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REMARKS ON A SUMMATION FORMULA FOR THREE-VARIABLES HYPERGEOMETRIC FUNCTION X_8 AND CERTAIN HYPERGEOMETRIC TRANSFORMATIONS

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Abstract. The first object of this note is to show that a summation formula due to Padmanabham for three-variables hypergeometric function X_8 introduced by Exton can be proved in a different (from Padmanabham's and his observation) yet, in a sense, conventional method, which has been employed in obtaining a variety of identities associated with hypergeometric series. The second purpose is to point out that one of two seemingly new hypergeometric identities due to Exton was already recorded and the other one is easily derivable from the first one. A corrected and a little more compact form of a general transform involving hypergeometric functions due to Exton is also given.

1. Introduction and preliminaries

In 1982, Exton [3] introduced a triple hypergeometric function of the second order X_8 whose series representation is given by

$$\begin{aligned} &X_8(a;\, b,\, c;\, d,\, e,\, f;\, x,\, y,\, z) \\ &= \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n+p}\,(b)_n\,(c)_p\,x^m\,y^n\,z^p}{(d)_m\,(e)_n\,(f)_p\,m!\,n!\,p!}, \end{aligned} \tag{1.1}$$

where $(z)_m$ denotes the Pochhammer symbol defined by

$$(z)_m = \frac{\Gamma(z+m)}{\Gamma(z)} \quad (z \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ m \in \mathbb{N}_0),$$

$\mathbb{Z}_0^-$ and $\mathbb{N}_0$ being the sets of nonpositive and nonnegative integers, respectively. The precise three-dimensional region of convergence of (1.1) is given by Srivastava and Karlsson [10, p. 101, Entry 41a]:

$$4r = (s+t-1)^2, \quad |x| < r, \quad |y| < s, \quad \text{and} \quad |z| < t$$

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where the positive quantities r, s and t are associated radii of convergence. For more details about this function and many other three-variables hypergeometric functions, we also refer to Srivastava and Karlsson [10].

Exton [3] gave the Laplace integral representation of (1.1) as follows:

$$\begin{aligned}
&X_8(a;\, b,\, c;\, d,\, e,\, f;\, x,\, y,\, z) \\
&= \frac{1}{\Gamma(a)} \int_0^\infty e^{-u}\, u^{a-1}\, {}_0F_1\left(-;\, d;\, u^2\, x\right)\, {}_1F_1\left(b;\, e;\, y\, u\right)\, {}_1F_1\left(c;\, f;\, z\, u\right)\, du,
\end{aligned} \tag{1.2}$$

provided $\Re(a) > 0$.

It is interesting to observe that X_8 reduces to the Horn's function H_4 and the Appell's function F_2 when $z \to 0$ and $x \to 0$, respectively.

The following interesting summation formula for X_8 whose terms are involved in the generalized hypergeometric functions ${}_pF_q$

$$\begin{aligned}
&X_8(a;\, b,\, c;\, d,\, e,\, f;\, x,\, y,\, z) \\
&= \sum_{n=0}^{\infty} \frac{(a)_n\, (b)_n}{(e)_n\, n!}\, {}_3F_2\left(\begin{matrix} -n,\ 1-n-e,\ c \\ 1-n-b,\ f \end{matrix} \;\middle|\; -\frac{z}{y} \right) \\
&\quad \cdot {}_2F_1\left(\frac{a+n}{2},\ \frac{a+n+1}{2};\ d \;\middle|\; 4x \right) y^n,
\end{aligned} \tag{1.3}$$

was presented by Padmanabham [7] who proved the identity (1.3) by using the Cauchy product for the two ${}_1F_1$ series in the integrand of (1.2), and applying term-by term integration, and then evaluating the resulting Gamma integral. Padmanabham [7] also observed that the identity (1.3) can be derived as a special case of a known result of Srivastava [11, p. 156, Eq. (58)] by making suitable substitutions and changing parameters:

$$N = 1,\ \Omega_{m,n} = (a)_{m+n},\ \gamma_n = 1,\ \delta_n = \frac{(c)_n}{(f)_n}\ \ (m,\, n \in \mathbb{N}_0)\,,$$

$$l = m = 1\ (c_1 = b \text{ and } d_1 = e);\ y \to -\frac{z}{y} \text{ and } t = y.$$

Here first we aim at proving the identity (1.3) by making use a rather *conventional* method (different from Padmanabham's one and his subsequent observation), which has usually been employed in obtaining a diversity of identities associated with hypergeometric series (or functions) (see, e.g., [8]). Secondly we point out that one of two seemingly new hypergeometric identities due to Exton was already recorded and the other one is easily derivable from the first one. A corrected and a little more compact form of a general transform involving hypergeometric functions due to Exton is also presented.

2. A conventional method of derivation of (1.3)

Let, for convenience, $X_8 := X_8(a;\, b,\, c;\, d,\, e,\, f;\, x,\, y,\, z)$. A use of the easily-derivable Pochhammer symbol identity

$$(a)_{2m+n+p} = (a+n+p)_{2m}\,(a)_{n+p}$$

and then

$$(a+n+p)_{2m} = 2^{2m}\left(\frac{a+n+p}{2}\right)_m\left(\frac{a+n+p+1}{2}\right)_m$$

is made in (1.1), and after a little simplification, a double series of X_8 whose terms are involved in ${}_2F_1$ is obtained:

$$\begin{aligned}
X_8 &= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{n+p}\,(b)_n\,(c)_p\,y^n\,z^p}{(e)_n\,(f)_p\,n!\,p!}\sum_{m=0}^{\infty}\frac{\left(\frac{a+n+p}{2}\right)_m\left(\frac{a+n+p+1}{2}\right)_m 2^{2m}}{(d)_m\,m!}\,x^m \\
&= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{n+p}\,(b)_n\,(c)_p\,y^n\,z^p}{(e)_n\,(f)_p\,n!\,p!}\,{}_2F_1\left(\frac{a+n+p}{2},\ \frac{a+n+p+1}{2};\ d\;\middle|\;4x\right),
\end{aligned}$$

which, upon using, for $(b)_n$ and $(e)_n$, the identity

$$\Gamma(\alpha-n)\,\Gamma(1-\alpha+n) = (-1)^n\,\Gamma(\alpha)\,\Gamma(1-\alpha)$$

yields

$$\begin{aligned}
X_8 =& \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{n+p}\,(b)_{n+p}\,(c)_p}{(e)_{n+p}\,(f)_p\,n!\,p!}\,\frac{\Gamma(1-e-n)\,\Gamma(1-b-p-n)}{\Gamma(1-e-n-p)\,\Gamma(1-b-n)}\,y^n\,z^p \\
&\cdot {}_2F_1\left(\frac{a+n+p}{2},\ \frac{a+n+p+1}{2};\ d\;\middle|\;4x\right).
\end{aligned}$$

Using an identity of formal manipulation of double series (for more identities, see [2])

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}A_{p,n} = \sum_{n=0}^{\infty}\sum_{p=0}^{n}A_{p,n-p}$$

in the last identity, after a little simplification, we get

$$\begin{aligned}
X_8 =& \sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{(a)_n\,(b)_n\,(c)_p}{(e)_n\,(f)_p\,(n-p)!\,p!}\,\frac{\Gamma(1-n-e+p)\,\Gamma(1-n-b)\,\left(\frac{z}{y}\right)^p\,y^n}{\Gamma(1-n-b+p)\,\Gamma(1-n-e)} \\
&\cdot {}_2F_1\left(\frac{a+n}{2},\ \frac{a+n+1}{2};\ d\;\middle|\;4x\right) \\
=& \sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{(a)_n\,(b)_n}{(e)_n\,(n-p)!}\,\frac{(1-n-e)_p\,(c)_p}{(1-n-b)_p\,(f)_p\,p!}\left(\frac{z}{y}\right)^p\,y^n \\
&\cdot {}_2F_1\left(\frac{a+n}{2},\ \frac{a+n+1}{2};\ d\;\middle|\;4x\right).
\end{aligned}$$

Using the identity

$$(n-p)! = \frac{(-1)^p\, n!}{(-n)_p},$$

we finally obtain the desired result (1.3) to complete the proof:

$$\begin{aligned} X_8 &= \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(e)_n\, n!}\, y^n \sum_{p=0}^{n} \frac{(-1)^p\,(-n)_p\,(c)_p\,(1-n-e)_p}{(f)_p\,(1-n-p)_p\, p!} \left(\frac{z}{y}\right)^p \\ &\quad \cdot {}_2F_1\left(\frac{a+n}{2},\ \frac{a+n+1}{2};\ d \,\middle|\, 4x\right) \\ &= \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(e)_n\, n!}\, y^n\, {}_3F_2\left(\begin{array}{c} -n,\ 1-n-e,\ c \\ 1-n-b,\ f \end{array} \,\middle|\, -\frac{z}{y}\right) \\ &\quad \cdot {}_2F_1\left(\frac{a+n}{2},\ \frac{a+n+1}{2};\ d \,\middle|\, 4x\right). \end{aligned}$$

3. Certain hypergeometric transformations

In 1998, by applying elementary manipulations of series and making use of Gauss's summation theorem, Exton [4, Eq.(1.6)] obtained a quite general transformation involving hypergeometric functions containing *misprint*, which is written here in a corrected and more compact form as follows:

$$\begin{aligned} &{}_{G+2}F_{H+2}\left[g_1,\, \ldots,\, g_G,\, d,\, e;\, h_1,\, \ldots,\, h_H,\, f-d,\, f-e;\, y\right] \\ &= \sum_{n} \frac{(d)_n\,(e)_n}{(f)_n\, n!}\, {}_{G+1}F_{H+1}\left[g_1,\, \ldots,\, g_G,\, -n;\, h_1,\, \ldots,\, h_H,\, f+n;\, y\right], \end{aligned} \tag{3.1}$$

where the generalized hypergeometric function ${}_AF_B$ is given by

$${}_AF_B\left[a_1,\, \ldots,\, a_A;\, b_1,\, \ldots,\, b_B;\, x\right] = \sum_{n} \frac{(a_1)_n \cdots (a_A)_n}{(b_1)_n \cdots (b_B)_n}\, \frac{x^n}{n!}. \tag{3.2}$$

As noted by Exton [4], Eq. (3.1) is a special case of a general result given in Slater [9, Eq. (2.4.10), p. 60].

In (3.1), by using two known hypergeometric identities [9, Appendix III]:

$$\begin{aligned} &{}_5F_4\left[a,\, 1+\frac{a}{2},\, b,\, c,\, -n;\, \frac{a}{2},\, 1+a-b,\, 1+a-c,\, 1+a+n;\, 1\right] \\ &\quad = \frac{(1+a)_n\,(1+a-b-c)_n}{(1+a-b)_n\,(1+a-c)_n} \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &{}_4F_3\left[a,\, 1+\frac{a}{2},\, b,\, -n;\, \frac{a}{2},\, 1+a-b,\, 1+a+n;\, 1\right] \\ &\quad = \frac{(1+a)_n\,\left(\frac{1}{2}+\frac{a}{2}-b\right)_n}{(1+a-b)_n\,\left(\frac{1}{2}+\frac{a}{2}\right)_n}, \end{aligned} \tag{3.4}$$

Exton thought that he deduced the following two *new* hypergeometric identities [4, Eqs. (2.3) and (2.4)]:

$$
\begin{aligned}
&{}_6F_5\left[a,\, 1+\frac{a}{2},\, b,\, c,\, d,\, e;\, \frac{a}{2},\, 1+a-b,\, 1+a-c,\, 1+a-d,\, 1+a-e;\, -1\right]\\
&\quad=\frac{\Gamma(1+a-d)\,\Gamma(1+a-e)}{\Gamma(1+a)\,\Gamma(1+a-d-e)}\,{}_3F_2\left[d,\, e,\, 1+a-b-c;\, 1+a-b,\, 1+a-c;\, 1\right]
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
&{}_5F_4\left[a,\, 1+\frac{a}{2},\, b,\, d,\, e;\, \frac{a}{2},\, 1+a-b,\, 1+a-d,\, 1+a-e;\, -1\right]\\
&\quad=\frac{\Gamma(1+a-d)\,\Gamma(1+a-e)}{\Gamma(1+a)\,\Gamma(1+a-d-e)}\,{}_3F_2\left[d,\, e,\, \frac{1}{2}+\frac{a}{2}-b;\, \frac{1}{2}+\frac{a}{2},\, 1+a-b;\, 1\right].
\end{aligned}
\tag{3.6}
$$

It should be *remarked* that Eq. (3.5) was recorded in Bailey [1, Eq. (2), p. 28], while Eq. (3.6) can be easily obtained by letting $c=\frac{1}{2}+\frac{a}{2}$ in Eq. (3.5).

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