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일반화된 Lyapunov 방정식을 이용한 descriptor 시스템의 안정석 해석

(Stability Analysis of Descriptor System Using Generalized Lyapunov Equation)

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요 약

이 논문에서는 특별한 형태의 일반화된 연속시간 Lyapunov 방정식과 해의 존재성에 대해 다룬다. 이것은 무한대의 고유치를 가지는 descriptor 시스템에 대해 시스템의 안정성을 분석하기 위해 필요하다. 주요결과로써 먼저 지수 1과 2를 가지는 경우의 descriptor시스템에 대해 안정성을 위한 필요충분조건을 먼저 제안하고 다음으로 일반적인 경우의 descriptor 시스템에 대하여 특별한 형태의 Lyapunov 방정식을 이용하여 비슷한 안정성 조건을 제안한다. 마지막으로 제안한 방법의 타당성을 보이기 위한 예제를 살펴본다.

Abstract

In this paper we consider the specific types of the generalized continuous-time Lyapunov equation and the existence of solution. This is motivated to analyze the system stability in situations where descriptor system has infinite eigenvalue. As main results, firstly the necessary and sufficient condition for stability of the descriptor system with index one or two will be proposed. Secondly, for the general case of any index, the similar condition for stability of descriptor system will be proposed with the specific type of the generalized Lyapunov equation. Finally some examples are used to show the validity of proposed methods.

Keywords: Generalized Lyapunov Equation, Descriptor system, Existence of Solution, Stability Condition

I. INTRODUCTION

For last decades there have been many studies on the generalized Lyapunov equation and stability analysis. Lyapunov equations arise not only in the stability analysis but also in many other applications such as system and control theory, eigenvalue problems[1~2]. Especially when the stability and control design problems of descriptor system are considered, the standard Lyapunov equation is extended to the generalized Lyapunov equation. Such descriptor systems arise naturally in many applications such as multibody dynamics, electrical circuit simulation, chemical engineering and semi-discretization of partial differential equations[3~4]. The generalized continuous-time algebraic Lyapunov equation (GCALE) $E^TXA+A^TXE=-G$ have been considered, where E,A,G are given matrices and X is an unknown matrix. By using the concept of controllability and observability of descriptor system, the descriptor system can be transformed to balanced system, and various model reduction techniques and

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its applications also received a lot of interests[5~6].

On the other hand many numerical algorithms were developed for the GCALE with nonsingular matrix E. However, for the case of singular matrix E only little attention has been paid to the generalized Lyapunov equations[1, 7]. It is known that the GCALE has a unique solution for every G if the matrix E is nonsingular and all the eigenvalues of the pencil $\lambda E-A$ have negative real part. However, if E is singular, then the GCALE may have no solutions even if all the finite eigenvalues of $\lambda E-A$ lie in the open left half-plane and if the equation has a solution, it is not unique. To overcome these difficulties various types of generalized Lyapunov equations have been proposed, however, these equations are mostly limited to the case of pencils of index at most one[8~9].

In this paper we consider the specific types of the generalized continuous-time Lyapunov equation and the existence of solution. This is motivated to analyze the system stability in situations where descriptor system has infinite eigenvalue. In the following chapter II the descriptor system and mathematical preliminaries will be discussed. In chapter III, as main results of this paper, firstly the necessary and sufficient condition for stability of the descriptor system with index one or two will be proposed. Secondly, for the general case of any index, the similar condition for stability of descriptor system will be proposed with the specific type of the generalized Lyapunov equation. Finally some examples are used to show the validity of proposed methods.

II. LINEAR DESCRIPTOR SYSTEM

Consider a linear time-invariant continuous-time system

$$E\dot{x}(t) = Ax(t) + Bu(t) \qquad (1)$$
$$y(t) = Cx(t),$$

where $E, A \in \mathbb{R}^{nxn}$, $B \in \mathbb{R}^{nxm}$, $C \in \mathbb{R}^{pxn}$, $x(t)$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the output. If $E=I$, then (1) is a standard state space system. Otherwise, (1) is a descriptor system or generalized state space system. Assume that the pencil $\lambda E-A$ is *regular*, i.e., $\det(\lambda E-A) \neq 0$ for some $\lambda \in \mathbb{C}$. In this case $\lambda E-A$ can be reduced to the Weierstrass canonical form[10]. There exist nonsingular matrices W and T such that

$$E = W\begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix}T, \qquad A = W\begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix}T. \qquad (2)$$

Where I_k is the identity matrix of order k, J is the Jordan block corresponding to the finite eigenvalues of $\lambda E-A$, N is nilpotent and corresponds to the infinite eigenvalues. The index of nilpotency of N, denoted by υ, is called the *index* of the pencil $\lambda E-A$. The nilpotent matrix N has the following form

$$N = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \qquad (3)$$

The index υ is the size of the nilpotent block. Clearly, $N^{\upsilon-1} \neq 0$ and $N^{\upsilon}=0$. If the matrix E is nonsingular, then $\lambda E-A$ is of index zero. The pencil $\lambda E-A$ is of index one if and only if it has exactly $n_f=\mathrm{rank}(E)$ finite eigenvalues. The pencil $\lambda E-A$ is called *c-stable* if it is regular and all the finite eigenvalues of $\lambda E-A$ lie in the open left half-plane. And clearly, the pencil $\lambda E-A$ has an eigenvalue at infinity if and only if the matrix E is singular. Representation (2) defines the decomposition of $\mathbb{R}^n$ into complementary deflating subspaces of dimensions n_f and n_∞ corresponding to the finite and infinite eigenvalues of the pencil $\lambda E-A$, respectively. The matrices

$$P_r = T^{-1}\begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix}T, \quad P_l = W\begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix}W^{-1} \qquad (4)$$

are the spectral projections onto the right and left deflating subspaces of $\lambda E-A$ corresponding to the finite eigenvalues. Descriptor system (1) can be changed to the system with the Weierstrass canonical form

$$\begin{aligned} \dot{x}_1(t) &= Jx_1(t)+B_1u(t) \\ N\dot{x}_2(t) &= x_2(t)+B_2u(t) \quad W^{-1}B=\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CT^{-1}=[C_1 \quad C_2]. \\ y(t) &= C_1x_1(t)+C_2x_2(t) \end{aligned} \tag{5}$$

It is well known that computing the Weierstrass canonical form in finite precision arithmetic is, in general, an ill-conditioned problem in the sense that small changes in the data may extremely change the canonical form. Now consider the GCALE

$$E^T XA + A^T XE = -G. \tag{6}$$

For the case of nonsingular E the following theorem is satisfied.

Theorem 1[1]: Let $\lambda E-A$ be a regular pencil. If all eigenvalues of $\lambda E-A$ are finite and lie in the open left half-plane, then for every positive (semi)definite matrix G, the GCALE (6) has a unique positive (semi)definite solution X. Conversely, if there exist positive definite matrices X and G satisfying (6), then all eigenvalues of the pencil $\lambda E-A$ are finite and lie in the open left half-plane. ■

However, many applications of descriptor systems lead to generalized Lyapunov equations with a singular matrix E, In the case of singular E the GCALE (6) may have no solutions even if all finite eigenvalues of the pencil $\lambda E-A$ have negative real part.

Ⅲ. GENERALIZED LYAPUNOV EQUATION AND STABILITY ANALYSIS

In this chapter we consider the specific types of the generalized continuous-time Lyapunov equation and the existence of solution. And the necessary and sufficient condition for stability of the descriptor system is proposed. The following lemma gives the necessary and sufficient conditions for unique solvability of the generalized Sylvester equation.

Lemma 1[11]: The generalized Sylvester equation

$$BXA - FXE = -G \tag{7}$$

has a unique solution X if and only if the pencils $\lambda B-F$ and $\lambda E-A$ are regular and they have no common eigenvalues. Where $A,B,E,F,G \in \mathbb{R}^{n\times n}$ are given matrices and $X \in \mathbb{R}^{n\times n}$ is an unknown matrix. ■

A consideration of the GCALE (6) with another special right-hand side is useful since for such an equation, the existence theorems can be stated.

Theorem 2: Let $\lambda E-A$ be a regular pencil of index at most two. If $\lambda E-A$ is c-stable, then for every matrix G, the GCALE

$$E^T XA + A^T XE = -E^T(E-A)^{-T}G(E-A)^{-1}E, \tag{8}$$

has a solution. Where it is assumed that $E-A$ is a nonsingular matrix.

Proof: Let the pencil $\lambda E-A$ be in Weierstrass canonical form (2), where the eigenvalues of J lie in the open left half-plane. Let the matrices

$$\begin{aligned} G_T &:= T^{-T}GT^{-1} = \begin{bmatrix} G_{T1} & G_{T2} \\ G_{T2}^T & G_{T4} \end{bmatrix}, \\ P &:= W^T XW = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_4 \end{bmatrix} \end{aligned} \tag{9}$$

be defined and partitioned in blocks conformably to E and A with Weierstrass canonical transform matrices T and W. Then we have

$$\begin{aligned} &(W^{-1}ET^{-1})^T W^T XW(W^{-1}AT^{-1}) + (W^{-1}AT^{-1})^T W^T XW(W^{-1}ET^{-1}) \\ &= -(W^{-1}ET^{-1})^T (W^{-1}ET^{-1} - W^{-1}AT^{-1})^{-T} T^{-T}GT^{-1} \\ &\quad \times (W^{-1}ET^{-1} - W^{-1}AT^{-1})^{-1}(W^{-1}ET^{-1}), \end{aligned} \tag{10}$$

$$\begin{aligned} &P_1 J + J^T P_1 = -(I_{n_\varepsilon} - J)^{-T} G_{T1} (I_{n_\varepsilon} - J)^{-1} \\ &P_2 + J^T P_2 N = -(I_{n_\varepsilon} - J)^{-T} G_{T2} (N - I_{n_\infty})^{-1} N \\ &N^T P_2^T J + P_2^T = -N^T (N - I_{n_\infty})^{-T} G_{T2}^T (I_{n_\varepsilon} - J)^{-1} \\ &N^T P_4 + P_4 N = -N^T (N - I_{n_\infty})^{-T} G_{T4} (N - I_{n_\infty})^{-1} N, \end{aligned} \tag{11}$$

$$(I - J)^{-T} G_{T1} (I - J)^{-1} > 0.$$

Since all eigenvalues of J have negative real part, the first Lyapunov equation of (11) has a unique positive definite solution P_1 for every positive definite G_{T1} using theorem 1. And let the left hand side matrices of the equation in Sylvester equation (7) $B=I$, $A=I$, $F=J^T$, and $E=-N$ then the Sylvester equation is same as the second equation of (11). ($\lambda I-J^T$) and ($-\lambda N-I$) are regular and have no common eigenvalues because ($\lambda I-J^T$) has only finite eigenvalues and ($-\lambda N-I$) has infinite eigenvalues. Therefore, from lemma 1, the second equation of (11) has a unique solution. We can see that

$$P_2 = -(I - J)^{-T} G_{T2} (N - I)^{-1} N \tag{12}$$

because $N=0$ or $N \times N = N^T \times N^T = 0$ by assumption. The solution of the last equation of (11) is not unique for every G_{T4}, for example the matrix

$$P_4 = \frac{1}{2}\left\{ N^T (N - I)^{-T} G_{T4} + G_{T4} (N - I)^{-1} N \right\} \tag{13}$$

can be a solution. For the case of index two we have

$$\begin{aligned} N^T P_4 + P_4 N &= \frac{1}{2} N^T \left\{ N^T (N - I)^{-T} G_{T4} + G_{T4} (N - I)^{-1} N \right\} \\ &\quad + \frac{1}{2}\left\{ N^T (N - I)^{-T} G_{T4} + G_{T4} (N - I)^{-1} N \right\} N \\ &= \frac{1}{2} N^T G_{T4} (N - I)^{-1} N + \frac{1}{2} N^T (N - I)^{-T} G_{T4} N \\ &= \begin{bmatrix} 0 & 0 \\ 0 & -g_{41} \end{bmatrix}, \quad \text{where } G_{T4} := \begin{bmatrix} g_{41} & g_{42} \\ g_{42} & g_{44} \end{bmatrix} \end{aligned} \tag{14}$$

from the left hand side of last equation of (11). And from the right hand side of the equation we can have the same matrix. G is positive definite, then also GT1 is positive definite and, hence, the solution P1 is positive definite. ■

Theorem 3: Let $\lambda E-A$ be a regular pencil of index at most two. Let G be a positive definite matrix. The solution of GCALE (8) is positive definite on the subspace $\mathrm{Im}(P_l)$ if and only if $\lambda E-A$ is c-stable. Where $E-A$ is a nonsingular matrix and $\mathrm{Im}(P_l)$ means the range space of P_l.

Proof: The symmetric matrix

$$X := W^{-T} \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_4 \end{bmatrix} W^{-1} \tag{15}$$

satisfies the GCALE (8). If $\lambda E-A$ is c-stable then P_1 is positive definite for positive definite G and G_{T1} Therefore X is positive definite on the subspace $\mathrm{Im}(P_l)$. That is, for $z \in \mathrm{Im}(P_l)$, we have

$$\begin{aligned} z^T X z &= x^T P_l^T X P_l x \\ &= x^T W^{-T} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} W^T W^{-T} \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_4 \end{bmatrix} W^{-1} W \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} W^{-1} x \\ &= x^T W^{-T} \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} W^{-1} x, \qquad W^{-1} x = \eta = \begin{bmatrix} \eta_\varepsilon \\ \eta_\infty \end{bmatrix} \\ &= \begin{bmatrix} \eta_\varepsilon^T & \eta_\infty^T \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_\varepsilon \\ \eta_\infty \end{bmatrix} = \eta_\varepsilon^T P_1 \eta_\varepsilon > 0. \end{aligned} \tag{16}$$

Conversely if X is positive definite on $\mathrm{Im}(P_l)$, the matrix

$$E^T X E = T^T \begin{bmatrix} P_1 & P_2 N \\ N^T P_2^T & N^T P_4 N \end{bmatrix} T \tag{17}$$

is positive definite on $\mathrm{Im}(P_r)$. And the matrix

$$\begin{aligned} &E^T (E - A)^{-T} G (E - A)^{-1} E \\ &= T^T \begin{bmatrix} (I_{n_\varepsilon} - J)^{-T} G_{T1} (I_{n_\varepsilon} - J)^{-1} & (I_{n_\varepsilon} - J)^{-T} G_{T2} (N - I_{n_\infty})^{-1} N \\ N^T (N - I_{n_\infty})^{-T} G_{T2}^T (I_{n_\varepsilon} - J)^{-1} & N^T (N - I_{n_\infty})^{-T} G_{T4} (N - I_{n_\infty})^{-1} N \end{bmatrix} T \end{aligned} \tag{18}$$

is also positive definite on $\mathrm{Im}(P_r)$ because $G_{T1} > 0$ and $(I-J)^{-T} G_{T1} (I-J)^{-1} > 0$. Let $\zeta \neq 0$ be an eigenvector of the pencil $\lambda E-A$ corresponding to a finite eigenvalue λ, that is, $\lambda E\zeta = A\zeta$, ζ is a vector in $\mathrm{Im}(P_r)$. Multiplication of (8) on the right and left by ζ and ζ^T, respectively, gives

$$\begin{aligned}&-\zeta^T\left\{E^T(E-A)^{-T}G(E-A)^{-1}E\right\}\zeta\\&=\zeta^T(E^TXA+A^TXE)\zeta=\lambda\zeta^TE^TXE\zeta+\bar{\lambda}\zeta^TE^TXE\zeta\\&=2\mathrm{Re}\{\lambda\}\zeta^TE^TXE\zeta<0,\end{aligned}\tag{19}$$

Since E^TXE is positive definite on $\mathrm{Im}(P_r)$, we obtain that $2\mathrm{Re}\{\lambda\}<0$, i.e., all finite eigenvalues of $\lambda E-A$ lie in the open left half-plane. ■

Corollary 1: Let $\lambda E-A$ be a regular pencil and the index of $\lambda E-A$ be at most one. Let G be a symmetric positive definite matrix. The GCALE (8) has a symmetric positive (semi)definite solution X if and only if $\lambda E-A$ is c-stable.

Proof: If the pencil $\lambda E-A$ is of index at most one and c-stable, then from the proof of theorem 3 we obtain the matrix

$$X=W^{-T}\begin{bmatrix}P_1 & 0\\ 0 & P_4\end{bmatrix}W^{-1}.\tag{20}$$

And it satisfies the GCALE (8). Here P_1 is a unique symmetric positive definite solution of the first equation of (11) and P_4 is an arbitrary symmetric positive (semi)definite matrix. In this case X is the symmetric positive (semi)definite solution of (8).

Now assume that $\lambda E-A$ is at most one and the GCALE (8) has a symmetric positive (semi)definite solution X with symmetric positive definite G. The matrix $P_2=0$ is a unique solution, and P_4 is an arbitrary matrix. Using Schur complement[12],

$$\begin{aligned}&\begin{bmatrix}P_1 & P_2\\ P_2^T & P_4\end{bmatrix}>0 \;\Leftrightarrow\; P_1>0,\; P_4-P_2^TP_1^{-1}P_2>0\\&\begin{bmatrix}P_1 & P_2\\ P_2^T & P_4\end{bmatrix}\ge 0 \;\Leftrightarrow\; P_1>0,\; P_4-P_2^TP_1^{-1}P_2\ge 0\end{aligned}\tag{21}$$

P_1 should be positive definite and P_4 should be positive (semi)definite because X is the symmetric positive (semi)definite solution. Therefore $\lambda E-A$ is c-stable. ■

In the following theorem 4 we consider specific type of the generalized Lyapunov equation and propose the necessary and sufficient condition for the c-stable descriptor system (1) with nonsingular E and any index.

Theorem 4: Let $\lambda E-A$ be a regular pencil and let G be a symmetric positive definite matrix. The GCALE

$$\begin{aligned}&E^TXA+A^TXE=-M^T(G+\Delta_o)M,\\&M=I \text{ or } M=(E-A)^{-1}E\end{aligned}\tag{22}$$

has a symmetric positive semidefinite solution X if and only if $\lambda E-A$ is c-stable with any index. Where $G_{\Delta o}:=G+\Delta_o$ is also a specifically defined positive semidefinite matrix such that

$$\begin{aligned}\Gamma&:=T^{-T}(G_{\Delta o})T^{-1}=T^{-T}(G+\Delta_o)T^{-1}\\&=\begin{bmatrix}\Gamma_1 & \Gamma_2\\ \Gamma_2^T & \Gamma_4\end{bmatrix}=\begin{bmatrix}G_{T1} & 0\\ 0 & 0\end{bmatrix}\ge 0,\quad G_{T1}>0\end{aligned}\tag{23}$$

Where W, T, are the Weierstrass canonical transformation matrices and it is assumed that $E-A$ is a nonsingular matrix.

Proof: Let the pencil $\lambda E-A$ be in Weierstrass canonical form (2). First of all, assume that the eigenvalues of J lie in the open left half-plane so $\lambda E-A$ is c-stable. Then we have

$$\begin{aligned}\Delta_o&=T^T\Gamma T-G\\&=T^T(\Gamma-G_T)T=T^T\begin{bmatrix}0 & -G_{T2}\\ -G_{T2}^T & -G_{T4}\end{bmatrix}T\end{aligned}\tag{24}$$

And $G_{\Delta o}=G+\Delta_o$ is positive semidefinite if and only if Γ is positive semidefinite for nonsingular T and symmetric $G_{\Delta o}$. So there exists the positive semidefinite matrix $G_{\Delta o}$ for every possible positive definite matrix G. Then from (22) and (10) we have

$$\begin{aligned}&(W^{-1}ET^{-1})^TW^TXW(W^{-1}AT^{-1})+(W^{-1}AT^{-1})^TW^TXW(W^{-1}ET^{-1})\\&=-T^{-T}(G+\Delta_o)T^{-1},\qquad M=I\\&=-H^TT^{-T}(G+\Delta_o)T^{-1}H,\qquad M=(E-A)^{-1}E\\&H:=(W^{-1}ET^{-1}-W^{-1}AT^{-1})^{-1}(W^{-1}ET^{-1})\end{aligned}\tag{25}$$

$$\begin{aligned} &P_1J + J^T P_1 = -G_{T1}, \quad M = I \\ &P_1J + J^T P_1 = -(I-J)^{-T} G_{T1}(I-J)^{-1}, \quad M = (E-A)^{-1}E \\ &P_2 + J^T P_2 N = 0, \quad N^T P_2^T J + P_2^T = 0 \\ &N^T P_4 + P_4 N = 0 \end{aligned} \tag{26}$$

Since all eigenvalues of J have negative real part, the first and second standard Lyapunov equation of (26) have a unique positive definite solution P_1 for every positive definite G_{T1} using theorem 1. The third two equations in (26) are uniquely solvable using lemma 1 and have the trivial solutions $P_2=0$ and the last equation of (26) is not uniquely solvable. But we can have simple trivial solution $P_4=0$. Therefore there exist positive semidefinite matrices P and $X=W^{-T}PW^{-1}$.

Conversely if the positive semidefinite solution of (22) exists with the positive semidefinite matrix $G_{\Delta o}$ for a positive definite G, it is obtained as a block diagonal form same to (20). Where $P_2=0$, and it is a unique solution, but P_4 is not necessarily zero matrix. From Schur complement equation (21) P_1 should be positive definite and P_4 should be positive semidefinite because X is the symmetric positive semidefinite solution. As a result the pencil $\lambda E-A$ is c-stable with any index because P_1 is positive definite with $G_{T1}>0$ and $(I-J)^{-T}G_{T1}(I-J)^{-1}>0$ and the proof is completed. ■

In the case that G is positive semidefinite, the existence of the GCALE solution can be considered as a necessary condition the descriptor system to be c-stable. The dual GCALE can be obtained as theorem 5.

Theorem 5: Let $\lambda E-A$ be a regular pencil and let F be a symmetric positive definite matrix. The GCALE

$$EYA^T + AYE^T = -M(F+\Delta_c)M^T, \quad M = I \text{ or } M = E(E-A)^{-1} \tag{27}$$

has a symmetric positive semidefinite solution Y if and only if $\lambda E-A$ is c-stable with any index. Where let the matrices

$$F_W := W^{-1}FW^{-T} = \begin{bmatrix} F_{W1} & F_{W2} \\ F_{W2}^T & F_{W4} \end{bmatrix}, \quad Q := TYT^T = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_4 \end{bmatrix} \tag{28}$$

are defined and partitioned in blocks conformably to E and A. And $F_{\Delta c}:=F+\Delta_c$ is also a specifically defined positive semidefinite matrix such that

$$\begin{aligned} \Lambda &:= W^{-1}(F_{\Delta c})W^{-T} = W^{-1}(F+\Delta_c)W^{-T} \\ &= \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_2^T & \Lambda_4 \end{bmatrix} = \begin{bmatrix} F_{W1} & 0 \\ 0 & 0 \end{bmatrix} \geq 0, \quad F_{W1} > 0. \end{aligned} \tag{29}$$

Where W, T, are the Weierstrass canonical transformation matrices and it is assumed that $E-A$ is a nonsingular matrix.

Proof: It is similar to the proof of theorem 4. First of all, assume that $\lambda E-A$ is c-stable, we have

$$\begin{aligned} \Delta_c &= W\Lambda W^T - F \\ &= W(\Lambda - F_W)W^T = W\begin{bmatrix} 0 & -F_{W2} \\ -F_{W2}^T & -F_{W4} \end{bmatrix} W^T \end{aligned} \tag{30}$$

And $F_{\Delta c}=F+\Delta_c$ is positive semidefinite if and only if Λ is positive semidefinite for nonsingular W and symmetric $F_{\Delta c}$. So there exists the positive semidefinite matrix $F_{\Delta c}$ for every possible positive definite matrix F. Then from (27) we have

$$\begin{aligned} &(W^{-1}ET^{-1})TYT^T(W^{-1}AT^{-1})^T + (W^{-1}AT^{-1})TYT^T(W^{-1}ET^{-1})^T \\ &= -W^{-1}(F+\Delta_c)W^{-T}, \quad M = I \\ &= -KW^{-1}(F+\Delta_c)W^{-T}K^T, \quad M = E(E-A)^{-1} \\ &K := (W^{-1}ET^{-1})(W^{-1}ET^{-1} - W^{-1}AT^{-1})^{-1} \end{aligned} \tag{31}$$

$$\begin{aligned} &Q_1J^T + JQ_1 = -F_{W1}, \quad M = I \\ &Q_1J^T + JQ_1 = -(I-J)^{-1}F_{W1}(I-J)^{-T}, \quad M = E(E-A)^{-1} \\ &Q_2 + JQ_2N^T = 0, \quad NQ_2^TJ^T + Q_2^T = 0 \\ &NQ_4 + Q_4N^T = 0 \end{aligned} \tag{32}$$

Since all eigenvalues of J have negative real part, the first and second standard Lyapunov equation of (32) have a unique positive definite solution Q_1 for every positive definite F_{W1} using theorem 1. The third two equations in (32) are uniquely solvable using lemma 1 and have the trivial solutions $Q_2=0$ and the last equation of (32) is not uniquely solvable. But we can have simple trivial solution $Q_4=0$. Therefore there exist positive semidefinite matrices Q and $Y=T^{-1}QT^{-T}$.

Conversely if the positive semidefinite solution of (27) exists with the positive semidefinite matrix $F_{\Delta c}$ for a positive definite F, it is obtained as

$$Y=T^{-1}\begin{bmatrix} Q_1 & 0 \\ 0 & Q_4 \end{bmatrix}T^{-T}. \quad (33)$$

Where $Q_2=0$, and it is a unique solution, but Q_4 is not necessarily zero matrix. From Schur complement equation (21) Q_1 should be positive definite and Q_4 should be positive semidefinite because Y is the symmetric positive semidefinite solution. As a result the pencil $\lambda E-A$ is c-stable with any index because Q_1 is positive definite with $F_{W1}>0$ and $(I-J)^{-1}F_{W1}(I-J)^{-T}>0$ and the proof is completed. ■

For the case of theorem 4 and 5 we can give the conditions $P_4=0$ and $Q_4=0$ to find the numerical solution uniquely. And these results which are proposed in this paper can be extended to the linear discrete-time descriptor system.

Ⅳ. NUMERICAL EXAMPLE

Consider the following c-stable continuous-time descriptor system as a numerical example. Where the index of nilpotent is two, we can easily see that there is no solution of standard Lyapunov equation (6).

$$A=\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ E=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ G=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ P=\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix}$$

From (6) and (8) we have

$$p_{11}(-1)+(-1)p_{11}=-2 \ \Rightarrow p_{11}=1 \qquad \text{for (6)},$$
$$p_{11}(-1)+(-1)p_{11}=-\frac{1}{4}(2)=-\frac{1}{2} \Rightarrow p_{11}=\frac{1}{4} \qquad \text{for (8)},$$
$$[p_{12}\ \ p_{13}]+(-1)[p_{12}\ \ p_{13}]\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}=[p_{12}\ \ p_{13}-p_{12}]$$
$$=[0\ \ 0] \quad \text{for (6) and (8)}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_{22} & p_{23} \\ p_{23} & p_{33} \end{bmatrix}+\begin{bmatrix} p_{22} & p_{23} \\ p_{23} & p_{33} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$=\begin{bmatrix} 0 & p_{22} \\ p_{22} & p_{23}+p_{23} \end{bmatrix}=-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{for (6)},$$
$$=-\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{for (8)}.$$

And the solution of (8) exists by Theorem 3. But it isn't a unique solution.

$$P=\begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0 & -0.5 \\ 0 & -0.5 & p_{33} \end{bmatrix},$$

We can get $p_{33}=0$ from (13). Where P_4 can not be a positive definite and it is indefinite for every p_{33}. Now consider the c-stable descriptor system with index one.

$$A=\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},\ E=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ G=\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},\ P=\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

Finding the solution of (6) and (8) by the similar method, we have

$$p_{11}(-1)+(-1)p_{11}=-2 \ \Rightarrow p_{11}=1 \qquad \text{for (6)},$$
$$p_{11}(-1)+(-1)p_{11}=-\frac{1}{4}(2)=-\frac{1}{2} \Rightarrow p_{11}=\frac{1}{4} \qquad \text{for (8)},$$
$$p_{12}+(-1)p_{12}(0)=p_{12}=0,$$
$$(0)p_{22}+p_{22}(0)=-1 \qquad \text{for (6)},$$
$$=0 \qquad \text{for (8)}.$$

In this example there is also no solution of (6), from the Corollary 1 if the system is c-stable, then P and X can be positive (semi)definite matrices as selecting the proper matrix to $(p_{22}\geq 0)p_{22}>0$. Conversely if the (semi)definite solution X exists, then P_1 is necessarily positive definite matrix for the X, and hence, the descriptor system is c-stable. Now consider the c-stable descriptor system with index three.

$$A=\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\quad E=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix},$$

$$G=\begin{bmatrix}2&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\quad P=\begin{bmatrix}p_{11}&p_{12}&p_{13}&p_{14}\\p_{12}&p_{22}&p_{23}&p_{24}\\p_{13}&p_{23}&p_{33}&p_{34}\\p_{14}&p_{24}&p_{34}&p_{44}\end{bmatrix}$$

From (6), (8) and (22) we have

$$p_{11}(-1)+(-1)p_{11}=-2 \;\Rightarrow p_{11}=1$$
$$\text{for (6) and (22) with } M=I,$$

$$p_{11}(-1)+(-1)p_{11}=-\frac{1}{4}(2)=-\frac{1}{2} \Rightarrow p_{11}=\frac{1}{4}$$
$$\text{for (8) and (22) with } M=(E-A)^{-1}E,$$

$$[p_{12}\;\; p_{13}\;\; p_{14}]+(-1)[p_{12}\;\; p_{13}\;\; p_{14}]\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$
$$=[p_{12}\;\; p_{13}-p_{12}\;\; p_{14}-p_{13}]=[0\;\;0\;\;0],$$

$$\begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}p_{22}&p_{23}&p_{24}\\p_{23}&p_{33}&p_{34}\\p_{24}&p_{34}&p_{44}\end{bmatrix}+\begin{bmatrix}p_{22}&p_{23}&p_{24}\\p_{23}&p_{33}&p_{34}\\p_{24}&p_{34}&p_{44}\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$
$$=\begin{bmatrix}0&p_{22}&p_{23}\\p_{22}&p_{23}+p_{23}&p_{24}+p_{33}\\p_{23}&p_{33}+p_{24}&p_{34}+p_{34}\end{bmatrix}=-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\quad\text{for (6)},$$
$$=-\begin{bmatrix}0&0&0\\0&1&1\\0&1&2\end{bmatrix}\quad\text{for (8)},$$
$$=-\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\quad\text{for (22)}.$$

For (6) and (8) there is no solution, in the case of (22),

$$P=\begin{bmatrix}p_{11}&0&0&0\\0&0&0&p_{24}\\0&0&p_{33}&0\\0&p_{24}&0&p_{44}\end{bmatrix},\quad p_{11}=\frac{1}{4},\quad p_{24}+p_{33}=0$$

In this equation if elements p_{33} and p_{44} are selected properly, then solution P and X can be positive semidefinite. Conversely if the semidefinite solution X exists, then P_1 is necessarily positive definite matrix for the X using Schur complement of (21), and hence, the descriptor system is c-stable.

V. CONCLUSIONS

In this paper we have proposed the specific types of the generalized Lyapunov equation and have found the solutions for stability of the descriptor system with singular E. Firstly The necessary and sufficient condition for stability of the descriptor system with index one and two were proposed. Secondly, for the general case of any index, the similar condition for stability of descriptor system was proposed with the specific type of the generalized continuous-time Lyapunov equation. In the future it needs to study for the applications of the generalized Lyapunov equation with singular E.

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