Honam Mathematical J. **31** (2009), No. 1, pp. 1–9

STICK NUMBER OF THETA-CURVES

Youngsik Huh and Seungsang Oh

Abstract. In this paper we establish strict lower bounds on number of sticks necessary to construct stick presentations of nontrivial or almost trivial θ-curves.

1. Introduction

A *knot* is a simple closed curve embedded into $\mathbb{R}^3$. Two knots K and K' are said to be *equivalent*, if there exists an orientation preserving homeomorphism of $\mathbb{R}^3$ which maps K to K', or to say roughly, K' is obtained from K by continuous moves without intersecting any strand of the knot. And the equivalence class of K is called the *knot type of* K. A knot is said to be *trivial*, if it is equivalent to another knot contained in a plane of the 3-space.

One natural way to represent knots is the stick presentation. A *stick knot* is a knot which consists of finite line segments, called *sticks*. Figure 1 shows the *right-handed trefoil knot* 3_1 and one of its stick presentations. The *stick number* $\mathfrak{s}(K)$ of a knot K is defined to be the minimal number of sticks for construction of the knot type into a stick knot. It is known that $\mathfrak{s}(3_1) = \mathfrak{s}(3_1^*) = 6$, where 3_1^*, called *left-handed trefoil*, is the mirror image of 3_1 [5, 7]. Since this presentation of knots has been considered to be a useful mathematical model of cyclic molecules or molecular chains, the stick number may be one of intrinsic properties which are interested not only in knot theory of mathematics, but also in biology, chemistry and physics. In general it is not easy to determine $\mathfrak{s}(K)$ precisely for arbitrary knot K. Numerous researches on this quantity have been done through experimental simulation using computer. But there are some literatures in which the range of $\mathfrak{s}(K)$ was theoretically investigated and

Received March 2, 2008. Accepted March 4, 2009.
Key words and phrases: knot, θ-curve, stick number.
This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund) (KRF-2006-312-C00469).

the quantity was determined precisely for some specific knots [1, 2, 4, 6, 7]. The following is one of the most elementary results on stick number.

Theorem 1. [5, 8]
If K is a nontrivial knot, then $\mathfrak{s}(K) \geq 6$. Furthermore, if $\mathfrak{s}(K) = 6$, then K is 3_1 or 3_1^*.

Note that an embedded graph in $\mathbb{R}^3$ consisting of sticks is also a usual model of molecules. Therefore it would be interesting to investigate its stick number. In this paper we investigate the stick number of θ-curves.

The *θ-graph* is the graph consisting of two vertices and three edges between them as seen in Figure 2. And a *θ-curve* is an embedding of the graph into $\mathbb{R}^3$. The equivalence, triviality and stick presentation of θ-curves can be defined in the same way as knots. The θ-graph contains three cycles. Therefore a θ-curve contains three knots as its subspace which are called *subknots*. A nontrivial θ-curve is said to be *almost trivial*, if every subknot is trivial. For example see Figure 2. The θ-curve $T_{\mathcal{K}}$, called *Kinoshita's θ-curve*, is almost trivial [9]. On contrary $T_{\mathcal{T}}$ is not almost trivial because it contains a trefoil knot as its subknot. The theorem below is the main result of this paper.

Theorem 2. *For any nontrivial θ-curve T, $\mathfrak{s}(T) \geq 7$. Furthermore, if T is almost trivial, then $\mathfrak{s}(T) \geq 8$.*

Figure 3 shows stick presentations of $T_{\mathcal{T}}$ and $T_{\mathcal{K}}$ each of which consists of 7 and 8 sticks respectively. Therefore our inequalities in the theorem are best possible. And we have that $\mathfrak{s}(T_{\mathcal{T}}) = 7$ and $\mathfrak{s}(T_{\mathcal{K}}) = 8$ as a corollary.

The rest of this paper is devoted to the proof of Theorem 2.

Remark. In a stick knot or stick θ-curve, a boundary point of any stick will be called a *vertex* of the stick presentation. This terminology should be distinguished from a vertex of a graph. Regarding stick presentation as a model for molecules, it is considered that vertices correspond to atoms in the molecule and sticks to chemical bonds among them. Therefore it is natural to make it a condition that the two vertices of the θ-graph should be sent to vertices of stick θ-curve each of which is shared by three sticks. This condition will be assumed throughout this paper.

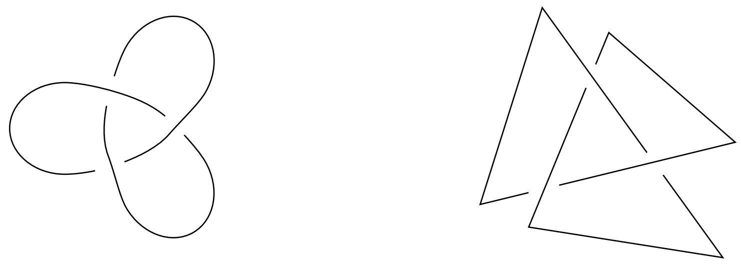

FIGURE 1. 3_1 knot (right-handed trefoil) and its stick presentation

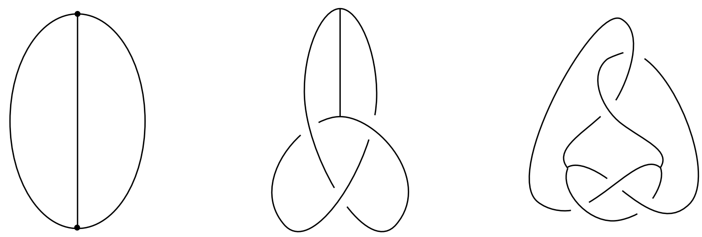

FIGURE 2. θ-graph and θ-curves $T_{\mathcal{T}}$, $T_{\mathcal{K}}$

2. Proof of Theorem 2

Before proving this theorem we introduce some notions. Let P be a stick knot or stick θ-curve. A triangle determined by two adjacent sticks is said to be *reducible*, if its interior is disjoint from the sticks of P. And $s(P)$ will denote the number of sticks of P.

Now we prove the theorem. Let T be a stick θ-curve which is not trivial. Then immediately we see that $s(T)$ should be more than 5. And throughout the proof it can be assumed that no four vertices are coplanar, by perturbing T slightly if necessary.

Firstly consider the case that T is not almost trivial. Then T should contain a nontrivial stick knot K as a subknot. By Theorem 1, $s(K)$ should be at least 6. Therefore

$$6 \leq s(K) < s(K) + 1 \leq s(T),$$

which is done. Now we consider the case that T is almost trivial.

Case 1: *T is almost trivial and $s(T) = 6$.*

Firstly suppose that T contains a subknot K of $s(K) = 3$. If K is reducible as a triangle of T, then two sticks of K can be isotoped

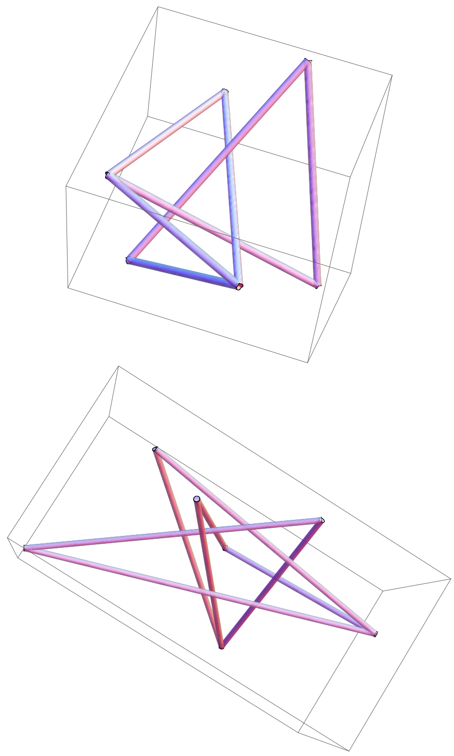

Figure 3. stick θ-curves

into a small tubular neighborhood of the third stick along the triangle. This implies that the triviality of T depends only on those of the other two subknots of T. But since every its subknot is trivial, we get a contradiction to the nontriviality of T. If K is irreducible, then we get an obvious configuration of T as depicted in Figure 4-(a) which corresponds to a trivial θ-curve.

Now consider possible numbers of sticks constituting subknots of T. By the observation in the above we have only one case $(4, 4, 4)$ which means each of three subknots has 4 sticks as depicted in Figure 4-(b). In such case, the triangle determined by the vertices $\{v_1, v_2, v_3\}$ is reducible, for none of four sticks $l_{1,5}$, $l_{5,3}$, $l_{1,4}$ and $l_{4,3}$ can penetrate the triangle. Therefore we can reduce the number of sticks of T by one as illustrated in Figure 5, which is contradictory to the nontriviality of T again.

Before dealing with the rest case, we introduce a necessary lemma.

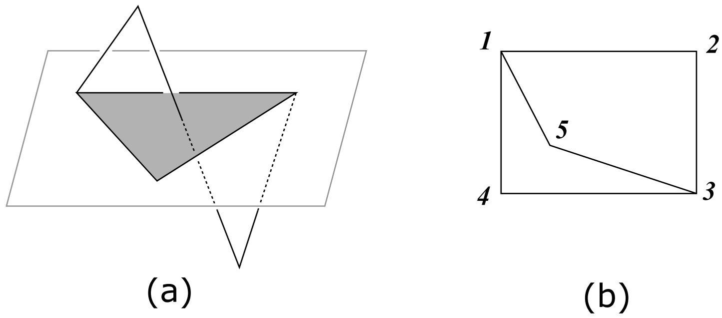

FIGURE 4

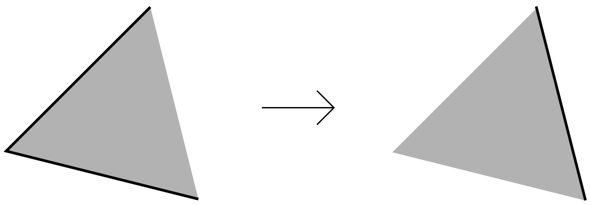

FIGURE 5. Reduction along a triangle

Lemma 1. *Let P be a stick knot of $s(P) = 6$ such that every triangle determined by two adjacent sticks is irreducible. Then P is 3_1 or 3_1^*.*

Proof. Since P is irreducible and $s(P) = 6$, by the arguments of Lemmas 4, 5, 6 and 7 in [3], it can be shown that each triangle between two adjacent sticks is penetrated only by one stick, and furthermore we can label the vertices of P so that

- $l_{1,2}$ penetrates $\triangle_{3,4,5}$ and $\triangle_{4,5,6}$,
- $l_{3,4}$ penetrates $\triangle_{5,6,1}$ and $\triangle_{6,1,2}$,
- $l_{5,6}$ penetrates $\triangle_{1,2,3}$ and $\triangle_{2,3,4}$.

The information in the above is enough to realize the knot type of P. If $l_{1,2}$ penetrates $\triangle_{3,4,5}$ in positive orientation, then P is 3_1 as depicted in Figure 6. Otherwise, P is 3_1^*. This lemma can be also proved by applying a method in the proof of Theorem 2 in [8]. □

Case 2: *T is almost trivial and $s(T) = 7$.*

For subknots of T, the possible distribution of numbers of sticks should be $(5, 5, 4)$, $(6, 4, 4)$ or $(6, 5, 3)$ as depicted in Figure 7. We observe each case and derive contradictions.

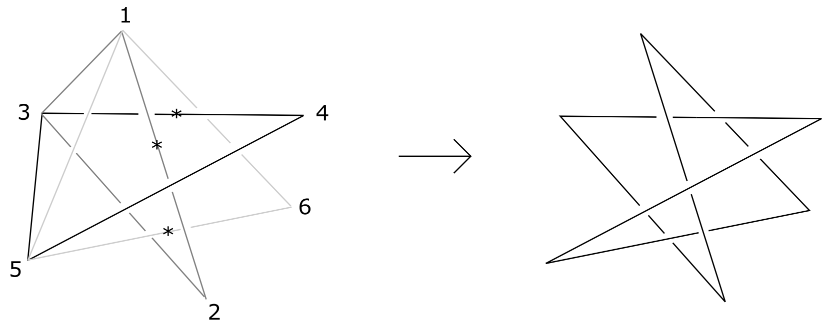

FIGURE 6

Case 2-1. (5, 5, 4): Suppose that the triangle $\triangle_{1,5,4}$ determined by the vertices $\{v_1, v_5, v_4\}$ is irreducible. The possible sticks which may penetrate $\triangle_{1,5,4}$ are $l_{2,3}$ and $l_{3,6}$. Without loss of generality we may assume that $l_{2,3}$ penetrate the triangle. Consider half lines each of which starts at v_2 and passes through a point of $\triangle_{1,5,4}$. Let $H^2_{1,5,4}$ be the union of all such half lines. Then the interior of $\triangle_{1,2,3}$ is contained in that of $H^2_{1,5,4}$ and $l_{4,5}$ is contained in the boundary of $H^2_{1,5,4}$ as depicted in Figure 8. Thus $l_{4,5}$ cannot penetrate $\triangle_{1,2,3}$. Since $l_{4,5}$ is the only possible stick that may penetrate $\triangle_{1,2,3}$, the triangle should be reducible. We conclude that at least one of $\triangle_{1,5,4}$ and $\triangle_{1,2,3}$ is reducible and hence we can reduce the number of sticks by one, which is a contradiction by Case 1.

Case 2-2. (6, 4, 4): Let K be the subknot with $s(K) = 6$. Note that $l_{4,5}$ and $l_{5,6}$ are only possible sticks of T which may penetrate $\triangle_{1,2,3}$. Therefore if $\triangle_{1,2,3}$ is reducible in K, then it should be reducible also in T and the number of sticks can be reduced. Hence $\triangle_{1,2,3}$, also similarly $\triangle_{2,3,4}$, $\triangle_{4,5,6}$ and $\triangle_{5,6,1}$, are irreducible in K. But, the almost-triviality of T implies that K should be trivial and, by Lemma 1, should have a reducible triangle. Consequently, $\triangle_{6,1,2}$ or $\triangle_{3,4,5}$ should be reducible in K. Without loss of generality assume that $\triangle_{6,1,2}$ is such a triangle. Then the triangle is reducible also in T, for $l_{3,4}$ and $l_{4,5}$ are only possible sticks of T which may penetrate it.

We observe other triangles. The triangle $\triangle_{4,1,2}$ should be irreducible. Otherwise, we can reduce the number of sticks along $\triangle_{4,1,2} \cup \triangle_{6,1,2}$ as depicted in Figure 9. Therefore $\triangle_{4,1,2}$ is penetrated by $l_{5,6}$ and similarly $\triangle_{4,1,6}$ by $l_{2,3}$.

As seen in the above $\triangle_{2,3,4}$ and $\triangle_{4,5,6}$ should be irreducible. We consider which sticks should penetrate them. Since $\triangle_{4,1,6}$ is penetrated

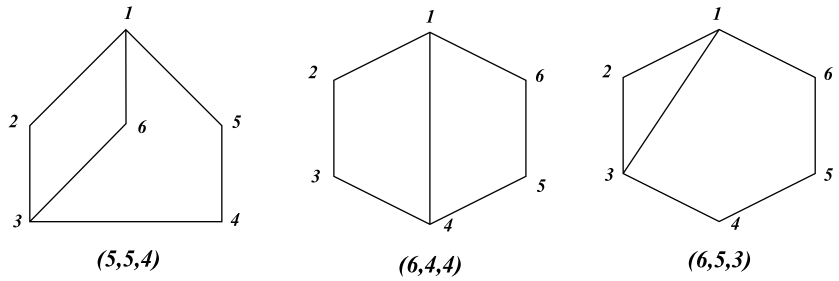

FIGURE 7. stick distributions of subknots

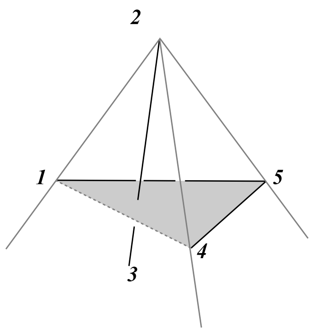

FIGURE 8

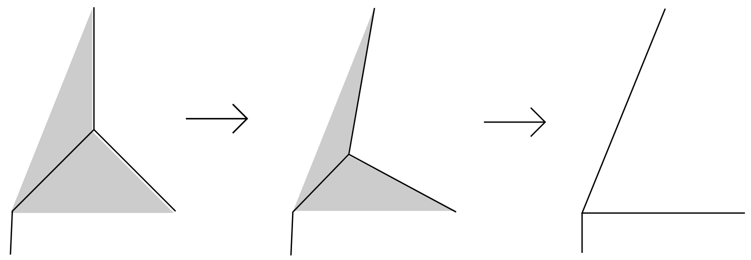

FIGURE 9. Reduction along two triangles near a trivalent vertex

by $l_{2,3}$, the interior of $\triangle_{2,3,4}$ is contained in that of $H^2_{4,1,6}$, which implies that $l_{1,6}$ does not penetrate the triangle. Therefore $\triangle_{2,3,4}$ should be penetrated by $l_{5,6}$. Then, considering $H^5_{2,3,4}$, we immediately see that $\triangle_{4,5,6}$ is not penetrated by $l_{2,3}$. On the other hand, the interior of $\triangle_{4,5,6}$ is contained in that of $H^6_{4,1,2}$, for $\triangle_{4,1,2}$ is penetrated by $l_{5,6}$. Hence $l_{1,2}$ can not penetrate $\triangle_{4,5,6}$. In conclusion we get a contradiction that $\triangle_{4,5,6}$ is reducible.

Case 2-3. (6, 5, 3): Firstly note that $\triangle_{1,2,3}$ should be irreducible. Otherwise, the triviality of T depends on the triviality of the other two subknots as mentioned in the proof of Case 1. Without loss of generality we may assume that $l_{4,5}$ penetrates the triangle. And also note that both $\triangle_{3,4,5}$ and $\triangle_{5,6,1}$ should be irreducible. Otherwise we can perform reduction along triangle. Considering $H^4_{1,2,3}$, we see that $l_{1,2}$ can not penetrate $\triangle_{3,4,5}$. Therefore the triangle should be penetrated by $l_{6,1}$. Again considering $H^6_{3,4,5}$, it is observed that $l_{5,6}$(*resp.* $l_{4,5}$) can not penetrate $\triangle_{1,3,4}$(*resp.* $\triangle_{6,1,3}$). Therefore $\triangle_{1,3,4}$ and $\triangle_{6,1,3}$ should be reducible, because $l_{5,6}$(*resp.* $l_{4,5}$) is only possible stick which may penetrate $\triangle_{1,3,4}$(*resp.* $\triangle_{6,1,3}$). Considering the possibility of reduction near the trivalent vertices v_1 and v_3, we see that the reducibility of $\triangle_{1,3,4}$ and $\triangle_{6,1,3}$ implies the irreducibility of $\triangle_{2,3,4}$ and $\triangle_{6,1,2}$ respectively.

Let K be the subknot of $s(K) = 6$. By Lemma 1, K contains a triangle which is reducible in K. From the observation in the above we see that only $\triangle_{4,5,6}$ is such a triangle. Note that although the triangle is reducible in K, it should be irreducible in T, that is, $l_{1,3}$ should penetrate $\triangle_{4,5,6}$. But considering $H^1_{3,4,5}$, we see that no such penetration can happen.

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Youngsik Huh
Department of Mathematics,
School of Natural Sciences,
Hanyang University, Seoul 133-791, Korea
E-mail: `yshuh@hanyang.ac.kr`

Seungsang Oh
Department of Mathematics,
Korea University, Anam-dong,
Sungbuk-ku, Seoul 136-701, Korea
E-mail: `seungsang@korea.ac.kr`