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Convolution Properties of Certain Class of Multivalent Meromorphic Functions

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Abstract. The purpose of the present paper is to introduce a new subclass of meromorphic multivalent functions defined by using a linear operator associated with the generalized hypergeometric function. Some properties of this class are established here by using the principle of differential subordination and convolution in geometric function theory.

1. Introduction

Let Σ_p denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k z^{k-p} \qquad (p \in \mathbf{N}), \tag{1.1}$$

which are analytic and $p-$valent in the punctured unit disk

$$D = \{z \ \in C : \ 0 < |z| < 1\} = \mathcal{U} \setminus \{0\},$$

where $\mathcal{U}$ is the open unit disk.

For $f(z)$ given by (1.1) and

$$g(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} b_k z^{k-p}$$

of the class Σ_p, the Hadamard product (or convolution) is defined by

$$f(z) * g(z) = (f * g)(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} = (g * f)(z). \tag{1.2}$$

In recent years, several families of derivative and integral operators which are closely related with the Hadamard product were introduced and investigated in the context of Geometric function theory. For example the Ruscheweyh derivative

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operator and its generalizations ([7], [8], [10], [15], [16]), the Carlson-Shaffer operator ([1], [2]), the Jung-Kim-Srivastava integral operator ([6], [9]), the Dziok-Srivastava operator ([4], [5]), the Noor integral operator ([11]-[14]), and so on. Motivated essentially by these works, we introduce here a novel family of integral operators defined on the space of mutivalent meromorphic functions in the class Σ_p. By using these integral operators, we define a subclass of meromorphic functions and investigate various inclusion relationships, coefficient estimates, structural formula and convolution with convex functions for the subclass introduced in this paper. Our results extend the results recently established by Cho et al. [2] and Sokŏl and Trojnar-Spelina [18] for subclasses of meromorphic multivalent functions.

Let ${}_q\mathcal{F}_s(a_1, \cdots, a_q; b_1, \cdots, b_s; z)$ be a function given by

$$ {}_q\mathcal{F}_s(a_1, \cdots, a_q; b_1, \cdots, b_s; z) = \frac{1}{z^p}\, {}_qF_s(a_1, \cdots, a_q; b_1, \cdots, b_s; z), \tag{1.3} $$

$(q \leq s+1,\ q, s \in \mathbf{N}_0 = \mathbf{N} \cup \{0\},\ z \in D,\ a_i, b_j \in C \setminus Z_0^-;\ Z_0^- = \{0, -1, \cdots\},$
$i = 1, \cdots, q \textit{ and } j = 1, \cdots, s)$

where ${}_qF_s(z)$ is the well-known generalized hypergeometric function [17].

Corresponding to ${}_q\mathcal{F}_s(a_1, \cdots, a_q; b_1, \cdots, b_s; z)$ defined by (1.3), we introduce a function ${}_q\mathcal{F}_s^{(-1)}(a_1, \cdots, a_q; b_1, \cdots, b_s; z)$ by

$$ \begin{aligned} &{}_q\mathcal{F}_s(a_1, \cdots, a_q; b_1, \cdots, b_s; z) \ *\ {}_q\mathcal{F}_s^{(-1)}(a_1, \cdots, a_q; b_1, \cdots, b_s; z) \\ &\quad = \frac{1}{z^p(1-z)^{\lambda+p}} \quad (\lambda > -p), \end{aligned} \tag{1.4} $$

We now define the linear operator

$$ {}_qI_s^{\lambda,p}(a_i; b_j) : \Sigma_p \to \Sigma_p. $$

by

$$ \begin{aligned} {}_qI_s^{\lambda,p}(a_i; b_j)f(z) &= {}_qI_s^{\lambda,p}(a_1, \cdots, a_q; b_1, \cdots, b_s)f(z) \\ &= {}_q\mathcal{F}_s^{(-1)}(a_1, \cdots, a_q; b_1, \cdots, b_s; z) * f(z) \end{aligned} \tag{1.5} $$

$(q \leq s+1,\ q, s \in \mathbf{N}_0 = \mathbf{N} \cup \{0\},\ z \in D,\ a_i, b_j \in C \setminus Z_0^-;\ Z_0^- = \{0, -1, \cdots\},$
$i = 1, \cdots, q \textit{ and } j = 1, \cdots, s)$

It is well-known that for $\lambda > -p$

$$ \frac{1}{(1-z)^{\lambda+p}} = \sum_{k=0}^{\infty} \frac{(\lambda+p)_k}{k!} z^k \quad (z \in \mathcal{U}). $$

Thus

$$ \frac{1}{z^p(1-z)^{\lambda+p}} = \sum_{k=0}^{\infty} \frac{(\lambda+p)_k}{k!} z^{k-p} \qquad (z \in D). \tag{1.6} $$

From (1.4) and (1.6), we get

$$\sum_{k=0}^{\infty}\frac{(a_1)_k\ \cdots\ (a_q)_k}{(b_1)_k\ \cdots\ (b_s)_k}\frac{z^{k-p}}{k!}*\ {}_q\mathcal{F}_s^{(-1)}(a_1,\cdots,a_q;b_1,\cdots,b_s;z)=\sum_{k=0}^{\infty}\frac{(\lambda+p)_k}{k!}z^{k-p}.$$

Therefore the function ${}_q\mathcal{F}_s^{(-1)}(a_1,\cdots,a_q;b_1,\cdots,b_s;z)$ has the following form

$$(1.7)\qquad {}_q\mathcal{F}_s^{(-1)}(a_1,\cdots,a_q;b_1,\cdots,b_s;z)=\sum_{k=0}^{\infty}\frac{(\lambda+p)_k(b_1)_k\ \cdots\ (b_s)_k}{(a_1)_k\ \cdots\ (a_q)_k}z^{k-p}.$$

Thus from (1.5), we have

$$(1.8)\qquad {}_qI_s^{\lambda,p}(a_i;b_j)f(z)=\frac{1}{z^p}+\sum_{k=1}^{\infty}\frac{(\lambda+p)_k(b_1)_k\ \cdots\ (b_s)_k}{(a_1)_k\ \cdots\ (a_q)_k}a_kz^{k-p}.$$

For convenience, we use the notation

$$\begin{aligned}&{}_qI_s^{\lambda,p}(a_i+m;b_j+n)f(z)\\&\quad=\frac{1}{z^p}+\sum_{k=1}^{\infty}\frac{(\lambda+p)_k(b_1)_k\ \cdots\ (b_{j-1})_k\ (b_j+n)_k\ (b_{j+1})_k\ \cdots\ (b_s)_k}{(a_1)_k\ \cdots\ (a_{j-1})_k\ (a_i+m)_k\ (a_{j+1})_k\ ..\ (a_q)_k}a_kz^{k-p}.\end{aligned}$$

$$(i=1,\cdots,q \text{ and } j=1,\cdots,s)$$

Obviously the operators studied recently by Noor [13] and Yuan et al. [19] are special cases of ${}_qI_s^{\lambda,p}$ - operator defined by (1.8).

It can easily be verified that

$$(1.9)\quad z[{}_qI_s^{\lambda,p}(a_i+1;b_j)f(z)]'=a_i\ {}_qI_s^{\lambda,p}(a_i;b_j)f(z)-(a_i+p)\ {}_qI_s^{\lambda,p}(a_i+1;b_j)f(z),$$

and

$$(1.10)\quad z[{}_qI_s^{\lambda,p}(a_i;b_j)f(z)]'=(\lambda+p)\ {}_qI_s^{\lambda+1,p}(a_i;b_j)f(z)-(\lambda+2p)\ {}_qI_s^{\lambda,p}(a_i;b_j)f(z).$$

If f and g are analytic and if there exists a Schwarz function w, analytic in $\mathcal{U}$ with

$$w(0)=0,\ |w(z)|<1\ \ z\in\mathcal{U},$$

such that $f(z)=g(w(z))$, then the function f is called *differential subordinate* to g and denoted by

$$f\prec g \text{ or } f(z)\prec g(z),\ z\in\mathcal{U}.$$

In particular, if the function g is univalent in $\mathcal{U}$ and $f(0)=g(0)$, then

$$f\prec g\Leftrightarrow f(\mathcal{U})\subset g(\mathcal{U}).$$

In this case we have $w(z)=g^{-1}[f(z)]$.

Let $\mathcal{N}$ be the class of functions h with the normalization $h(0) = 1$, which are convex and univalent in $\mathcal{U}$ and satisfy the condition $Re[h(z)] > 0$ for $z \in \mathcal{U}$. Now by using the operator ${}_qI_s^{\lambda,p}(a_i; b_j)$ for $0 \le \eta < p,\ p \in \mathbf{N},\ h \in \mathcal{N}$, we define the following subclass of meromorphic functions Σ_p,

(1.11)

$$\mathcal{T}_{a_i,b_j}^{\lambda}(\eta, p, h) = \left\{ f \in \Sigma_p : \frac{1}{p-\eta}\left[\frac{-z({}_qI_s^{\lambda,p}(a_i; b_j)f(z))'}{{}_qI_s^{\lambda,p}(a_i; b_j)f(z)} - \eta\right] \prec h(z), \quad z \in \mathcal{U} \right\}.$$

For $q = 2,\ s = 1,\ p = 1,\ a_1 = \lambda + 1,\ b_1 = a_2$ the above class of functions reduces to the class $\mathcal{MS}(\eta; \phi)$ studied by Cho et al. [3].

2. Inclusion properties

The following lemmas will be used in our investigation.

Lemma 1([15]). *Let $f \in K,\ g \in S^*$, where S^* and K denote the subclasss of univalent functions consisting of starlike and convex functions in $\mathcal{U}$. Then for each analytic function h in $\mathcal{U}$,*

$$\frac{(f * hg)(\mathcal{U})}{(f * g)(\mathcal{U})} \subseteq \overline{co}h(\mathcal{U}), \tag{2.1}$$

where $\overline{co}h(\mathcal{U})$ denotes the closed convex hull of $h(\mathcal{U})$.

Lemma 2([15]). *Let either $0 < a \le c$ and $c \ge 2$ when a, c are real, or $Re(a+c) \ge 3$, $Re(a) \le Re(c)$ and $Im(a) = Im(c)$ when a,c are complex. Then the function*

$$f(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \quad (z \in \mathcal{U}), \tag{2.2}$$

belongs to the class K of convex functions.

Now we give two inclusion relationships for the class of meromorphic functions defined by (1.11).

Theorem 1. *Let $0 < a_i \le \alpha_i,\ \forall a_i,\ \alpha_i,\ b_j \in\ C \setminus Z_0^-,\ (i = 1, \cdots, q,\ j = 1, \cdots, s),\ h \in \mathcal{N}$ and*

$$\max_{z \in \mathcal{U}} (Re[h(z)]) < 1 + \frac{1}{p-\eta} \quad (z \in \mathcal{U}). \tag{2.3}$$

Further suppose that either a_i, α_i are real such that $\alpha_i \ge 2$ or a_i, α_i are complex such that $Re(a_i + \alpha_i) \ge 3$, $Re(a_i) \le Re(\alpha_i)$ and $Im(a_i) = Im(\alpha_i)$. Then

$$\mathcal{T}_{a_i,b_j}^{\lambda}(\eta, p, h) \subset \mathcal{T}_{\alpha_i,b_j}^{\lambda}(\eta, p, h). \tag{2.4}$$

Proof. Let $f \in \mathcal{T}_{a_i,b_j}^{\lambda}(\eta, p, h)$. Then from the definition of the class $\mathcal{T}_{a_i,b_j}^{\lambda}(\eta, p, h)$ we have

$$\frac{1}{p-\eta}\left[\frac{-z({}_qI_s^{\lambda,p}(a_i;b_j)f(z))'}{{}_qI_s^{\lambda,p}(a_i;b_j)f(z)}-\eta\right]=h(w(z)), \tag{2.5}$$

where h is convex univalent in $\mathcal{U}$ with $Re(h(z))>0$ and $|w(z)|<1$ in $\mathcal{U}$ with $w(0)=0=h(0)-1$. Therefore

$$-\left(\frac{z({}_qI_s^{\lambda,p}(a_i;b_j)f(z))'}{{}_qI_s^{\lambda,p}(a_i;b_j)f(z)}\right)=(p-\eta)h(w(z))+\eta, \tag{2.6}$$

and

$$\frac{z[z^{1+p}({}_qI_s^{\lambda,p}(a_i;b_j)f(z))]'}{z^{1+p}({}_qI_s^{\lambda,p}(a_i;b_j)f(z))}=-(p-\eta)h(w(z))+p-\eta+1\prec\frac{1+z}{1-z}. \tag{2.7}$$

Now

$$\begin{aligned}\left[\frac{z({}_qI_s^{\lambda,p}(\alpha_i;b_j)f(z))'}{{}_qI_s^{\lambda,p}(\alpha_i;b_j)f(z)}\right] &= \left[\frac{z({}_q\mathcal{F}_s^{(-1)}(\alpha_i;b_j)*f(z))'}{{}_q\mathcal{F}_s^{(-1)}(\alpha_i;b_j)*f(z)}\right]\\ &= \left[\frac{z\left\{\sum_{k=0}^{\infty}\frac{(a_i)_k}{(\alpha_i)_k}z^{k-p}*\ {}_q\mathcal{F}_s^{(-1)}(a_i;b_j)*f(z)\right\}'}{\sum_{k=0}^{\infty}\frac{(a_i)_k}{(\alpha_i)_k}z^{k-p}*\ {}_q\mathcal{F}_s^{(-1)}(a_i;b_j)*f(z)}\right]\\ &= \left[\frac{\sum_{k=0}^{\infty}\frac{(a_i)_k}{(\alpha_i)_k}z^{k-p}*z\ \left\{{}_q\mathcal{F}_s^{(-1)}(a_i;b_j)*f(z)\right\}'}{\sum_{k=0}^{\infty}\frac{(a_i)_k}{(\alpha_i)_k}z^{k-p}*\ \left\{{}_q\mathcal{F}_s^{(-1)}(a_i;b_j)*f(z)\right\}}\right]\\ &= \left[\frac{\sum_{k=0}^{\infty}\frac{(a_i)_k}{(\alpha_i)_k}z^{k-p}*z({}_q\mathcal{I}_s^{\lambda,p}(a_i;b_j)f(z))'}{\sum_{k=0}^{\infty}\frac{(a_i)_k}{(\alpha_i)_k}z^{k-p}*\ {}_q\mathcal{I}_s^{\lambda,p}(a_i;b_j)f(z)}\right].\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{1}{p-\eta}\left[\frac{-z({}_qI_s^{\lambda,p}(\alpha_i;b_j)f(z))'}{{}_qI_s^{\lambda,p}(\alpha_i;b_j)f(z)}-\eta\right]\\ &\quad=\frac{1}{p-\eta}\left[\frac{-\sum_{k=0}^{\infty}\frac{(a_i)_k}{(\alpha_i)_k}z^{k-p}*z({}_q\mathcal{I}_s^{\lambda,p}(a_i;b_j)f(z))'}{\sum_{k=0}^{\infty}\frac{(a_i)_k}{(\alpha_i)_k}z^{k-p}*\ {}_q\mathcal{I}_s^{\lambda,p}(a_i;b_j)f(z)}-\eta\right]\\ &\quad=\frac{1}{p-\eta}\left[\frac{\sum_{k=0}^{\infty}\frac{(a_i)_k}{(\alpha_i)_k}z^{k-p}*[(p-\eta)h(w(z))+\eta]_q\mathcal{I}_s^{\lambda,p}(a_i;b_j)f(z)}{\sum_{k=0}^{\infty}\frac{(a_i)_k}{(\alpha_i)_k}z^{k-p}*\ {}_q\mathcal{I}_s^{\lambda,p}(a_i;b_j)f(z)}-\eta\right]. \qquad (2.8)\end{aligned}$$

It follows from Lemma 2 that

$$z^{p+1}\sum_{k=0}^{\infty}\frac{(a_i)_k}{(\alpha_i)_k}z^{k-p}\in\mathcal{K}. \tag{2.9}$$

Hence from (2.7) we get

$$z^{p+1}\ {}_q\mathcal{I}_s^{\lambda,p}(a_i;b_j)f(z)\in\mathcal{S}^*.$$

Let $s(w(z))=(p-\eta)h(w(z))+\eta$. Then applying Lemma 1 we get

$$\frac{\left\{[z^{p+1}\sum_{k=0}^{\infty}\frac{(a_i)_k}{(\alpha_i)_k}z^{k-p}]*s(w)z^{p+1}\ {}_q\mathcal{I}_s^{\lambda,p}(a_i;b_j)f\right\}(\mathcal{U})}{\left\{[z^{p+1}\sum_{k=0}^{\infty}\frac{(a_i)_k}{(\alpha_i)_k}z^{k-p}]*z^{p+1}\ {}_q\mathcal{I}_s^{\lambda,p}(a_i;b_j)f\right\}(\mathcal{U})}\subseteq\overline{co}sw(\mathcal{U}),$$

because s is convex univalent function. Therefore we conclude that

$$\frac{1}{p-\eta}\left[\frac{\left\{\sum_{k=0}^{\infty}\frac{(a_i)_k}{(\alpha_i)_k}z^{k-p}*s(w)\ {}_q\mathcal{I}_s^{\lambda,p}(a_i;b_j)f\right\}(\mathcal{U})}{\left\{\sum_{k=0}^{\infty}\frac{(a_i)_k}{(\alpha_i)_k}z^{k-p}*\ {}_q\mathcal{I}_s^{\lambda,p}(a_i;b_j)f\right\}(\mathcal{U})}-\eta\right]\subseteq h(\mathcal{U}),$$

and hence that (2.8) is subordinate to the convex univalent function h, and finally that $f\in\mathcal{T}^{\lambda}_{\alpha_i,b_j}(\eta,p,h)$. The proof of Theroem 1 is completed. □

Theorem 2. *Let* $0<b_j\leq\beta_j,\forall a_i,\ \alpha_i,\ b_j,\ \beta_j\in C\setminus Z_0^-,(\ i=1,\cdots,q,\ j=1,\cdots,s)$,
$h\in\mathcal{N}$ *and* h *satisfies* (2.3). *If* $b_j,\ \beta_j$ *are real such that* $\beta_j\geq 2$ *or if* b_j,β_j *are complex such that* $Re(b_j+\beta_j)\geq 3,\ Re(b_j)\leq Re(\beta_j)$ *and* $Im(b_j)=Im(\beta_j)$. *Then*

$$\mathcal{T}^{\lambda}_{a_i,\beta_j}(\eta,p,h)\subset\mathcal{T}^{\lambda}_{a_i,b_j}(\eta,p,h).$$

Proof. Let $f\in\mathcal{T}^{\lambda}_{a_i,\beta_j}(\eta,p,h)$. In the same way as we have obtained (2.7) we get

$$\frac{z[z^{1+p}({}_qI_s^{\lambda,p}(a_i;\beta_j)f(z))]'}{z^{1+p}({}_qI_s^{\lambda,p}(a_i;\beta_j)f(z))}=-(p-\eta)h(w(z))+p-\eta+1\prec\frac{1+z}{1-z}. \tag{2.10}$$

Again we get

$$\begin{aligned}\left[\frac{z({}_qI_s^{\lambda,p}(a_i;b_j)f(z))'}{{}_qI_s^{\lambda,p}(a_i;b_j)f(z)}\right] &= \left[\frac{z({}_q\mathcal{F}_s^{(-1)}(a_i;b_j)*f(z))'}{{}_q\mathcal{F}_s^{(-1)}(a_i;b_j)*f(z)}\right]\\ &= \left[\frac{z\left\{(\sum_{k=0}^{\infty}\frac{(b_j)_k}{(\beta_j)_k}z^{k-p}*\ {}_q\mathcal{F}_s^{(-1)}(a_i;\beta_j))*f(z)\right\}'}{\sum_{k=0}^{\infty}\frac{(b_j)_k}{(\beta_j)_k}z^{k-p}*\ {}_q\mathcal{F}_s^{(-1)}(a_i;\beta_j)*f(z)}\right]\\ &= \left[\frac{\sum_{k=0}^{\infty}\frac{(b_j)_k}{(\beta_j)_k}z^{k-p}*z\left\{{}_q\mathcal{F}_s^{(-1)}(a_i;\beta_j)*f(z)\right\}'}{\sum_{k=0}^{\infty}\frac{(b_j)_k}{(\beta_j)_k}z^{k-p}*\ \left\{{}_q\mathcal{F}_s^{(-1)}(a_i;\beta_j))*f(z)\right\}}\right]\\ &= \left[\frac{\sum_{k=0}^{\infty}\frac{(b_j)_k}{(\beta_j)_k}z^{k-p}*z({}_q\mathcal{I}_s^{\lambda,p}(a_i;\beta_j)f(z))'}{\sum_{k=0}^{\infty}\frac{(b_j)_k}{(\beta_j)_k}z^{k-p}*\ ({}_q\mathcal{I}_s^{\lambda,p}(a_i;\beta_j)f(z))}\right].\end{aligned}$$

Therefore

$$\frac{1}{p-\eta}\left[\frac{-z({}_qI_s^{\lambda,p}(a_i;b_j)f(z))'}{{}_qI_s^{\lambda,p}(a_i;b_j)f(z)}-\eta\right]$$

$$=\frac{1}{p-\eta}\left[\frac{-\sum_{k=0}^{\infty}\frac{(b_j)_k}{(\beta_j)_k}z^{k-p}*z({}_q\mathcal{I}_s^{\lambda,p}(a_i;\beta_j)f(z))'}{\sum_{k=0}^{\infty}\frac{(b_j)_k}{(\beta_j)_k}z^{k-p}*{}_q\mathcal{I}_s^{\lambda,p}(a_i;\beta_j)f(z)}-\eta\right]$$

$$=\frac{1}{p-\eta}\left[\frac{\sum_{k=0}^{\infty}\frac{(b_j)_k}{(\beta_j)_k}z^{k-p}*[(p-\eta)h(w(z))+\eta]\,{}_q\mathcal{I}_s^{\lambda,p}(a_i;\beta_j)f(z)}{\sum_{k=0}^{\infty}\frac{(b_j)_k}{(\beta_j)_k}z^{k-p}*{}_q\mathcal{I}_s^{\lambda,p}(a_i;\beta_j)f(z)}-\eta\right]. \tag{2.11}$$

It follows from Lemma 2 that

$$z^{p+1}\sum_{k=0}^{\infty}\frac{(b_j)_k}{(\beta_j)_k}z^{k-p}\in\mathcal{K}.$$

Hence from (2.10) we get

$$z^{p+1}\,{}_q\mathcal{I}_s^{\lambda,p}(a_i;\beta_j)f(z)\in\mathcal{S}^*.$$

Thus, by virtue of Lemma 1, we conclude that (2.11) is subordinate to h and consequently $f\in\mathcal{T}^{\lambda}_{a_i,b_j}(\eta,p,h)$. We thus complete the proof of theorem 2. □

By taking $h(z)=\dfrac{1+Az}{1+Bz}$ where $-1\le B<A\le 1$ in Theorems 1 and 2 we have the following interesting corollary

Corollary 3. *Let* $0<a_i\le\alpha_i$ *and* $0<b_j\le\beta_j$, $\forall a_i,\ \alpha_i,\ b_j\ \beta_j\in\ C\setminus Z_0^-$, $(i=1,\cdots,q,\ j=1,\cdots,s)$, $h(z)=\dfrac{1+Az}{1+Bz}$ *and* $\dfrac{1+A}{1+B}<1+\dfrac{1}{p-\eta}$ *where* $-1\le B<A\le 1$. *Further suppose that either* a_i,α_i,b_j,β_j *are real such that* $\alpha_i\ge 2$*and* $\beta_j\ge 2$ *or* a_i,α_i,b_j,β_j *are complex such that* $Re(a_i+\alpha_i)\ge 3$, $Re(b_j+\beta_j)\ge 3$, $Re(a_i)\le Re(\alpha_i)$, $Im(a_i)=Im(\alpha_i)$, $Re(b_j)\le Re(\beta_j)$ *and* $Im(b_j)=Im(\beta_j)$. *Then*

$$\mathcal{T}^{\lambda}_{a_i,\beta_j}(\eta,p,\frac{1+Az}{1+Bz})\subset\mathcal{T}^{\lambda}_{a_i,b_j}(\eta,p,\frac{1+Az}{1+Bz})\subset\mathcal{T}^{\lambda}_{\alpha_i,b_j}(\eta,p,\frac{1+Az}{1+Bz}).$$

3. Coefficients Estimates

Theorem 3. *Let* $f\in\mathcal{T}^{\lambda}_{a_i,b_j}(\eta,p,h)$ *and the function* h *satisfies* (2.3). *Then*

$$|a_k|\le\frac{(k+1)\Pi_{i=1}^{q}(a_i)_k}{(\lambda+p)_k\Pi_{j=1}^{s}(b_j)_k},\ k=0,1,\cdots. \tag{3.1}$$

Proof. Since $f(z)\in\mathcal{T}^{\lambda}_{a_i,b_j}(\eta,p,h)$ and $\max_{z\in\mathcal{U}}(Re[h(z)])<1+\dfrac{1}{p-\eta}$, therefore by (2.7) we have

$$z^{p+1}\,{}_q\mathcal{I}_s^{\lambda,p}(a_i;b_j)f(z)\in\mathcal{S}^*.$$

Hence

$$\sum_{k=o}^{\infty} \frac{(\lambda+p)_k \Pi_{j=1}^{s}(b_j)_k}{\Pi_{i=1}^{q}(a_i)_k} a_k z^{k+1} \in \mathcal{S}^*.$$

Now by using the estimation of $(k+1)^{th}$ coefficient of starlike function we obtain

$$\left| \frac{(\lambda+p)_k \Pi_{j=1}^{s}(b_j)_k}{\Pi_{i=1}^{q}(a_i)_k} a_k \right| \leq k+1, \quad k = 0, 1, \cdots,$$

which completes the proof. □

Remark. If $h(z) = \frac{1}{p-\eta}(p - \eta - \frac{2z}{1-z})$, then the above estimates of coefficients become sharp. The extremal function is

$$f_0(z) = \sum_{k=0}^{\infty} \frac{(k+1)\Pi_{i=1}^{q}(a_i)_k}{(\lambda+p)_k \Pi_{j=1}^{s}(b_j)_k} z^{k-p}.$$

Then we have

$$\begin{aligned} \frac{1}{p-\eta}\left[\frac{z({}_qI_s^{\lambda,p}(a_i;b_j)f_0(z))'}{{}_qI_s^{\lambda,p}(a_i;b_j)f_0(z)} + \eta\right] &= \frac{1}{p-\eta}\left[\frac{z[\sum_{k=0}^{\infty}(k+1)z^{k-p}]'}{\sum_{k=0}^{\infty}(k+1)z^{k-p}} + \eta\right] \\ &= \frac{1}{p-\eta}\left[\frac{2z}{1-z} - (p-\eta)\right]. \end{aligned}$$

4. Structural formula

Theorem 4. *A function f belongs to the class $\mathcal{T}_{a_i,b_j}^{\lambda}(\eta, p, h)$ if and only if there exists a Schwarz function $w(z)$ such that*

$$f(z) = \left[\sum_{k=0}^{\infty} \frac{\Pi_{i=1}^{q}(a_i)_k}{(\lambda+p)_k \Pi_{j=1}^{s}(b_j)_k} z^{k-p}\right] * \left[\frac{1}{z^p} exp \int_0^z \frac{(p-\eta)\{1-h(w(t))\}}{t} dt\right]. \tag{4.1}$$

Proof. Let $f \in \mathcal{T}_{a_i,b_j}^{\lambda}(\eta, p, h)$. Then from the definition of the class $\mathcal{T}_{a_i,b_j}^{\lambda}(\eta, p, h)$ we have

$$\frac{1}{p-\eta}\left[\frac{-z({}_qI_s^{\lambda,p}(a_i;b_j)f(z))'}{{}_qI_s^{\lambda,p}(a_i;b_j)f(z)} - \eta\right] = h(w(z)),$$

where $h \in \mathcal{N}$ and $|w(z)| < 1$ in $\mathcal{U}$ with $w(0) = 0 = h(0) - 1$.
Therefore

$$\frac{({}_qI_s^{\lambda,p}(a_i;b_j)f(z))'}{{}_qI_s^{\lambda,p}(a_i;b_j)f(z)} + \frac{p}{z} = \frac{(p-\eta)\{1-h(w(z))\}}{z}.$$

Thus

$$log\left[z^p.\ {}_qI_s^{\lambda,p}(a_i;b_j)f(z)\right]=\int_0^z\frac{(p-\eta)\left\{1-h(w(t))\right\}}{t}dt.$$

$${}_qI_s^{\lambda,p}(a_i;b_j)f(z)=\frac{1}{z^p}\ exp\int_0^z\frac{(p-\eta)\left\{1-h(w(t))\right\}}{t}dt.$$

Therefore, we obtain

$$f(z)*\sum_{k=o}^{\infty}\frac{(\lambda+p)_k\Pi_{j=1}^s(b_j)_k}{\Pi_{i=1}^q(a_i)_k}z^{k-p}=\frac{1}{z^p}\ exp\int_0^z\frac{(p-\eta)\left\{1-h(w(t))\right\}}{t}dt.$$

Which gives the result (4.1) easily. □

Remark. If we apply the Theorem 4 to the function

$$h(z)=\frac{1-[1+2(p-\eta)^{-1}]z}{1-z},w(z)=z,$$

then from the structural formula contained in Theorem 4 we obtain the function f_0.

5. Convolutions with convex functions

Theorem 5. *Let* $\lambda\geq 0,\phi\in\mathcal{K},h\in\mathcal{N}$ *and* h *satisfies* (2.3). *Then*

$$f\in\mathcal{T}_{a_i,b_j}^{\lambda}(\eta,p,h)\Rightarrow[z^{-(p+1)}\phi]*f\in\mathcal{T}_{a_i,b_j}^{\lambda}(\eta,p,h).$$

Proof. Let $f\in\mathcal{T}_{a_i,b_j}^{\lambda}(\eta,p,h)$ and $\phi\in\mathcal{K}$. By applying the properties of convolution we obtain

$$-\left[\frac{z({}_qI_s^{\lambda,p}(a_i;b_j)((z^{-(p+1)}\phi)*f)(z))'}{{}_qI_s^{\lambda,p}(a_i;b_j)((z^{-(p+1)}\phi)*f)(z)}\right]$$

$$=-\left[\frac{z({}_q\mathcal{F}_s^{(-1)}(a_i;b_j)*(z^{-(p+1)}\phi)*f(z))'}{{}_q\mathcal{F}_s^{(-1)}(a_i;b_j)*(z^{-(p+1)}\phi)*f(z)}\right]$$

$$=-\left[\frac{(z^{-(p+1)}\phi(z))*z({}_q\mathcal{F}_s^{(-1)}(a_i;b_j)*f(z))'}{(z^{-(p+1)}\phi(z))*\ {}_q\mathcal{F}_s^{(-1)}(a_i;b_j)*f(z)}\right]$$

$$=-\left[\frac{(z^{-(p+1)}\phi(z))*z({}_q\mathcal{I}_s^{\lambda,p}(a_i;b_j)f(z))'}{(z^{-(p+1)}\phi(z))*\ {}_q\mathcal{I}_s^{\lambda,p}(a_i;b_j)f(z)}\right]$$

$$=\left[\frac{(z^{-(p+1)}\phi(z))*\left[(p-\eta)h(w(z))+\eta\right]\ {}_q\mathcal{I}_s^{\lambda,p}(a_i;b_j)f(z)}{(z^{-(p+1)}\phi(z))*\ {}_q\mathcal{I}_s^{\lambda,p}(a_i;b_j)f(z)}\right].\tag{5.1}$$

Let us put

$$F(z) = \frac{1}{p-\eta}\left[\frac{-z({}_qI_s^{\lambda,p}(a_i;b_j)((z^{-(p+1)}\phi)*f)(z))'}{{}_qI_s^{\lambda,p}(a_i;b_j)((z^{-(p+1)}\phi)*f)(z)} - \eta\right].$$

Then, by using (5.1), we obtain

$$F(z) = \frac{1}{p-\eta}\left[\frac{\phi(z)*[(p-\eta)h(w(z))+\eta]z^{p+1}\,{}_q\mathcal{I}_s^{\lambda,p}(a_i;b_j)f(z)}{\phi(z)*z^{p+1}\,{}_q\mathcal{I}_s^{\lambda,p}(a_i;b_j)f(z)} - \eta\right].$$

From (2.7) it follows that $z^{p+1}\,{}_q\mathcal{I}_s^{\lambda,p}(a_i;b_j)f(z) \in \mathcal{S}^*$. Hence by applying the arguments similar to those used in the proof of Theorem 1, we conclude that $F \prec h$ and $z^{-(p+1)}\phi * f \in \mathcal{T}_{a_i,b_j}^{\lambda}(\eta,p,h)$.
This completes the proof. □

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