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VIRTUAL LINKS WITH NORMAL DIAGRAMS

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Abstract. We show that if two virtual link diagrams which are normal are equivalent under generalized Reidemeister moves, then they are equivalent under generalized Reidemeister moves preserving the normality of diagrams.

1. Introduction

L. H. Kauffman [8] introduced virtual knot theory as a generalization of classical knot theory. A virtual link diagram is a link diagram in $\mathbb{R}^2$ possibly with some encircled crossings without over/under informaton, called *virtual crossings.* A virtual link is the equivalence class of such a link diagram by generalized Reidemeister moves in Figure 1.

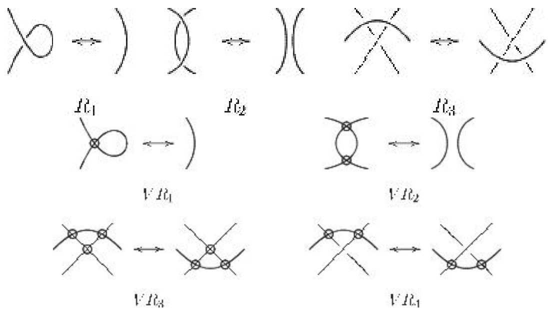

Figure 1. generalized Reidemeister Moves

It is known [1, 8] that if two classical knot diagrams are equivalent under generalized Reidemeister moves, then they are equivalent under the classical Reidemeister moves. In this sense classical knot theory is properly embedded in virtual knot theory and any invariant of virtual links is a generalization of

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a classical invariant. In [6], Kauffman showed that the group of an oriented classical link, the bracket polynomial and hence the Jones polynomial are generalized naturally in virtual links.

On the other hand, N. Kamada[2] introduced the notion of a *normal* or *checkerboard colorable* virtual link diagram. It is well known[2] that every calssical link diagram and alternating virtual link diagram are normal. But not every virtual link diagram is normal. It has been shown that many results on classical knots and links can be extended to checkerboard colorable virtual links[2, 4, 5].

The purpose of this paper is to show that if two virtual link diagrams which are *normal* are equivalent under generalized virtual Reidemeister moves, then they are equivalent under generalized Reidemeister moves preserving the normality.

2. Normal diagrams of virtual links

In this section, we show that if two virtual link diagrams which are *normal* are equivalent under generalized virtual Reidemeister moves, then they are equivalent under generalized Reidemeister moves preserving the normality. For example, any classical link diagram is always *normal* but the *virtual trefoil knot diagram* D_v in Figure 2 is not *normal*.

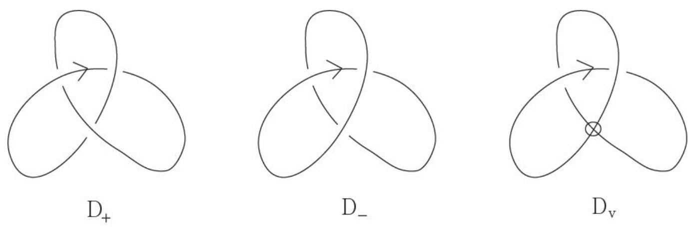


FIGURE 2

We start basic definitions and results which are needed throughout this paper.

A *state* of a virtual link diagram D is a union of immersed loops in $\mathbb{R}^2$ with only virtual crossings, which is obtained by splicing all classical crossings of D. At each spliced crossing we attach a chord labeled A or B to represent the splicing direction as shown in Figure 3.

A state σ of a virtual link diagram D is *normal* if for any classical crossing x of D, the loops of σ spliced at x are of type (1) or (2) in Figure 4. A virtual link diagram D is *normal* if every state of D is *normal*. A virtual link K is *normal* if K is the equivalence class of a normal diagram under equivalence relation

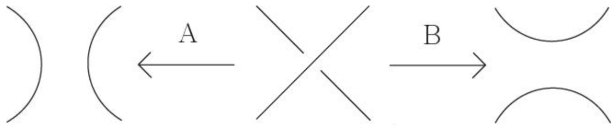
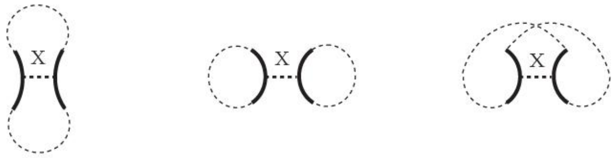


FIGURE 3



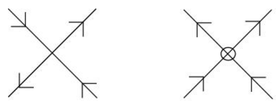
(1) (2) (3)

FIGURE 4

generated by *generalized Reidemeister moves*. This concept is eventually the same as *checkerboard colorable* virtual link introduced in [2].

For a virtual link diagram D, we denote by $\bar{D}$ the union of immersed circles in $\mathbb{R}^2$ obtained by ignoring the over- and under-information at classical crossings of D and leaving the virtual information unchanged so that the edges of $\bar{D}$ are oriented alternately at each vertex, which corresponds to a real crossing of D.

Definition. [5] $\bar{D}$ admits an alternate orientation if all edges (when regarding $\bar{D}$ as a 4-valent planar graph) can be oriented as shown in Figure 5.



(1) (2)

FIGURE 5

Proposition 2.1. [5] *A virtual link diagram D is normal if and only if $\bar{D}$ admits an alternate orientation.*

Proposition 2.2. [5] *Let $D = D_1 \cup \cdots \cup D_\mu$ be a virtual link diagram of a μ-component virtual link. If D is normal, then the number of all classical crossings between D_i and $D \setminus D_i$ is even.*

Remark 2.3. We assign an orientation alternately for vertices of $\bar{D}$ corresponding to real crossings of D and assign an orientation for virtual crossings as shown in (2) of Figure 5. If $\bar{D}$ does not admit an alternate orientation, either the edges of $\bar{D}$ at some vertex $\bar{p}$ can be oriented as in Figure 6 or we cannot assign a suitable orientation as shown in the right of Figure 7 because of Proposition 2.2. For example, there are the virtual Hopf link and the virtual trefoil knot diagram which do not admit an alternate orientation, see Figure 7.

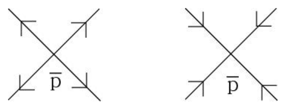


FIGURE 6

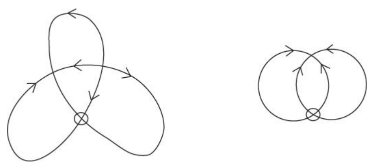


FIGURE 7

A local move for a diagram such as a Reidemeister move is applied in a local disk. We call such a disk *a stage* for the move. To show the main result of this section, we begin with the following lemmas which illustrate the behavior of generalized Reidemeister moves.

Lemma 2.4. *Let D be a normal diagram and D' be a virtual link diagram. Suppose D' is obtained from D by applying a single generalized Reidemeister move Ω, where Ω is not $\mathbf{R_2}$-move. Then D' is a normal diagram.*

Proof. Let B be a stage for a given generalized Reidemeister move Ω. Since D is a normal diagram, $\bar{D}$ admits an alternate orientation. In the complement of B, $\bar{D}$ and $\bar{D}'$ admit the same alternate orientation. It is easy to check that D' is a normal diagram as shown in Figure 8. □

Lemma 2.5. *Let D be a normal diagram and D' be a virtual link diagram. Suppose D' is obtained from D by applying a single $\mathbf{R_2}$-move.*

(1) If the stage B is Figure 9, then D' is a normal diagram.

(2) If the stage B is Figure 10, then D' is not a normal diagram.

Proof. (1) Since D is a normal diagram, $\bar{D}$ admits an alternate orientation. In the complement of B, there is an alternate orientation of $\bar{D}'$. According to Figure 9, $\bar{D}'$ admits an alternate orientation.

(2) As shown in Figure 10, $\bar{D}'$ does not admit an alternate orientation. □

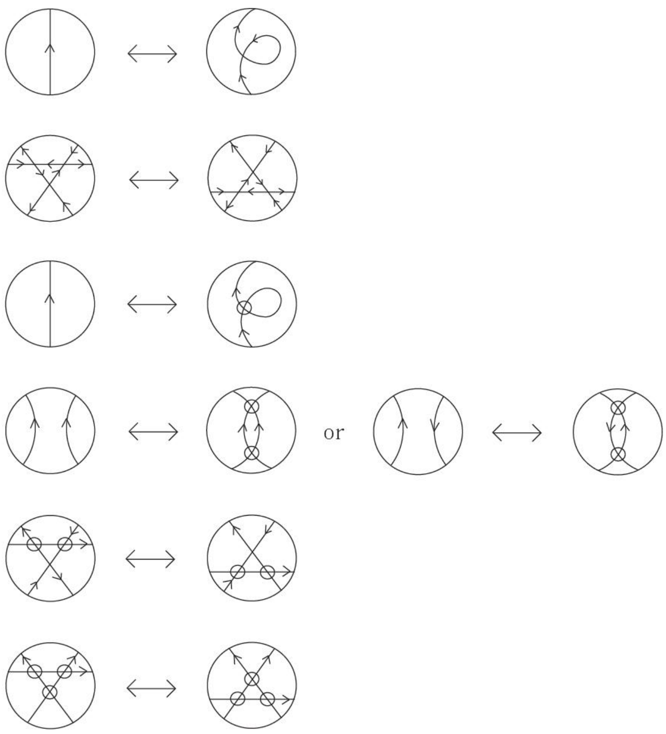


FIGURE 8

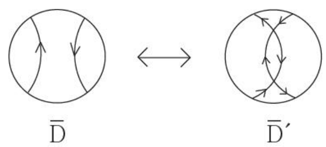


FIGURE 9

Lemma 2.6. *Let D be a normal diagram and D' be a virtual link diagram which is not normal. Suppose D' is obtained from D by applying a $\mathbf{R_2}$-move Ω. Then there is a normal diagram D'', which is equivalent to D', obtained from D by replacing Ω with a $\mathbf{VR_2}$-move Γ.*

Proof. Let B be a stage for a $\mathbf{R_2}$-move Ω. Since D is a normal diagram, $\bar{D}$ admits an alternate orientation. Being D' a virtual link diagram which is not normal, we assume that the two arcs of $\bar{D}$ with an alternate orientation in B have parallel orientations as in Figure 10. Since D and D' are the same in the complement of B, $\bar{D}'$ does not admit an alternate orientation as in Figure 10.

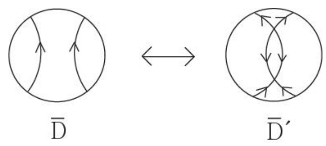


FIGURE 10

Applying a $\mathbf{VR_2}$-move Γ instead of $\mathbf{R_2}$-move Ω, we get a normal diagram D'' by Lemma 2.4, which is equivalent to D'. □

Lemma 2.7. *Let D_1 and D_2 be virtual link diagrams which are equivalent to a virtual normal link L. Suppose D_2 is obtained from D_1 by applying a $\mathbf{R_3}$-move Ω in a stage B and $\bar{D}_1$ does not admit an alternate orientation, so that the orientation in a stage B is like as shown in Figure 6. Then there are virtual link diagrams $D_i'(i=1,2)$ so that D_2' is obtained from D_1' by applying a $\mathbf{VR_4}$-move Γ, instead of $\mathbf{R_3}$-move Ω, where D_1' is a virtual link diagram obtained from D_1 by replacing two real crossings in a stage B with two virtual crossings.*

Proof. Let p_i be real crossings of D_1 in B and $\bar{p_i}$ the corresponding vertices of $\bar{D_1}$ for $i = 1, 2, 3$. Since $\bar{D_1}$ does not admit an alternate orientation, the edges of $\bar{D_1}$ at some vertex $\bar{p_i}$ in a stage B can be oriented as in Figure 6. By traveling along the components of $\bar{D}_1$, we can assume that any alternate orientation fails at two points $\bar{p_1}$ and $\bar{p_2}$. Replacing real crossings p_1 and p_2 of D_1 by virtual crossings, we obtain a virtual link diagram D_1'. Now, by applying $\mathbf{VR_4}$-move Γ, we have a virtual diagram D_2'. □

Lemma 2.8. *Let D_1 and D_2 be virtual link diagrams which are equivalent to a virtual normal link K. Suppose D_2 is obtained from D_1 by applying a $\mathbf{VR_4}$-move Ω in a stage B and D_1 does not admit an alternate orientation, so that the orientation in a stage B is like as shown in Figure 6. Then there are virtual link diagrams $D_i'(i=1,2)$ so that D_2' is obtained from D_1' by applying a $\mathbf{VR_3}$-move Γ, instead of $\mathbf{VR_4}$-move Ω, where D_1' is a virtual link diagram obtained from D_1 by replacing a real crossing in a stage B with a virtual crossing.*

Proof. Let p be a real crossing of D_1 in B and $\bar{p}$ the corresponding vertex of $\bar{D_1}$. Since $\bar{D_1}$ does not admit an alternate orientation, the edges of $\bar{D_1}$ at the vertex $\bar{p}$ in a stage B can be oriented as in Figure 6. By traveling along the components of $\bar{D_1}$, we can assume that any alternate orientation fails at $\bar{p}$. Replacing the real crossing p of D_1 by a virtual crossing, we obtain a virtual link diagram D_1'. Now, by applying $\mathbf{VR_3}$-move Γ, we have a virtual diagram D_2'. □

The following is the main theorem of this section.

Theorem 2.9. *If D and D' are normal diagrams of virtual links such that D and D' are equivalent under generalized virtual Reidemeister moves, then*

D and D' are equivalent under generalized Reidemeister moves preserving the normality.

Proof. Since D and D' are normal diagrams of virtual links such that D and D' are equivalent under generalized virtual Reidemeister moves, there is a finite sequence of generalized Reidemeister moves $\{\Omega_i\}_{i=1,\cdots,n}$ and a sequence of virtual link diagrams $\{D_i\}_{i=0,1,\cdots,n}$ such that

$$D = D_0 \xrightarrow{\Omega_1} D_1 \xrightarrow{\Omega_2} D_2 \rightarrow, \cdots, \rightarrow D_{n-1} \xrightarrow{\Omega_n} D_n = D'.$$

Since D_0 is a normal diagram, we can assume that D_1 is not a normal diagram. By Lemma 2.4 and 2.5, D_1 is obtained from D_0 by applying a single $\mathbf{R_2}$-move Ω_1 in a stage B_1 as $\overline{D_0}$ and $\overline{D_1}$ are in Figure 10. According to Lemma 2.5, we obtain a normal diagram D_1' by replacing Ω_1 with a $\mathbf{VR_2}$-move Γ_1.

For the following step, we consider two cases. First, we assume that the stage B_2 for Ω_2 does not contain any crossing in B_1 for Ω_1. If Ω_2 is not a $\mathbf{R_2}$-move as in Figure 10, then we choose $\Gamma_2 = \Omega_2$. If Ω_2 is a $\mathbf{R_2}$-move as in Figure 10, we choose a $\mathbf{VR_2}$-move Γ_2 so that we have a normal diagram D_2'.

Next, we assume that the stage B_2 for Ω_2 contains at least one crossing in B_1 for Ω_1, which is replaced by a virtual crossing via Γ_1. According to lemma 2.7 or 2.8, we obtain a normal diagram D_2' by replacing Ω_2 with Γ_2 preserving the normality.

By proceeding step by step, we construct a finite sequence of normal diagrams $\{D_i'\}_{i=0,1,\cdots,n}$ and a finite sequence of generalized Reidemeister moves $\{\Gamma_i\}_{i=1,\cdots,n}$ preserving the normality such that

$$D = D_0 \xrightarrow{\Gamma_1} D_1' \xrightarrow{\Gamma_2} D_2' \rightarrow, \cdots, \rightarrow D_{n-1}' \xrightarrow{\Gamma_n} D_n = D'.$$

□

Remark 2.10. If D is a classical link diagram, the stage B_1 in Figure 10 cannot be happened but it is possible for the case of virtual link diagrams.

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