that

$$f(N(p,q),u_0,v) + h(u_0,v) - h(u_0,u_0) \notin -\operatorname{int} C(u_0),$$
$$\text{for all } v \in D,\ p \in A(v) \text{ and } q \in T(v).$$

□

For the remainder of the proof, we can derive the conclusion of Theorem 2.2 by following the same proof as in Theorem 2.1.

Remark 2.1. (1) If we take $N(p,q) = N(p_0)$ and $h(u,u) = h(u)$, then we have the following problem: Find $u_0 \in D$, there exists $p_0 \in A(u_0)$ such that,

$$f(N(p_0),u_0,v) + h(v) - h(u_0) \notin -\operatorname{int} C(u_0) \text{ for all } v \in D,$$

which is considered by Ceng *et al.* [3].

(2) In particular, if we put $f(z,x,y) = \langle z, \eta(y,x)\rangle$ for all $(z,x,y) \in L(X,Y) \times D \times D$, where $\eta : D \times D \to X$ then the above problem reduces to

the *generalized mixed vector variational type inequality problem.* Find $u_0 \in D$, there exist $p_0 \in A(u_0)$, $q_0 \in T(u_0)$ such that

$$\langle N(p_0, q_0), \eta(v, u_0)\rangle + h(u_0, v) - h(u_0, u_0) \notin -\operatorname{int} C(u_0) \quad \text{for all } v \in D,$$

which is the variant for of Zhao and Zia [15] and Ahmad and Salahuddin [1].

(3) If we take A as single-valued mapping, then we have the following *generalized vector variational type inequality problem*: Find $u_0 \in D$, there exists $q_0 \in T(u_0)$ such that

$$\langle N(u_0, q_0), \eta(v, u_0)\rangle + h(u_0, v) - h(u_0, u_0) \notin -\operatorname{int} C(u_0) \quad \text{for all } v \in D,$$

considered by Lee *et al.* [10].

(4) Again if we take $N(u_0, q_0) \cong N(q_0)$ and $h(u, u) \cong h(u)$, then we have the following problem of finding $u_0 \in D$ such that

$$\langle N(q_0), \eta(v, u_0)\rangle + h(v) - h(u) \notin -\operatorname{int} C(u_0) \text{ for all } v \in D,$$

which is the *generalized vector variational like inequality problem* considered by Khan and Salahuddin [7].

(5) If $h \cong 0$, N is an identity mapping, then we have the finding $u_0 \in D$ such that $q_0 \in T(u_0)$ and

$$\langle q_0, \eta(v, u_0)\rangle \notin -\operatorname{int} C(u_0) \text{ for all } v \in D,$$

which is the vector variational like inequality.

(6) If T is a single-valued mapping, then we have the finding $u_0 \in D$ such that

$$\langle T(u_0), \eta(v, u_0)\rangle \notin -\operatorname{int} C(u_0) \text{ for all } v \in D,$$

which is called a *generalized vector variational like inequality* considered and studied by Siddiqi *et al.* [13].

(7) If $\eta(v, u_0) = v - u_0$, then we have the finding $u_0 \in D$ such that

$$\langle T(u_0), v - u_0\rangle \notin -\operatorname{int} C(u_0) \text{ for all } v \in D,$$

which is called *vector variational inequality* considered and studied by Lee *et al.* [10].

(8) If $Y = R$, $L(X, Y) = X^*$ (the dual of X), $C(u) = R^+$ for $u \in D$ then we have the finding $u_0 \in D$ such that

$$\langle T(u_0), v - u_0\rangle \;\geq\; 0 \text{ for all } v \in D,$$

which is called *classical variational inequality*, considered by Hartman and Stampacchia [6].

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