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HYBRID MONOTONE PROJECTION ALGORITHMS FOR ASYMPTOTICALLY QUASI-PSEUDOCONTRACTIVE MAPPINGS

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Abstract. In this paper, we consider the hybrid monotone projection algorithm for asymptotically quasi-pseudocontractive mappings. A strong convergence theorem is established in the framework of Hilbert spaces. Our results mainly improve the corresponding results announced by [H. Zhou, Demiclosedness principle with applications for asymptotically pseudo-contractions in Hilbert spaces, Nonlinear Anal. 70 (2009) 3140-3145] and also include Kim and Xu [T.H. Kim, H.K. Xu, Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups, Nonlinear Anal. 64 (2006) 1140-1152; Convergence of the modified Mann's iteration method for asymptotically strict pseudo-contractions, Nonlinear Anal. 68 (2008) 2828-2836] as special cases.

1. Introduction and Preliminaries

Throughout this paper, we assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Assume that C is a nonempty closed convex subset of H and $T : C \to C$ is a nonlinear mapping. We use $F(T)$ to denote the set of fixed points of T.

Recall that the mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

The mapping $T : C \to C$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of positive real numbers with $\lim_{n\to\infty} k_n = 1$ and such that

$$\|T^n x - T^n y\| \le k_n \|x - y\| \quad \forall n \ge 1 \text{ and } \forall x, y \in C. \tag{1.2}$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] in 1972. They proved that if C is a nonempty bounded closed

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convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping T of C has a fixed point. Further, the set $F(T)$ of fixed points of T is closed and convex.

The mapping $T : C \to C$ is said to be pseudo-contractive if

$$\langle Tx - Ty, x - y\rangle \le \|x - y\|^2, \quad \forall x, y \in C. \tag{1.3}$$

The mapping $T : C \to C$ is said to be asymptotically pseudo-contractive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ for which the following inequality holds:

$$\langle T^n x - T^n y, x - y\rangle \le k_n\|x - y\|^2, \quad \forall x, y \in C. \tag{1.4}$$

The class of asymptotically pseudo-contractive mappings which was introduced by Schu [15] in 1991 contains properly the class of asymptotically nonexpansive mappings as a subclass, which can be seen from the following example.

Example 1.1. For $x \in [0, 1]$, define a mapping $T : [0, 1] \to [0, 1]$ by

$$Tx = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}.$$

Then T is asymptotically pseudocontractive but it is not asymptotically nonexpansive.

Recall that the mapping T is said to be asymptotically quasi-pseudocontractive if $F(T) \neq \emptyset$ and (1.4) holds for all $x \in C$ but $y \in F(T)$.

We remark that every asymptotically pseudo-contractive mapping with a nonempty fixed point set is asymptotically quasi-pseudocontractive, but the converse may be not true, which can be seen from the following example, see Zhou [18].

Example 1.2. Let H be a real line. We define a mapping $T : H \to H$ by

$$Tx = \begin{cases} \frac{\pi}{2}\sin\frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

Then T is asymptotically quasi-pseudocontractive with the constant sequence $\{1\}$ but not asymptotically pseudocontractive.

Recall that the normal Mann's iterative process was introduced by Mann [6] in 1953. Since then, construction of fixed points for nonexpansive mappings and pseudo-contractions via the normal Mann's iterative process has been extensively investigated by many authors. The normal Mann's iterative process generates a sequence $\{x_n\}$ in the following manner:

$$\forall x_1 \in C, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad \forall n \ge 1, \tag{1.5}$$

where the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval $(0, 1)$.

If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by normal Mann's iterative process (1.5) converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [14]). It is well known that (1.5) has only weak convergence, in general (see [2] for an example). Attempts to modify the normal Mann iteration method (1.5) by hybrid projection algorithms so that strong convergence is guaranteed have recently been made; see, e.g., [3-5,7-13,16-18] and the references therein.

Kim and Xu [4] adapted the normal Mann's iterative process for asymptotically nonexpansive mappings to have strong convergence theorem in Hilbert spaces. More precisely, they gave the following result.

Theorem KX1. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ such that $k_n \to 1$ as $n \to \infty$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in [0,1] such that $\limsup_{n\to\infty} \alpha_n < 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where

$$\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0, \quad \textit{as } n \to \infty.$$

Then $\{x_n\}$ defined by above iterative algorithm converges strongly to $P_{F(T)}x_0$.

Recently, Kim and Xu [5] improved Theorem KX1 from asymptotically nonexpansive mappings to asymptotically strict pseudocontractions. To be more precise, they proved the following theorem.

Theorem KX2. *Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically k-strict pseudo-contraction for some $0 \le k < 1$. Assume that the fixed point set $F(T)$ of T is nonempty and bounded. Let $\{x_n\}$ be the sequence generated by the following (CQ) algorithm:*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 \\ \qquad\qquad\qquad + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where

$$\theta_n = \Delta_n(1-\alpha_n)\gamma_n \to 0, \quad \Delta_n = \sup\{\|x_n - z\| : z \in F(T)\} < \infty.$$

Assume that the control sequence $\{\alpha_n\}$ is chosen so that $\limsup_{n\to\infty} \alpha_n < 1-k$. Then $\{x_n\}$ defined by above iterative algorithm converges strongly to $P_{F(T)}x_0$.

Very recently, Zhou [18] improved the results of Kim and Xu [5] from asymptotically strict pseudo-contractions to asymptotically pseudo-contractive mappings. To be more precise, he proved the following theorem.

Theorem Z. *Let C be a bounded and closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a uniformly L-Lipschitzian and asymptotically pseudo-contraction with a fixed point. Assume the control sequence $\{\alpha_n\}$ is chosen so that $\alpha_n \in [a, b]$ for some $a, b \in (0, \frac{1}{1+L})$. Let a sequence $\{x_n\}$ be generated by the following manner:*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \alpha_n[1-(1+L)\alpha_n]\|x_n - T^n x_n\|^2 \leq \langle x_n - z, y_n - T^n y_n\rangle \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (k_n - 1)(diamC)^2\}, \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \geq 0. \end{cases}$$

Then the sequence $\{x_n\}$ generated by above sequence converges strongly to a $P_{F(T)}x_0$.

In this paper, motivated by Theorem KX1, Theorem KX2, Theorem Z, we modify the normal Mann's iterative scheme to obtain strong convergence for asymptotically quasi-pseudocontractive mappings in the framework of Hilbert spaces without any compact assumption. The results presented in this paper improved the corresponding results announced in Kim and Xu [4], Kim and Xu [5], Qin, Su and Shang [10] and Zhou [18].

In order to prove our main results, we need the following lemmas.

Lemma 1.1 can be deduced from Zhou [18]. For the sake of completeness, we still give the proof.

Lemma 1.1. *Let C be a nonempty bounded and closed convex subset of H and T a uniformly L-Lipschitzian and asymptotically quasi-pseudocontractive mapping. Then $F(T)$ is a closed convex subset of C.*

Proof. From the continuity of T, one has that $F(T)$ is closed. Next, we show $F(T)$ is convex. Let $p_1, p_2 \in F(T)$. We prove $p \in F(T)$, where $p = tp_1 + (1-t)p_2$, for $t \in (0, 1)$. Put $y_\alpha = (1-\alpha)p + \alpha T^n p$, where $\alpha \in (0, \frac{1}{1+L})$. For $\forall w \in F(T)$, one sees

$$\begin{aligned} &\|p - T^n p\|^2 \\ &= \langle p - T^n p, p - T^n p\rangle = \frac{1}{\alpha}\langle p - y_\alpha, p - T^n p\rangle \\ &= \frac{1}{\alpha}\langle p - y_\alpha, p - T^n p - (y_\alpha - T^n y_\alpha)\rangle + \frac{1}{\alpha}\langle p - y_\alpha, y_\alpha - T^n y_\alpha\rangle \\ &= \frac{1}{\alpha}\langle p - y_\alpha, p - T^n p - (y_\alpha - T^n y_\alpha)\rangle + \frac{1}{\alpha}\langle p - w + w - y_\alpha, y_\alpha - T^n y_\alpha\rangle \end{aligned}$$

$$\leq \frac{1+L}{\alpha}\|p-y_\alpha\|^2 + \frac{1}{\alpha}\langle p-w, y_\alpha - T^n y_\alpha\rangle + \frac{1}{\alpha}\langle w-y_\alpha, y_\alpha - T^n y_\alpha\rangle$$
$$\leq (1+L)\alpha\|p-T^np\|^2 + \frac{1}{\alpha}\langle p-w, y_\alpha - T^n y_\alpha\rangle + \frac{1}{\alpha}(k_n-1)\|y_\alpha - w\|^2$$
$$\leq (1+L)\alpha\|p-T^np\|^2 + \frac{1}{\alpha}\langle p-w, y_\alpha - T^n y_\alpha\rangle + \frac{1}{\alpha}(k_n-1)(diamC)^2$$

which yields that

$$\alpha[1-(1+L)\alpha]\|p-T^np\|^2 \leq \langle p-w, y_\alpha - T^n y_\alpha\rangle + (k_n-1)(diamC)^2, \quad \forall w \in F(T). \tag{1.4}$$

Taking $w = p_i$ $i = 1, 2$ in (1.4), multiplying t and $(1-t)$ on the both sides of (1.4), respectively and adding up, one has

$$\alpha[1-(1+L)\alpha]\|p-T^np\|^2 \leq (k_n-1)(diamC)^2.$$

Let $n \to \infty$ in (1.4) yields that $\lim_{n\to\infty}\|p-T^np\| = 0$. Since T is continuous, we have $Tp = p$. This shows that $F(T)$ is convex. This completes the proof. □

Lemma 1.2. *Let C be a closed convex subset of real Hilbert space H and let P_C be the metric projection from H onto C(i.e., for $x \in H$, P_Cx is the only point in C such that $\|x - P_Cx\| = \inf\{\|x-z\| : z \in C\}$). Given $x \in H$ and $z \in C$. Then $z = P_Cx$ if and only if there holds the relations: $\langle x-z, y-z\rangle \leq 0$, for any $y \in C$.*

2. Main results

Theorem 2.1. *Let C be a nonempty bounded and closed convex subset of a Hilbert space H and T a uniformly L-Lipschitz and asymptotically quasi-pseudocontractive mapping from C into itself with a nonempty fixed point set. Let $\{x_n\}$ be a sequence generated in the following algorithm:*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = (1-\alpha_n)x_n + \alpha_n T^n x_n, \\ C_n = \{z \in C : \alpha_n[1-(1+L)\alpha_n]\|x_n - T^nx_n\|^2 \leq \langle x_n - z, y_n - T^n y_n\rangle + \theta_n\}, \\ Q_0 = C; \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}; \\ x_{n+1} = P_{C_n\cap Q_n}x_0, \end{cases}$$

where

$$\theta_n = (k_n-1)(diamC)^2 \to 0, \quad \textit{as } n \to \infty.$$

If the control sequence satisfies the restriction:

$$0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}\alpha_n \leq a < 1,$$

where $a \in (0, \frac{1}{1+L})$, then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Proof. From the definition of C_n and Q_n, one can easily see that C_n and Q_n are closed and convex for all $n \geq 0$. Next, we prove that $F(T) \subset C_n \cap Q_n$ for all $n \geq 0$. For $\forall w \in F(T)$, one has

$$\begin{aligned}
&\|x_n - T^n x_n\|^2 \\
&= \langle x_n - T^n x_n, x_n - T^n x_n\rangle \\
&= \frac{1}{\alpha_n}\langle x_n - y_n, x_n - T^n x_n\rangle \\
&= \frac{1}{\alpha_n}\langle x_n - y_n, x_n - T^n x_n - (y_n - T^n y_n)\rangle + \frac{1}{\alpha_n}\langle x_n - y_n, y_n - T^n y_n\rangle \\
&= \frac{1}{\alpha_n}\langle x_n - y_n, x_n - T^n x_n - (y_n - T^n y_n)\rangle + \frac{1}{\alpha_n}\langle x_n - w + w - y_n, y_n - T^n y_n\rangle \\
&\leq \frac{1+L}{\alpha_n}\|x_n - y_n\|^2 + \frac{1}{\alpha_n}\langle x_n - w, y_n - T^n y_n\rangle + \frac{1}{\alpha_n}\langle w - y_n, y_n - T^n y_n\rangle \\
&\leq (1+L)\alpha_n\|x_n - T^n x_n\|^2 + \frac{1}{\alpha_n}\langle x_n - w, y_n - T^n y_n\rangle + \frac{1}{\alpha_n}\theta_n.
\end{aligned}$$

It follows that

$$\alpha_n[1 - (1+L)\alpha_n]\|x_n - T^n x_n\|^2 \leq \langle x_n - w, y_n - Ty_n\rangle + \theta_n,$$

which shows that $w \in C_n$. This implies that $F(T) \subset C_n$ for all $n \geq 0$. Next, we prove $F(T) \subset Q_n$ for all $n \geq 0$ by induction. For $n = 0$, we have $F(T) \subset C = Q_0$. Assume that $F(T) \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, we have

$$\langle x_0 - x_{n+1}, x_{n+1} - w\rangle, \quad \forall w \in C_n \cap Q_n,$$

as $F(T) \subset C_n \cap Q_n$; the last inequality holds, in particular, for all $w \in F(T)$. This together with the definition of Q_{n+1} implies that $F(T) \subset Q_{n+1}$. Hence $F(T) \subset C_n \cap Q_n$ holds for all $n \geq 0$. Noticing that

$$x_{n+1} = P_{C_n \cap Q_n} x_0 \in Q_n$$

and $x_n = P_{Q_n} x_0$, one arrives at

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|,$$

which shows that the sequence $\{\|x_0 - x_n\|\}$ is nondecreasing.

On the other hand, from $x_n = P_{Q_n} x_0$, one has

$$\langle x_0 - x_n, x_n - w\rangle \geq 0, \quad \forall w \in Q_n. \tag{2.1}$$

From Lemma 1.1, we have that $P_{F(T)}x_0$ is well defined. There exists a unique p such that $p = P_{F(T)}x_0$. It follows from (2.1) that

$$\begin{aligned}
0 &\leq \langle x_0 - x_n, x_n - p\rangle \\
&= \langle x_0 - x_n, x_n - x_0 + x_0 - p\rangle \\
&\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - p\|,
\end{aligned}$$

which yields that

$$\|x_0 - x_n\| \leq \|x_0 - p\|. \tag{2.2}$$

It follows from (2.2) that the sequence $\{x_n\}$ is bounded. Therefore, we have that $\lim_{n\to\infty} \|x_0 - x_n\|$ exists.

On the other hand, by the construction of Q_n, one has that $Q_m \subset Q_n$ and $x_m = P_{Q_m} x_0 \in Q_n$ for any positive integer $m \geq n$. From (2.1), we have

$$\langle x_0 - x_n, x_n - x_m \rangle \geq 0. \tag{2.3}$$

It follows that

$$\begin{aligned} \|x_m - x_n\|^2 &= \|x_m - x_0 + x_0 - x_n\|^2 \\ &= \|x_m - x_0\|^2 + \|x_0 - x_n\|^2 - 2\langle x_0 - x_n, x_0 - x_m \rangle \\ &\leq \|x_m - x_0\|^2 - \|x_0 - x_n\|^2 - 2\langle x_0 - x_n, x_n - x_m \rangle \\ &\leq \|x_m - x_0\|^2 - \|x_0 - x_n\|^2. \end{aligned} \tag{2.4}$$

Letting $m, n \to \infty$ in (2.4), one has $\lim_{m,n\to\infty} \|x_n - x_m\| = 0$, $\forall m \geq n$. Hence, $\{x_n\}$ is a Cauchy sequence. Since H is a Hilbert space and C is closed and convex, one can assume that

$$x_n \to q \in C \quad as\ n \to \infty. \tag{2.5}$$

Finally, we show that $q = P_{F(T)} x_0$. To end this, we first show $q \in F(T)$. By taking $m = n + 1$ in (2.4), one arrives at

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0, \tag{2.6}$$

Noticing that $x_{n+1} = P_{C_n \cap Q_n} x_0 \in C_n$, we obtain

$$\alpha_n[1 - (1 + L)\alpha_n] \|x_n - T^n x_n\|^2 \leq \|x_n - x_{n+1}\| \|y_n - T^n y_n\| + \theta_n.$$

It follows from the assumptions on the control sequence $\{\alpha_n\}$ and (2.6) that

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0. \tag{2.7}$$

Since T is a uniformly L-Lipschitz, one has

$$\begin{aligned} \|x_n - Tx_n\| = &\|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1} x_{n+1}\| + \|T^{n+1} x_{n+1} - T^{n+1} x_n\| \\ &+ \|T^{n+1} x_n - Tx_n\| \\ &\leq (1 + L)\|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1} x_{n+1}\| + L\|T^n x_n - x_n\|. \end{aligned}$$

It follows from (2.6) and (2.7) that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{2.8}$$

Notice that

$$\|Tx_n - q\| \leq \|x_n - Tx_n\| + \|q - x_n\|.$$

From (2.5) and (2.8), one obtains

$$\lim_{n\to\infty} \|Tx_n - q\| = 0.$$

From the closedness of T, one has $q \in F(T)$. By using the definition of Q_n and noting the fact that $F(T) \subset Q_n$, we have

$$\langle x_0 - x_n, x_n - w\rangle \geq 0, \quad \forall w \in F(T) \subset Q_n. \tag{2.9}$$

Letting $n \to \infty$ in (2.9), one gets

$$\langle x_0 - q, q - w\rangle \geq 0, \quad \forall w \in F(T)$$

In view of Lemma 1.2, one sees that $q = P_{F(T)}x_0$. This completes the proof. □

Remark 2.2. Theorem 2.1 improves Theorem 3.5 of Zhou [18] from asymptotically pseudo-contractive mappings to asymptotically quasi-pseudocontractive mappings.

Remark 2.3. The hybrid monotone projection algorithm studied in this paper is also different from Zhou [18]'s. We do not require that the mapping $I - T$ is demi-closed at zero.

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