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# On Quasi-Baer and p.q.-Baer Modules

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Abstract. For an endomorphism $\alpha$ of $R$, in [1], a module $M_R$ is called $\alpha$-compatible if, for any $m \in M$ and $a \in R$, $ma = 0$ iff $m\alpha(a) = 0$, which are a generalization of $\alpha$-reduced modules. We study on the relationship between the quasi-Baerness and *p.q.*-Baer property of a module $M_R$ and those of the polynomial extensions (including formal skew power series, skew Laurent polynomials and skew Laurent series). As a consequence we obtain a generalization of [2] and some results in [9]. In particular, we show: for an $\alpha$-compatible module $M_R$ (1) $M_R$ is *p.q.*-Baer module iff $M[x;\alpha]_{R[x;\alpha]}$ is *p.q.*-Baer module. (2) for an automorphism $\alpha$ of $R$, $M_R$ is *p.q.*-Baer module iff $M[x, x^{-1};\alpha]_{R[x,x^{-1};\alpha]}$ is *p.q.*-Baer module.

## 1. Introduction

Throughout this work all rings $R$ are associative with identity and modules are unital right $R$-modules and $\alpha : R \to R$ is an endomorphism of the ring $R$. In [7] Clark called a ring $R$ *quasi-Baer ring* if the right annihilator of each right ideal of $R$ is generated (as a right ideal) by an idempotent. Recently, Birkenmeier et al. [5] called a ring $R$ *right* (resp. *left*) *principally quasi-Baer* [or simply *right* (resp. *left*) *p.q.-Baer*] if the right (resp. left) annihilator of a principal right (resp. left) ideal of $R$ is generated by an idempotent. $R$ is called *p.q.-Baer* if it is both right and left *p.q.*-Baer. A ring is called *reduced ring* if it has no nonzero nilpotent elements and $M_R$ is called *$\alpha$-reduced module* by Lee-Zhou [9] if, for any $m \in M$ and $a \in R$,

(1) $ma = 0$ implies $mR \cap Ma = 0$,

(2) $ma = 0$ iff $m\alpha(a) = 0$,

where $\alpha : R \to R$ is a ring endomorphism with $\alpha(1) = 1$. The module $M_R$ is called *a reduced module* if $M$ is $1_R$-reduced, where $1_R$ is the identity endomorphism of $R$. It is clear that $R$ is a reduced ring iff $R_R$ is a reduced module.

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According to Annin [1], a module $M_R$ is called *$\alpha$-compatible* if $ma = 0$ if and only if $m\alpha(a) = 0$ (i.e., only the second condition is satisfied in the definition of $\alpha$-reduced modules). It is clear that, if $M_R$ is $\alpha$-compatible then, $ma = 0$ if and only if $m\alpha^k(a) = 0$ for all $k$ and every $\alpha$-reduced modules are $\alpha$-compatible. We write $R[x], R[[x]], R[x, x^{-1}]$ and $R[[x, x^{-1}]]$ for the polynomial ring, the power series ring, the Laurent polynomial ring and the Laurent power series ring over $R$, respectively. In [9] Lee-Zhou introduced the following notation. For a module $M_R$, we consider
$M[x;\alpha] = \{\sum_{i=0}^{s} m_i x^i \ : \ s \geq 0,\ m_i \in M\}$,
$M[[x;\alpha]] = \{\sum_{i=0}^{\infty} m_i x^i \ : m_i \in M\}$,
$M[x, x^{-1};\alpha] = \{\sum_{i=-s}^{t} m_i x^i \ : \ s \geq 0,\ t \geq 0,\ m_i \in M\}$,
$M[[x, x^{-1};\alpha]] = \{\sum_{i=-s}^{\infty} m_i x^i \ : \ s \geq 0,\ m_i \in M\}$.
Each of these is an Abelian group under an obvious addition operation. Moreover $M[x;\alpha]$ becomes a module over $R[x;\alpha]$ under the following scalar product operation: For $m(x) = \sum_{i=0}^{s} m_i x^i \ \in M[x;\alpha]$ and $f(x) = \sum_{i=0}^{t} a_i x^i \ \in R[x;\alpha]$;

$$m(x)f(x) = \sum_{k=0}^{s+t} \left( \sum_{i+j=k} m_i \alpha^i(a_j) \right) x^k.$$

Similarly, $M[[x;\alpha]]$ is a module over $R[[x;\alpha]]$. The modules $M[x;\alpha]$ and $M[[x;\alpha]]$ are called the *skew polynomial extension* and the *skew power series extension of* $M$ respectively. If $\alpha \in Aut(R)$, then with a similar scalar product, $M[[x, x^{-1};\alpha]]$ (resp. $M[x, x^{-1};\alpha]$) becomes a module over $R[[x, x^{-1};\alpha]]$ (resp. $R[x, x^{-1};\alpha]$). The modules $M[x, x^{-1};\alpha]$ and $M[[x, x^{-1};\alpha]]$ are called the *skew Laurent polynomial extension* and the *skew Laurent power series extension of* $M$, respectively.

Following Lee-Zhou [9], a module $M_R$ is called *Armendariz* if, whenever $m(x)f(x) = 0$ where $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{t} a_j x^j \in R[x]$, we have $m_i a_j = 0$ for all $i, j$. By [9, Lemma 1.5], every reduced module is Armendariz. In [3], we define a module $M_R$ to be *quasi-Armendariz* if whenever these polynomials satisfy $m(x)R[x]f(x) = 0$, we have $m_i R a_j = 0$ for all $i, j$.

For a subset $X$ of a module $M_R$, let $r_R(X) = \{r \in R : Xr = 0\}$. In [9] Lee-Zhou introduced *quasi-Baer module* as follows: $M_R$ is called *quasi-Baer* if, for any submodule $N$ of $M$, $r_R(N) = eR$ where $e^2 = e \in R$. Clearly $R$ is a quasi-Baer ring iff $R_R$ is quasi-Baer module; if $R$ is a quasi-Baer ring then, for any right ideal $I$ of $R$, $I_R$ is a quasi-Baer module. Following [3], $M_R$ is called *principally quasi-Baer* (or simply *p.q.-Baer*) module if, for any $m \in M$, $r_R(mR) = eR$ where $e^2 = e \in R$. It is clear that $R$ is a right *p.q.*-Baer ring iff $R_R$ is a *p.q.*-Baer module. If $R$ is a *p.q.*-Baer ring, then for any right ideal $I$ of $R$, $I_R$ is a *p.q.*-Baer module. Every submodule of a *p.q.*-Baer module is *p.q.*-Baer module. Moreover, every quasi-Baer module is *p.q.*-Baer.

Motivated by results in Lee-Zhou [9], [3] and [2] we investigate a generalization of $\alpha$-reduced modules and introduce skew quasi-Armendariz types module which are skew polynomials versions of the quasi-Armendariz modules.

## 2. Skew polynomial and power series modules over quasi-Baer and p.q.-Baer modules

In this section we investigate a generalization of $\alpha$-reduced modules and introduce skew quasi-Armendariz and skew quasi-Armendariz of power series type modules, which are skew polynomial versions of the quasi-Armendariz modules. We then extend our previous results in [2] to non $\alpha$-reduced $\alpha$-compatible modules. Assume that $M_R$ is an $\alpha$-compatible module. Then we will show that:

(1) $M_R$ is p.q.-Baer module if and only if $M[x;\alpha]_{R[x;\alpha]}$ is p.q.-Baer module.

(2) $M_R$ is quasi-Baer module if and only if $M[x;\alpha]_{R[x;\alpha]}$ is quasi-Baer module if and only if $M[[x;\alpha]]_{R[[x;\alpha]]}$ is quasi-Baer module.

(3) If $M[[x;\alpha]]_{R[[x;\alpha]]}$ is p.q-Baer module then $M_R$ is p.q.-Baer module.

**Definition 2.1.** A module $M_R$ is called,
(i) *skew quasi-Armendariz*, if whenever $m(x)R[x;\alpha]f(x) = 0$ for $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^{t} a_j x^j \in R[x;\alpha]$, then $m_i R a_j = 0$ for all $i, j$.
(ii) *skew quasi-Armendariz of power series type*, if whenever $m(x)R[[x;\alpha]]f(x) = 0$ for $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x;\alpha]]$ and $f(x) = \sum_{j=0}^{\infty} a_j x^j \in R[[x;\alpha]]$, then $m_i R a_j = 0$ for all $i, j$.

Note that if $M_R$ is assumed to be $\alpha$-reduced, then it is clear that $M_R$ is skew quasi-Armendariz and skew quasi-Armendariz of power series type. To see that, let $m(x)R[[x;\alpha]]f(x) = 0$ for $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x;\alpha]]$ and $f(x) = \sum_{j=0}^{\infty} a_j x^j \in R[[x;\alpha]]$. Then $m(x)Rf(x) = 0$ and so $m(x)cf(x) = 0$ for all $c \in R$. Hence $0 = (\sum_{i=0}^{\infty} m_i x^i)c(\sum_{j=0}^{\infty} a_j x^j) = (\sum_{i=0}^{\infty} m_i x^i)(\sum_{j=0}^{\infty} c a_j x^j)$. Since $M_R$ is $\alpha$-reduced, $M_R$ satisfies all the hypothesis of [9, Lemma 1.5] by [9, Lemma 1.2]. Hence, we have $m_i \alpha^i(c a_j) = 0$ and so $m_i c a_j = 0$ for all $i, j$, since $M_R$ is $\alpha$-compatible. Then $m_i R a_j = 0$ for all $i, j$ and therefore, $M_R$ is skew quasi-Armendariz of power series type.

Following [8], for a module $M_R$, $\mathrm{rAnn}_R(\mathrm{sub}(M_R))=\{r_R(U) \mid U$ is a submodule of $M_R\}$.

**Proposition 2.2.** *Let $M_R$ be an $\alpha$-compatible module. Then the following statements are equivalent:*
(1) *$M_R$ is skew quasi-Armendariz.*
(2) $\psi : \mathrm{rAnn}_R(\mathrm{sub}(M_R)) \to \mathrm{rAnn}_{R[x;\alpha]}(\mathrm{sub}(M[x;\alpha]_{R[x;\alpha]}))$;
$I \mapsto I[x;\alpha]$ *is bijective.*

*Proof.* (1)⇒(2) Let $I \in \mathrm{rAnn}_R(\mathrm{sub}(M_R))$. Then there exists a submodule $U$ of $M_R$ such that $I = r_R(U)$. Then we have $r_R(U)[x;\alpha] = r_{R[x;\alpha]}(U[x;\alpha])$ since $M_R$ is $\alpha$-compatible. So $\psi$ is well-defined. Obviously $\psi$ is injective. Now, for a submodule $V$ of $M[x;\alpha]_{R[x;\alpha]}$, let $r_{R[x;\alpha]}(V) \in \mathrm{rAnn}_{R[x;\alpha]}(\mathrm{sub}(M[x;\alpha]_{R[x;\alpha]}))$. Let $C_V$ denote the set of coefficients of elements of $V$. Then $C_V R$ is a submodule of $M_R$. We claim that $\psi(r_R(C_V R)) = r_R(C_V R)[x;\alpha] = r_{R[x;\alpha]}(V)$. Let $f(x) = a_0 + a_1 x + \cdots + a_t x^t \in r_R(C_V R)[x;\alpha]$. Then $a_i \in r_R(C_V R)$ and hence $(C_V R)a_i = 0$ and in particular $C_V a_i = 0$ for all $i$. Since $M_R$ is $\alpha$-compatible

$C_V\alpha^k(a_i) = 0$ for all $k$. Then $v(x)f(x) = 0$ for all $v(x) \in V$. Thus $Vf(x) = 0$ and hence $f(x) \in r_{R[x;\alpha]}(V)$. Therefore $r_R(C_VR)[x;\alpha] \subseteq r_{R[x;\alpha]}(V)$. Conversely, let $g(x) = b_0 + b_1x + \cdots + b_nx^n \in r_{R[x;\alpha]}(V)$. Then $Vg(x) = 0$ and since $V$ is a submodule of $M[x;\alpha]_{R[x;\alpha]}$, $v(x)R[x;\alpha]g(x) = 0$ for all $v(x) \in V$. Since $M_R$ is skew quasi-Armendariz, $vRb_j = 0$ for all $v \in C_V$ and $j = 0, 1, \cdots, n$. Hence $(C_VR)b_j = 0$ for all $j$. Therefore $g(x) \in r_R(C_VR)[x;\alpha]$. Thus $r_{R[x;\alpha]}(V) \subseteq r_R(C_VR)[x;\alpha]$. Consequently, $\psi$ is surjective.
(2)⇒(1) Suppose $m(x)R[x;\alpha]f(x) = 0$ for $m(x) = \sum_{i=0}^s m_ix^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^t a_jx^j \in R[x;\alpha]$. Then $f(x) \in r_{R[x;\alpha]}(m(x)R[x;\alpha]) = r_R(CR)[x;\alpha]$, where $C$ is denote the set of coefficients of elements of $m(x)R[x;\alpha]$. Then $a_j \in r_R(CR)$ and so $(CR)a_j = 0$. In particular $m_iRa_j = 0$ for all $i, j$. Therefore $M_R$ is skew quasi-Armendariz. □

**Proposition 2.3.** *Let $M_R$ be an $\alpha$-compatible module. Then the following statements are equivalent:*
(1) *$M_R$ is skew quasi-Armendariz of power series type.*
(2) $\psi^{'}$ : $\mathrm{rAnn}_R(\mathrm{sub}(M_R)) \to \mathrm{rAnn}_{R[[x;\alpha]]}(\mathrm{sub}(M[[x;\alpha]]_{R[[x;\alpha]]}))$;
$J \mapsto J[[x;\alpha]]$ *is bijective.*

*Proof.* Similar to the proof of Proposition 2.2. □

**Definition 2.4.** A submodule $N$ of a left $R$-module $M$ is called a *pure submodule* if $L \otimes_R N \to L \otimes_R M$ is a monomorphism for every right $R$-module $L$.

Following Tominaga [11], an ideal $I$ of $R$ is said to be *left s-unital* if for each $a \in I$ there exists an $x \in I$ such that $xa = a$. If an ideal $I$ of $R$ is left s-unital, then for any finite subset $F$ of $I$, there exists an element $e \in I$ such that $ex = e$ for all $x \in F$. By [10, Proposition 11.3.13], for an ideal $I$, the following conditions are equivalent:
(1) $I$ is pure as a right ideal in $R$,
(2) $R/I$ is flat as a right $R$-module,
(3) $I$ is left s-unital.

**Theorem 2.5.** *Let $M_R$ be an $\alpha$-compatible module. Then the following are equivalent:*
(1) *$r_R(mR)$ is pure as a right ideal in $R$ for any element $m \in M_R$.*
(2) *$r_{R[x;\alpha]}(m(x)R[x;\alpha])$ is a pure as a right ideal in $R[x;\alpha]$ for any element $m(x) \in M[x;\alpha]$. In this case $M_R$ is skew quasi-Armendariz.*

*Proof.* (1)⇒(2) Assume that condition (1) holds. First we shall prove that $M_R$ is skew quasi-Armendariz. Suppose $m(x)R[x;\alpha]f(x) = 0$ for $m(x) = \sum_{i=0}^s m_ix^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^t a_jx^j \in R[x;\alpha]$. Then $(\sum_{i=0}^s m_ix^i)R(\sum_{j=0}^t a_jx^j) = 0$. Let $c$ be an arbitrary element of $R$. Then we have the following equation:

$$(1)\quad 0 = m_0ca_0 + \cdots + (m_s\alpha^s(ca_{t-2}) + m_{s-1}\alpha^{s-1}(ca_{t-1}) + m_{s-2}\alpha^{s-2}(ca_t))x^{s+t-2} + (m_s\alpha^s(ca_{t-1}) + m_{s-1}\alpha^{s-1}(ca_t))x^{s+t-1} + m_s\alpha^s(ca_t)x^{s+t}.$$

Then $m_s\alpha^s(ca_t) = 0$ and hence $m_sca_t = 0$ since $M_R$ is $\alpha$-compatible. Thus $m_sRa_t = 0$ and so $a_t \in r_R(m_sR)$. By hypothesis, $r_R(m_sR)$ is left s-unital, and hence there exists $e_s \in r_R(m_sR)$ such that $e_sa_t = a_t$. Replacing $c$ by $ce_s$ in Eq.(1), we obtain

$$(2)\quad 0 = m_0ce_sa_0 + \cdots + (m_s\alpha^s(ce_sa_{t-2}) + m_{s-1}\alpha^{s-1}(ce_sa_{t-1}) + m_{s-2}\alpha^{s-2}(ce_sa_t)) x^{s+t-2} + (m_s\alpha^s(ce_sa_{t-1}) + m_{s-1}\alpha^{s-1}(ce_sa_t))x^{s+t-1} + m_s\alpha^s(ce_sa_t)x^{s+t}.$$

Since $e_s \in r_R(m_sR)$, $m_sRe_s = 0$ and $m_s\alpha^k(Re_s) = 0$ for all $k$ since $M_R$ is $\alpha$-compatible. Using $e_sa_t = a_t$ and $m_s\alpha^k(Re_s) = 0$, we obtain from Eq.(2)

$$0 = m_0ce_sa_0 + \cdots + (m_{s-1}\alpha^{s-1}(ce_sa_{t-1}) + m_{s-2}\alpha^{s-2}(ca_t))x^{s+t-2} + m_{s-1}\alpha^{s-1}(ca_t)x^{s+t-1}.$$

Then we obtain $m_{s-1}\alpha^{s-1}(ca_t) = 0$ and hence $m_{s-1}ca_t = 0$ and so $m_{s-1}Ra_t = 0$ since $M_R$ is $\alpha$-compatible. Thus $a_t \in r_R(m_{s-1}R)$ and hence $a_t \in r_R(m_sR) \cap r_R(m_{s-1}R)$. Since $r_R(m_{s-1}R)$ is left s-unital, there exists $f \in r_R(m_{s-1}R)$ such that $fa_t = a_t$. If we put $e_{s-1} = fe_s$ then $e_{s-1}a_t = a_t$ and $e_{s-1} \in r_R(m_sR) \cap r_R(m_{s-1}R)$. Next, replacing $c$ by $ce_{s-1}$ in Eq.(1), we obtain $m_{s-2}ca_t = 0$ in the same way as above. Hence we have $a_t \in r_R(m_sR) \cap r_R(m_{s-1}R) \cap r_R(m_{s-2}R)$. Continuing this process, we obtain $m_iRa_t = 0$ for all $i = 0, 1, \cdots, s$. Thus we get $(\sum_{i=0}^{s} m_ix^i)R[x;\alpha](\sum_{j=0}^{t-1} a_jx^j) = 0$, since $M_R$ is $\alpha$-compatible. Using induction on $s + t$, we obtain $m_iRa_j = 0$ for all $i, j$. Thus we proved that $M_R$ is skew quasi-Armendariz. Now, let $m(x) = \sum_{i=0}^{s} m_ix^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^{t} a_jx^j \in r_{R[x;\alpha]}(m(x)R[x;\alpha])$. Then $m(x)R[x;\alpha]f(x) = 0$ and so $m_iRa_j = 0$ for all $i, j$ since $M_R$ is skew quasi-Armendariz. Since $r_R(m_iR)$ is left s-unital, there exists $e_i \in r_R(m_iR)$ such that $a_j = e_ia_j$ for $j = 0, 1, \cdots, t$. Put $e = e_0e_1 \cdots e_s$, then $a_j = ea_j$ for $j = 0, 1, \cdots, t$. Hence $ef(x) = f(x)$ and $e \in r_{R[x;\alpha]}(m(x)R[x;\alpha])$ since $e_i \in r_R(m_iR)$ and $M_R$ is $\alpha$-compatible. Therefore $r_{R[x;\alpha]}(m(x)R[x;\alpha])$ is left s-unital.

(2)⇒(1) Suppose that condition (2) holds. Let $m$ be an element of $M_R$. Since $M_R$ is $\alpha$-compatible, $r_R(mR) \subseteq r_{R[x;\alpha]}(mR[x;\alpha])$. Hence for any $b \in r_R(mR)$, there exists a polynomial $f(x) = \sum_{j=0}^{t} a_jx^j \in r_{R[x;\alpha]}(mR[x;\alpha])$ such that $f(x)b = b$. Then $a_0b = b$ and $a_0 \in r_R(mR)$. This implies that $r_R(mR)$ is left s-unital. □

**Corollary 2.6.** *Let $M_R$ be an $\alpha$-compatible module. Then $M_R$ is p.q.-Baer if and only if $M[x;\alpha]_{R[x;\alpha]}$ is p.q.-Baer. In this case $M_R$ is skew quasi-Armendariz.*

*Proof.* Let $M_R$ be a *p.q.*-Baer module. Then for each $m \in M$, there exists $e^2 = e \in R$, such that $r_R(mR) = eR$. Thus $r_R(mR)$ is left s-unital for each $m \in M$. It follows from Theorem 2.5 that $M_R$ is skew quasi-Armandariz. If $r_R(mR) = eR$ for some $e^2 = e \in R$, then we see that $ere = re$ holds for each $r \in R$. Thus if $r_R(m_iR) = e_iR$, for $i = 0, 1, \cdots, n$, then we get $r_R(m_0R + m_1R + \cdots + m_nR) = eR$, where $e = e_0e_1 \cdots e_n$ and $e^2 = e \in R$. Now, let $m(x) = m_0 + m_1x + \cdots + m_nx^n$ and $f(x) = a_0 + a_1x + \cdots + a_kx^k$ such that $f(x) \in r_{R[x;\alpha]}(m(x)R[x;\alpha])$. Then $m_iRa_j = 0$

since $M_R$ is skew quasi-Armendariz. Let $r_R(m_iR) = e_iR$, for $i = 0, 1, \cdots, n$ and $e = e_0e_1 \cdots e_n$. Then $r_{R[x;\alpha]}(m(x)R[x;\alpha]) = eR[x;\alpha]$ and hence we learn that $M[x;\alpha]_{R[x;\alpha]}$ is *p.q.*-Baer.
The proof for the converse part can be done similarly, and therefore is omitted. □

**Remark 2.7.** Since $\alpha$-reduced modules are $\alpha$-compatible, Corollary 2.6 extends [2, Theorem 7(1)(a)].

**Corollary 2.8.** *Let $M_R$ be a module. Then $M_R$ is p.q.-Baer if and only if $M[x]_{R[x]}$ is p.q.-Baer. In this case $M_R$ is quasi-Armendariz.*

**Corollary 2.9**([6, Theorem 3.1])**.** *$R$ is a right p.q.-Baer ring if and only if $R[x]$ is a right p.q.-Baer ring.*

**Proposition 2.10.** *Let $M_R$ be an $\alpha$-compatible module. Then* (1)⇒ (2)⇒ (3)*.*
(1) *$r_{R[[x;\alpha]]}(m(x)R[[x;\alpha]])$ is a pure as a right ideal in $R[[x;\alpha]]$ for any element $m(x) \in M[[x;\alpha]]$.*
(2) *$r_R(mR)$ is pure as a right ideal in $R$ for any element $m \in M_R$.*
(3) *$M_R$ is skew quasi-Armendariz of power series type.*

*Proof.* (1) ⇒(2) The proof is similar to that of Theorem 2.5.
(2)⇒(3) Assume that $m(x)R[[x;\alpha]]f(x) = 0$ for $m(x) = \sum_{i=0}^{\infty} m_ix^i \in M[[x;\alpha]]$ and $f(x) = \sum_{j=0}^{\infty} a_jx^j \in R[[x;\alpha]]$. Then $m(x)Rf(x) = 0$ and so we have the following equation for an arbitrary $c \in R$:

$$\sum_{k=0}^{\infty}\Big(\sum_{i+j=k} m_ix^ica_jx^j\Big) = \sum_{k=0}^{\infty}\Big(\sum_{i+j=k} m_i\alpha^i(ca_j)x^{i+j}\Big) = 0. \tag{3}$$

We will show that $m_iRa_j = 0$ for all $i, j$. We proceed by induction on $i + j$. From Eq.(3), we obtain, $m_0Ra_0 = 0$. This proves $i + j = 0$. Now suppose that $m_iRa_j = 0$ for $i + j \leq n - 1$. Hence $a_j \in r_R(m_iR)$ for $j = 0, 1, \cdots, n - 1$ and $i = 0, 1, \cdots, n - 1 - j$. Since $r_R(m_iR)$ is left s-unital, there exists $e_{ji} \in r_R(m_iR)$ such that $e_{ji}a_j = a_j$ for $j = 0, 1, \cdots, n-1$ and $i = 0, 1, \cdots, n-1-j$. From Eq.(3), we have:

$$\sum_{i+j=k} m_i\alpha^i(ca_j) = 0 \quad \text{for all} \quad k \geq 0. \tag{4}$$

If we put $f_j = e_{j1}e_{j2} \cdots e_{j(n-1-j)}$ for $j = 0, 1, \cdots, n - 1$, then $f_ja_j = a_j$ and $f_j \in r_R(m_0R) \cap r_R(m_1R) \cap \cdots \cap r_R(m_{n-1-j}R)$. For $k = n$ replacing $c$ by $cf_0$ in Eq.(4), we obtain $m_nca_0 = m_ncf_0a_0 = 0$. Hence $m_nRa_0 = 0$. Continuing this process (replacing $c$ by $cf_j$ in Eq.(4), for $j = 0, 1, \cdots, n-1$ and using $\alpha$-compatibility of $M_R$), we obtain $m_iRa_j = 0$ for $i+j = n$. Therefore $M_R$ is skew quasi-Armendariz of power series type. □

Since quasi-Baer modules satisfy the hypothesis of Theorem 2.5 and Proposition 2.10 we have,

**Corollary 2.11**([9, Theorem 2.13(1)]). *Let $M_R$ be an $\alpha$-compatible module. Then $M_R$ is quasi-Baer iff $M[x;\alpha]_{R[x;\alpha]}$ is quasi-Baer iff $M[[x;\alpha]]_{R[[x;\alpha]]}$ is quasi-Baer.*

*Proof.* The proof follows very similar to that of Corollary 2.6. □

**Corollary 2.12**([2, Theorem 7(1)(b)]). *Let $M_R$ be an $\alpha$-compatible module. If $M[[x;\alpha]]_{R[[x;\alpha]]}$ is p.q.-Baer then $M_R$ is p.q.-Baer.*

### 3. Skew Laurent polynomial and power series modules over quasi-Baer and p.q.-Baer modules

In this section we introduce skew quasi-Armendariz of Laurent type modules ans skew quasi-Armendariz of Laurent power series type modules, which are skew Laurent polynomial version of the quasi-Armendariz modules and then study on the relationship between the quasi-Baerness and p.q.-Baer property of a module $M_R$ and those of the skew Laurent polynomials and skew Laurent series. As a consequence we obtain a generalization of [2] and some result in [9].

**Definition 3.1.** Let $\alpha$ be an automorphism of $R$. A module $M_R$ is called:
(i) *skew quasi-Armendariz of Laurent type*, if whenever $m(x)R[x,x^{-1};\alpha]f(x)=0$ for $m(x)=\sum_{i=-s}^{t} m_i x^i \in M[x,x^{-1};\alpha]$ and $f(x)=\sum_{j=-p}^{q} a_j x^j \in R[x,x^{-1};\alpha]$, then $m_i R a_j = 0$ for all $i,j$.
(ii) *skew quasi-Armendariz of Laurent power series type* if whenever $m(x)R[[x,x^{-1};\alpha]]f(x)=0$ for $m(x)=\sum_{i=-s}^{\infty} m_i x^i \in M[[x,x^{-1};\alpha]]$ and $f(x)=\sum_{j=-p}^{\infty} a_j x^j \in R[[x,x^{-1};\alpha]]$, then $m_i R a_j = 0$ for all $i,j$.

Note that if $M_R$ is assumed to be $\alpha$-reduced, then it is clear that $M_R$ is skew quasi-Armendariz of Laurent type and skew quasi-Armendariz of Laurent power series type. In a similar way as in the proof of Proposition 2.2 and Theorem 2.5, we can prove the following results.

**Proposition 3.2.** *Let $\alpha$ be an automorphism of a ring $R$ and $M_R$ be an $\alpha$-compatible module. Then the following statements are equivalent:*
(1) $M_R$ is skew quasi-Armendariz of Laurent type.
(2) $\psi : \mathrm{rAnn}_R(\mathrm{sub}(M_R)) \to \mathrm{rAnn}_{R[x,x^{-1};\alpha]}(\mathrm{sub}(M[x,x^{-1};\alpha]_{R[x,x^{-1};\alpha]}))$;
$I \mapsto I[x,x^{-1};\alpha]$ is bijective.

**Proposition 3.3.** *Let $\alpha$ be an automorphism of a ring $R$ and $M_R$ be an $\alpha$-compatible module. Then the following statements are equivalent:*
(1) $M_R$ is skew quasi-Armendariz of Laurent power series type.
(2) $\psi' : \mathrm{rAnn}_R(\mathrm{sub}(M_R)) \to \mathrm{rAnn}_{R[[x,x^{-1};\alpha]]}(\mathrm{sub}(M[[x,x^{-1};\alpha]]_{R[[x,x^{-1};\alpha]]}))$;
$I \mapsto I[[x,x^{-1};\alpha]]$ is bijective.

**Theorem 3.4.** *Let $\alpha$ be an automorphism of a ring $R$ and $M_R$ be an $\alpha$-compatible module. Then the following are equivalent:*
(1) *$r_R(mR)$ is pure as a right ideal in $R$ for any element $m \in M_R$.*
(2) *$r_{R[x,x^{-1};\alpha]}(m(x)R[x,x^{-1};\alpha])$ is a pure as a right ideal in $R[x,x^{-1};\alpha]$ for any*

*element$m(x) \in M[x, x^{-1}; \alpha]$. In this case $M_R$ is skew quasi-Armendariz of Laurent type.*

**Corollary 3.5.** *Let $\alpha$ be an automorphism of a ring $R$ and $M_R$ be an $\alpha$-compatible module. Then $M_R$ is p.q.-Baer if and only if $M[x, x^{-1}; \alpha]_{R[x,x^{-1};\alpha]}$ is p.q.-Baer. In this case $M_R$ is skew quasi-Armendariz of Laurent type.*

**Remark 3.6.** Since $\alpha$-reduced modules are $\alpha$-compatible modules, the Corollary 3.5 extends [2, Theorem 7(2)(a)].

**Proposition 3.7.** *Let $\alpha$ be an automorphism of a ring $R$ and $M_R$ be an $\alpha$-compatible module. Then* (1)⇒(2)⇒(3).
(1) $r_{R[[x,x^{-1};\alpha]]}(m(x)R[[x, x^{-1}; \alpha]])$ is a pure as a right ideal in $R[[x, x^{-1}; \alpha]]$ for any element $m(x) \in M[[x, x^{-1}; \alpha]]$.
(2) $r_R(mR)$ is pure as a right ideal in $R$ for any element $m \in M_R$.
(3) $M_R$ is skew quasi-Armendariz of Laurent power series type.

**Corollary 3.8**([2, Theorem 7(2)(b)])**.** *Let $\alpha$ be an automorphism of a ring $R$ and $M_R$ be an $\alpha$-compatible module. If $M[[x, x^{-1}; \alpha]]_{R[[x,x^{-1};\alpha]]}$ is p.q.-Baer then $M_R$ is p.q.-Baer.*

Since quasi-Baer modules satisfy the hypothesis of Theorem 3.4 and Proposition 3.7 we have;

**Corollary 3.9**([9, Theorem 2.13(2)]**.** *Let $\alpha$ be an automorphism of a ring $R$ and $M_R$ be an $\alpha$-compatible module. Then $M_R$ is quasi-Baer iff $M[x, x^{-1}; \alpha]_{R[x,x^{-1};\alpha]}$ is quasi-Baer iff $M[[x, x^{-1}; \alpha]]_{R[[x,x^{-1};\alpha]]}$ is quasi-Baer.*

**Corollary 3.10**([9, Corollary 2.14])**.** *$M_R$ is quasi-Baer iff $M[x]_{R[x]}$ is quasi-Baer iff $M[[x]]_{R[[x]]}$ is quasi-Baer iff $M[x, x^{-1}]_{R[x,x^{-1}]}$ is quasi-Baer iff $M[[x, x^{-1}]]_{R[[x,x^{-1}]]}$ is quasi-Baer.*

**Corollary 3.11**([4, Theorem 1.8])**.** *$R$ is quasi-Baer iff $R[x]$ is quasi-Baer iff $R[[x]]$ is quasi-Baer iff $R[x, x^{-1}]$ is quasi-Baer iff $R[[x, x^{-1}]]$ is quasi-Baer.*

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