A Closed-Form BER Expression for Decode-and-Forward Cooperative Communication Protocol

Hyung-Yun Kong

Abstract

The decode-and-forward cooperative communication protocol allows single-antenna users in wireless environments to obtain the powerful benefits of multi-antenna systems without the need for physical arrays. Evaluating the performance of this protocol through simulations is time-consuming and therefore, a need exists for an analytical BER expression to serve as a reference. This paper proposed such an expression for coherently BPSK-modulated data.

Key words : Cooperative Communications, Decode-and-Forward Protocol, Rayleigh Fading, AWGN, BER Expression.

Ⅰ. Introduction

Signal fading due to multi-path propagation is a serious problem in wireless communications. Using a diversified signal in which information related to the same data appears in multiple time instances, frequencies, or antennas that are independently faded can reduce considerably this effect of the channel[1]. Among well-known diversity techniques, spatial diversity has received a great deal of attention in recent years because of the feasibility of deploying multiple antennas at both transmitter and receiver[2]. However, wireless mobiles may not be able to support multiple antennas due to size or other constraints[3], so they may not be able to take advantage of spatial diversity. To overcome this limitation, a new technique was born, called cooperative communications, which allows single-antenna mobiles to gain some benefits of transmit diversity. Among three basic cooperative protocols[3], the decode-and-forward(DF) protocol demonstrates the most reasonable trade-off between implementation complexity and BER performance.

However, until now, the performance measure for analyzing BER for DF has been limited to the outage probability[4]. In [5], the authors investigated the feasibility of using the DF protocol with BPSK for direct transmission and QPSK for the relay scheme. However, they only showed the upper bound of the BER expression. Thus, our goal in this paper is to derive its closed-form BER expression in the condition of a variance channel link.

Ⅱ. BER Expression Formulation

Consider a wireless network consisting of single-antenna terminals: a source(S), a relay(R) and a destination (D). This study investigates all terminals equipped with single-antenna transceivers and sharing the same frequency band. In addition, to mitigate implementation complexity, terminals do not transmit and receive signal at the same time, since considerable attenuation over wireless channels and insufficient electrical isolation between transmit and receive circuitry make a terminal's transmitted signal dominate the signals of other terminals at its receiver input. Towards this end, channel allocation based on the time-division approach is investigated.

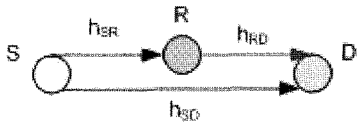


Fig. 1. A simple cooperation model.

Assuming that channels between terminals experience independent slow frequency-flat Rayleigh fading, i.e., they are constant during an N-symbol block but change independently to the next. Without loss of generality, we only illustrate the analysis for the first symbol of each block. In addition, we assume perfect channel-state information at all the respective receivers but not at the transmitters.

To capture the effect of path loss on BER performance, we use the same model as discussed in [5] where the variance of α_{ij} is given by $\lambda_{ij}=(d_{SD}/d_{ij})^{\beta}(d_{SD}/d_{ij})$ with d_{ij} and α_{ij} being the distance and the channel coefficient between transmitter i and receiver j, respectively, and being the path loss exponent. For convenience of presentation, we use discrete-time complex equivalent base-band models to express all the signals.

Manuscript received March 17, 2009 ; revised June 11, 2009. (ID No. 20090317-013J)
Department of Electrical Engineering, University of Ulsan, Ulsan, Korea.

The DF protocol includes two phases. In the first phase, S broadcasts a BPSK-modulated symbol "a" and so, the signals received at Rand D are given by:

$$y_{SR}=\alpha_{SR}\sqrt{E_S}a+n_{SR} \quad (1)$$

$$y_{SD}=\alpha_{SD}\sqrt{E_S}a+n_{SD} \quad (2)$$

where denotes a signal received at the terminal j from the terminal i, n_{ij} is a zero-mean unit-variance complex additive noise sample at the node j, and E_S is an average bit energy of S.

At the end of this phase, R recovers the original data by maximum likelihood(ML) decoding as:

$$\hat{a}=sign(Re(\alpha^*_{SR}y_{SR})) \quad (3)$$

Here, $sign(.)$ is a sign function and $Re(.)$ is a real part.

In the second phase, the relay sends $\hat{a}$ to D with an average bit energy E_R. The signal arriving at D is of the form

$$y_{RD}=\alpha_{RD}\sqrt{E_R}\hat{a}+n_{RD} \quad (4)$$

Now D combines the received signals from both phases based on maximum ratio combining and then detects the transmitted signal "a":

$$\bar{a}=sign(Re(\sqrt{E_S}\alpha^*_{SD}y_{SD}+\sqrt{E_R}\alpha^*_{RD}y_{RD})) \quad (5)$$

Using (2) and (4), we can rewrite (5) as

$$\bar{a}=sign((E_S|\alpha_{SD}|^2a+E_R|\alpha_{RD}|^2\hat{a})+n)$$
$$=sign((E_S|\alpha_{SD}|^2+\varepsilon E_R|\alpha_{RD}|^2)a+n) \quad (6)$$

Here $n=Re(\sqrt{E_S}\alpha^*_{SD}n_{SD}+\sqrt{E_R}\alpha^*_{RD}n_{RD})$ is a Gaussian r.v. with zeromean and variance $(E_S|\alpha_{SD}|^2+E_R|\alpha_{RD}|^2)/2$, given channel coefficients; $\varepsilon=-1$ means that the relay made the wrong decision on the symbol "a" otherwise, $\varepsilon=-1$.

Based on (6), the ML detection offers the minimum error probability, conditioned on the channel coefficients as

$$P_e=\Pr[\bar{a}=1|a=-1]$$
$$=\Pr[-(E_S|\alpha_{SD}|^2+E_R|\alpha_{RD}|^2)+n>0]\Pr[\varepsilon=1]+$$
$$\Pr[-(E_S|\alpha_{SD}|^2-E_R|\alpha_{RD}|^2)+n>0]\Pr[\varepsilon=-1]$$
$$=P_{e1}(1-\Pr[\varepsilon=-1])+P_{e2}\Pr[\varepsilon=-1] \quad (7)$$

The average BER can be found by averaging the above over distributions of channel coefficients as

$$\overline{P_e}=\overline{P_{e1}}(1-\overline{\Pr[\varepsilon=-1]})+\overline{P_{e2}}\,\overline{\Pr[\varepsilon=-1]} \quad (8)$$

Since $\Pr[\varepsilon=-1]$ is the instantaneous error probability of the BPSK-modulated symbol over Rayleigh fading channel S-R plus AWGN with zero mean and unit variance, its average BER is easily established.

Rewrite the expression of $\overline{P_{e1}}$ in the explicit form

$$\overline{\Pr[\varepsilon=-1]}=\frac{1}{2}\left[1-\sqrt{\frac{E_S\lambda_{SR}}{1+E_S\lambda_{SR}}}\right] \quad (9)$$

Rewrite the expression of $\overline{P_{e1}}$ in the explicit form

$$\overline{P_{e1}}=\overline{\Pr[n>(E_S|\alpha_{SD}|^2+E_R|\alpha_{RD}|^2)]}$$
$$=\overline{Q(\sqrt{2(E_S|\alpha_{SD}|^2+E_R|\alpha_{RD}|^2)})} \quad (10)$$

Here $Q(.)$ is a Q-function.

Let $x=E_S|\alpha_{SD}|^2$ and $y=E_R|\alpha_{RD}|^2$. Since α_{ij} are zero-mean complex Gaussian r.v.'s with variance λ_{ij}, x and y have exponential distributions with mean values of $E_S\lambda_{SD}$ and $E_R\lambda_{RD}$, that is, $f_x(x)=\lambda_x\exp(-\lambda_x x)$ and $f_y(y)=\lambda_y\exp(-\lambda_y y)$ where $\lambda_x=1/(E_S\lambda_{SD})$ and $\lambda_y=1/(E_R\lambda_{RD})$, and $x, y\geq 0$; $f_x(x)$, $f_y(y)$ are pdfs of r.v.s x and y, respectively.

Also, we denote $w=x+y$. The pdf of w, hence, is expressed as

$$f_w(w)=\int_0^w f_x(x)f_y(w-x)dx$$
$$=\begin{cases}\dfrac{\lambda_x\lambda_y}{\lambda_x-\lambda_y}[e^{-\lambda_y w}-e^{-\lambda_x w}] & \lambda_x=\lambda_y\\ b^2e^{-bw}w & \lambda_x=\lambda_y=b\end{cases} \quad (11)$$

Now we establish the BER according to two cases of (10).

2-1 Case of $\lambda_x=\lambda_y$

From (9), we have

$$\overline{P_{e1}}=\int_0^\infty Q(\sqrt{2w})b^2we^{-bw}dw$$
$$=\frac{1}{4}\left(1-\sqrt{\frac{1}{1+b}}\right)^2\left(2+\sqrt{\frac{1}{1+b}}\right) \quad (12)$$

The last equality in (11) is taken from [9, (12)]. Also in this case, it is easy to see that

$$\overline{P_{e2}}=\overline{\Pr[-(E_S|\alpha_{SD}|^2-E_R|\alpha_{RD}|^2)+n>0]}=0.5 \quad (13)$$

Substituting (8), (11), and (12) into (7), we obtain $\overline{P_e}$.

2-2 Case of $\lambda_x \# \lambda_y$

In such a case, (9) is of the form

$$
\begin{aligned}
\overline{P_{e1}} &= \int_0^\infty Q(\sqrt{2w})\frac{\lambda_x\lambda_y}{\lambda_x-\lambda_y}[e^{-\lambda_y w}-e^{-\lambda_x w}]dw \\
&= \frac{\lambda_x}{\lambda_x-\lambda_y}\int_0^\infty Q(\sqrt{2w})\lambda_y e^{-\lambda_y w}dw \\
&\quad -\frac{\lambda_y}{\lambda_x-\lambda_y}\int_0^\infty Q(\sqrt{2w})\lambda_x e^{-\lambda_x w}dw \\
&= \frac{\lambda_x}{2(\lambda_x-\lambda_y)}\left[1-\sqrt{\frac{1}{1+\lambda_y}}\right] \\
&\quad -\frac{\lambda_y}{2(\lambda_x-\lambda_y)}\left[1-\sqrt{\frac{1}{1+\lambda_x}}\right]
\end{aligned}
\tag{14}
$$

If we let $z=x-y$, then P_{e2} and $\overline{P_{e2}}$ are written as:

$$
\begin{aligned}
P_{e2} &= \Pr[-(E_S|\alpha_{SD}|^2-E_R|\alpha_{RD}|^2)+n>0] \\
&= Q\left(\sqrt{\frac{2z^2}{w}}\right)\Pr[z\geq 0]+\left[1-Q\left(\sqrt{\frac{2z^2}{w}}\right)\right]\Pr[z\leq 0] \\
&= P_Q P_G+(1-P_Q)(1-P_G)
\end{aligned}
\tag{15}
$$

And

$$
\overline{P_{e2}}=\overline{P_Q}P_G+(1-\overline{P_Q})(1-P_G) \tag{16}
$$

Considering the case of $z\geq 0$ in the sequel, we have [6, (6-55)]

$$
f_z(z)=\int_0^\infty f_{xy}(z+y,y)dy=\frac{\lambda_x\lambda_y e^{-\lambda_x z}}{\lambda_x+\lambda_y} \tag{17}
$$

So

$$
P_G=\Pr[z\geq 0]=\int_0^\infty f_z(z)dz=\frac{\lambda_y}{\lambda_x+\lambda_y} \tag{18}
$$

Moreover, the pdf of $v=z^2$ is easily found as [6, (5-22)]

$$
f_v(v)=\frac{1}{2\sqrt{v}}f_x(\sqrt{v})=\frac{1}{2\sqrt{v}}\frac{\lambda_x\lambda_y e^{-\lambda_x\sqrt{v}}}{\lambda_x+\lambda_y} \tag{19}
$$

Now we compute the pdf of $u=v/w$ as follows [6, (6-60)]:

$$
\begin{aligned}
f_u(u) &= \int_0^\infty w f_v(wu)f_w(w)dw= \\
&\int_0^\infty w\frac{1}{2\sqrt{wu}}\frac{\lambda_x\lambda_y e^{-\lambda_x\sqrt{wu}}}{\lambda_x+\lambda_y}\frac{\lambda_x\lambda_y}{\lambda_x-\lambda_y}\cdot \\
&\quad [e^{-\lambda_y w}-e^{-\lambda_x w}]
\end{aligned}
\tag{20}
$$

By changing the variable $k=\sqrt{w}$ and using [7, (7) on page 361], the above is reduced to

$$
f_u(u)=\int_0^\infty \frac{1}{\sqrt{u}}\frac{\lambda_x\lambda_y}{\lambda_x+\lambda_y}\frac{\lambda_x\lambda_y}{\lambda_x-\lambda_y}\cdot k^2[e^{-(\lambda_y k^2+\lambda_x k\sqrt{u})}-e^{-(\lambda_x k^2+\lambda_x k\sqrt{u})}]dk \tag{21}
$$

Finally $\overline{P_Q}$ is given by:

$$
\begin{aligned}
\overline{P_Q} &= \int_0^\infty Q(\sqrt{2u})f_u(u)du= \\
&\frac{\lambda_x^2\lambda_y^2}{\lambda_x^2-\lambda_y^2}\times\int_0^\infty Q(\sqrt{2u})\left(\int_0^\infty \begin{matrix}\frac{1}{\sqrt{u}}k^2e^{-(\lambda_y k^2+\lambda_x k\sqrt{u})}- \\ \frac{1}{\sqrt{u}}k^2e^{-(\lambda_x k^2+\lambda_x k\sqrt{u})}dk\end{matrix}\right)du \\
&= \frac{\lambda_x^2\lambda_y^2}{\lambda_x^2-\lambda_y^2}[f(\lambda_y,\lambda_x)-f(\lambda_x,\lambda_x)]
\end{aligned}
\tag{22}
$$

The last equality in (16) is obtained from (A4) in the Appendix.

Using (15) and (16), we find (14). In addition, from (8), (13), and (14), we calculate the BER in (7).

Ⅲ. Numerical Result

An asymmetrical network geometry is examined where the relay is located on an S-D straight line. The direct path length S-D is normalized to be 1. We also denote d as the distance between S and R. For the case of the symmetrical network, all distances between the two terminals are set to 1. Additionally, we only consider $\beta=3$ and $k=1$.

For a fair comparison, it is essential that the total energy consumed by the cooperative system does not exceed that of the corresponding direct transmission system. Therefore, the total energy of power at the source and the relay equals the power of direct transmission $P_S+P_T=P_{Direct}$.

We verify the accuracy of the BER expression in (7) by comparison with Monte-Carlo simulations. The results are depicted in Fig. 1. We can see that the simulation results match the analytical ones. This shows that the theoretical BER expression is completely exact.

In the Fig. 2, 3, we give BER versus distance and SNR. The smaller distance from the source to the relay,

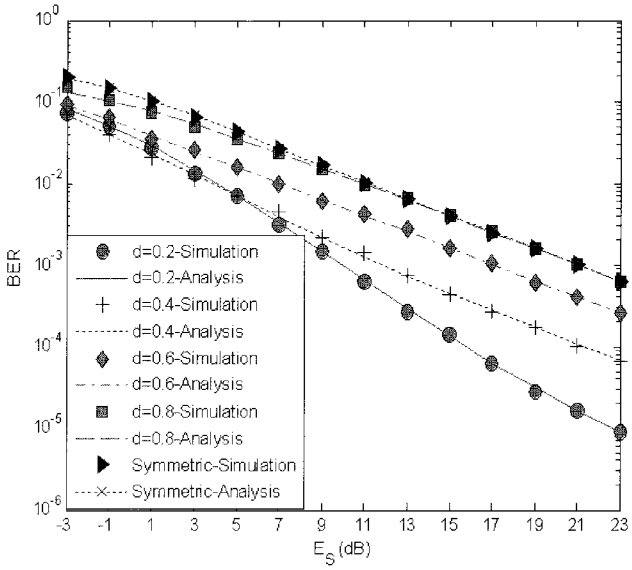


Fig. 2. BER comparison between theory and simulation.

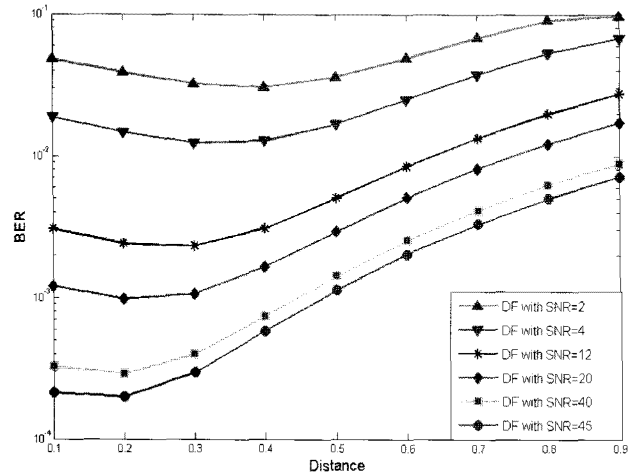


Fig. 3. BER vs distance of source relay destination.

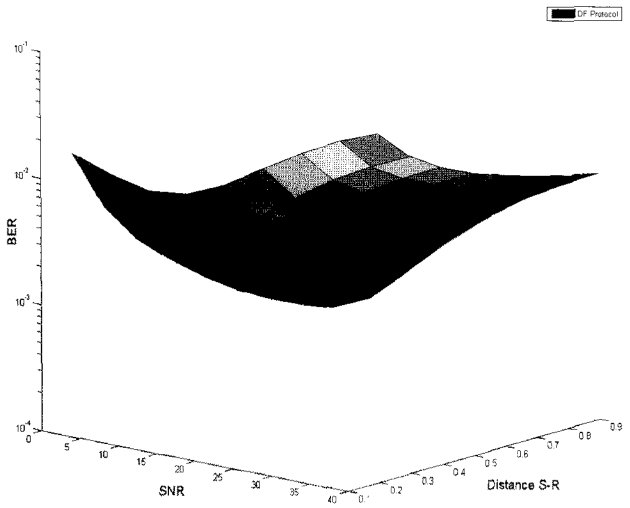


Fig. 4. The relationship among BER distance-SNR.

the better performance we can get from the system. It is also closed for both the theoretical value and the simulated result.

Ⅳ. Conclusion

A closed-form BER expression for the decode-and-forward cooperative communication protocol was established. Its validity was demonstrated by a number of Monte-Carlo simulations. This expression significantly contributes to the performance evaluation of the decode-and-forward protocol under different transmission conditions, such as fading and path-loss, without time-consuming simulations. Additionally, it can be employed in optimum energy allocation for the relay and the source in order to improve the BER performance.

Appendix

Applying [7, (7) on page 361 and (4) on page 880] and [8, (14)], we obtain

$$f_{\varepsilon}(\varepsilon,\lambda_1,\lambda_2)=\int_0^{\infty} g^2 e^{-\lambda_1 g-\lambda_2\sqrt{\varepsilon}g}dg$$
$$=-\frac{\lambda_2\sqrt{\varepsilon}}{4\lambda_1^2}+\sqrt{\frac{\Pi}{\lambda_1^5}}\frac{\lambda_2^2\varepsilon/2+\lambda_1}{4}\left(\frac{1}{6}+\frac{1}{2}e^{-\frac{\lambda_2^2\varepsilon}{12\lambda_1}}\right) \tag{23}$$

and

$$f(\lambda_1,\lambda_2)=\int_0^{\infty} Q(\sqrt{2\varepsilon})\cdot \int_0^{\infty}\frac{1}{\sqrt{\varepsilon}}g^2 e^{-\lambda_1 g^2-\lambda_2\sqrt{\varepsilon}g}dgd\varepsilon$$
$$=\frac{1}{2}\int_0^{\infty} erfc(\sqrt{\varepsilon})\frac{1}{\sqrt{\varepsilon}}f(\varepsilon,\lambda_1,\lambda_2)d\varepsilon$$
$$=\frac{1}{2}\int_0^{\infty}\left[\frac{1}{6}e^{-\varepsilon}+\frac{1}{2}e^{-4\varepsilon/3}\right]\frac{1}{\sqrt{\varepsilon}}f_{\varepsilon}(\varepsilon,\lambda_1,\lambda_2)d\varepsilon \tag{24}$$

By substituting (23) into (24) and changing the variable $L=\sqrt{\varepsilon}$, (24) is rewritten as

$$f(\lambda_1,\lambda_2)=\frac{1}{2}\int\left[\begin{array}{l}\left(\frac{1}{6}e^{-L^2}+\frac{1}{2}e^{-4L^2/3}\right)\times\\ \left(\begin{array}{l}\frac{-\lambda_2 L}{2\lambda_1^2}+\sqrt{\frac{\Pi}{\lambda_1^5}}\frac{\lambda_2^2L^2/2+\lambda_1}{2}\\ \cdot\left(\frac{1}{6}+\frac{1}{2}e^{-\frac{\lambda_2^2L^2}{12\lambda_1}}\right)\end{array}\right)\end{array}\right]dL \tag{25}$$

Applying the results in [7, (2)-(3) on page 360], we can compute (25) as

$$f(\lambda_1,\lambda_2)=\frac{1}{2}\times\left[\begin{array}{l}-\frac{13\lambda_2}{96\lambda_1^2}\\ +\sqrt{\frac{\pi}{\lambda_1^5}}\left(\begin{array}{l}\sqrt{\pi}\left(\frac{\lambda_2^2+4\lambda_1}{576}\right)\\ +\sqrt{\frac{12\lambda_1\pi}{12\lambda_1+\lambda_2^2}}\left(\frac{\lambda_1(\lambda_2^2+34\lambda_1)}{12(12\lambda_1+\lambda_2^2)}\right)+\\ \sqrt{\frac{3\pi}{4}}\frac{3\lambda_2^2+16\lambda_1}{768}+\\ \sqrt{\frac{12\lambda_1\pi}{16\lambda_1+\lambda_2^2}}\frac{\lambda_1(\lambda_2^2+4\lambda_1)}{4(16\lambda_1+\lambda_2^2)}\end{array}\right)\end{array}\right] \tag{26}$$

References

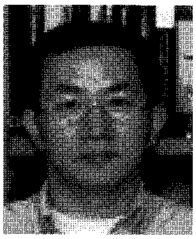
[1] J. G. Proakis, *Digital Communications*, Fourth Edition, McGraw-Hill, 2001.

[2] V. Tarokh, H. Jafarkhani, and A. R. Calderbank, "Space-time block coding for wireless communications: performance results", *IEEE Trans. Commun.*, 17, pp. 451-460, 1999.

[3] A. Nosratinia, T. E. Hunter, and A. Hedayat, "Cooperative communication in wireless networks", *IEEE Commun. Mag.*, vol. 42, pp. 74-80, Oct. 2004.

[4] J. N. Laneman, D. N. C. Tse, and G. W. Wornell, "Cooperative diversity in wireless networks: Efficient protocols and outage behavior", *IEEE Trans. Inform. Theory*, vol. 50, pp. 3062-3080, Dec. 2004.
[5] P. Herhold, E. Zimmermann, and G. Fettweis, "A simple cooperative extension to wireless relaying", *IEEE Int. Zurich Seminar on Commun.*, Zurich, Switzerland, Feb. 2004.
[6] A. Papoulis, S. U. Pillai, *Probability, Random Wariables and Stochastic Process*, Fourth Edition, McGraw Hill, 2002.
[7] I. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, 2000.
[8] M. Chiani, D. Dardari, and M. K. Simon, "New exponential bounds and approximations for the computation of error probability in fading channels", *IEEE Trans. Wireless Commun.*, vol. 2, pp. 840-845, Jul. 2003.
[9] H. Y. Kong, H. V. Khuong, "Closed-form BER expressions for performance of Alamouti STC", *JKIPS (Journal of Korea Information Processing Society) Trans. on Commun.*, vol. 12-C, pp. 673-678, Oct. 2005.

Hyung-Yun Kong



received the M.E. and Ph.D. degrees in electrical engineering from Ploytechnic University, Brooklyn, New York, USA, in 1991 and 1996. And he received the BE in electrical engineering from New York Institute of Technology, New York, in 1989. Since 1996, he was with LG electronics Co., Ltd., in the multimedia research lab developing PCS mobile phone systems, and from 1997 the LG chairman's office planning future satellite communication systems. Currently he is in Associate Professor in electrical engineering at University of Ulsan, Korea. He performs in several government projects supported by ITRC, Korean Science and Engineering Foundation(KOSEF), etc. His research area includes high data rate modulation, channel coding, detection and estimation, cooperative communications, and sensor networks. He is a member of IEEK, KICS, KIPS, and IEICE.