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Pseudo-Rank Functions on Rickart *-rings

Dedicated to Dr. K. Anjaneyulu on the occasion of his 75th birthday.

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Abstract. Pseudo-rank functions on Rickart *-rings are introduced and their properties are studied.

1. Introduction

A real valued function D on a lattice L is called a *dimension function* if the range of D has either an upper bound or a lower bound and for all $a, b \in L$, $D(a \vee b) + D(a \wedge b) = D(a) + D(b)$, see; von Neumann [12] p.58. The theory of dimension functions is studied in various structures. von Neumann [12] introduced dimensionality in continuous geometries by using perspectivity, whereas Iwamura [6] used the concept of a relation called the p-relation .
Kaplansky [8], Murray and von Neumann [11] and others have introduced dimensionality in rings of operators by using equivalence of projections. Maeda [10] generalized the work of von Neumann [12] and Kaplansky [8] for a certain class of lattices. At the same time Loomis [9] gave an abstract setting to the Murray, von Neumann dimension theory by using complete orthocomplemented lattices. Berberian [2] has developed theory of dimension functions on the lattice of projections of a finite Baer *-ring. Goodearl [4] developed the dimension theory for a certain class of modules. von Neumann [12], p.231 has introduced the concept of a rank-function on a regular ring which generalizes the dimension function. Goodearl [3], [5] has introduced and developed the study of *pseudo-rank functions* on regular rings, which is a generalization of rank functions.
In this paper we introduce and study the concept of a *pseudo-rank function* on a Rickart *-ring R. We obtain some basic properties of pseudo-rank functions and the set of all pseudo-rank functions on R, on the lines of Goodearl [5] for Rickart *-rings. The undefined terms are from Berberian [2] and Birkhoff [1].

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2. Preliminaries

A **-ring* is a ring R with an *involution* "$*$"(i.e. an antiautomorphism of period two) such that $x^{**} = x$, $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$. Throughout we denote by R, a *-ring. An element $e \in R$ is called a *projection* if it is *self-adjoint* (i.e. $e = e^*$) and *idempotent* (i.e. $e = e^2$). The set of projections in R can be partially ordered by $e \leq f$ if and only if $e = ef$, see; Berberian [2]. If for two projections $e, f \in R$, $ef = fe$, then $inf\{e, f\} = e \wedge f = ef$ and $sup\{e, f\} = e \vee f = e + f - ef$. Two projections $e, f \in R$ are called *equivalent*, in notation $e \sim f$, if there exists some $w \in R$ such that $w^*w = e$ and $ww^* = f$. Then w is a *partial isometry*, (i.e. $ww^*w = w$) and $we = w = fw$. For projections $e, f \in R$, we say that f *dominates* e, in notation $e \precsim f$, if $e \sim g \leq f$ for some projection $g \in R$. Two elements $x, y \in R$ are said to be *orthogonal*, in notation $x \perp y$, if $x^*y = xy^* = 0$, see; Loomis [9] p.26. A *-ring A is called a *Rickart *-ring* if for each $x \in A$, the *right annihilator* of x, $R(\{x\}) = \{y \in A : xy = 0\}$, is a right ideal generated by a projection. i. e. $R(\{x\}) = gA$ for some projection $g \in A$. A *-ring A is called a *Baer *-ring* if the right annihilator of any nonempty subset S of A is the right ideal generated by a projection $e \in A$ i. e. $R(S) = eA$. In this case, the projection $1 - e$ is called the *right projection of S*. Similarly the *left projection* of S is defined. The right projection (respectively, left projection) of an element x in a Rickart *-ring is denoted by $RP(x)$ (respectively, by $LP(x)$) and it is the smallest projection e such that $xe = x$ ($ex = x$) and $xy = 0$ is equivalent to $RP(x)y = 0$ ($yLP(x) = 0$). It is known that a *-ring with proper involution (i.e. $x^*x = 0$ implies $x = 0$)is a poset under the partial order (called the **-order*) $x \leq y$ iff $x^*x = x^*y$ and $xx^* = xy^*$, see; Janowitz [7]. This partial order generalizes the partial order defined on the set of projections. A Rickart *-ring has proper involution.

3. Pseudo-rank function

A *pseudo-rank function* f on a *-ring R is a mapping $f : R \to [0, 1]$ such that

(1) $f(1) = 1$,

(2) $f(xy) \leq f(x), f(y)$ for all $x, y \in R$,

(3) $f(x + y) = f(x) + f(y)$ for all orthogonal $x, y \in R$,

(4) $f(x) = f(x^*) = f(RP(x)) = f(LP(x))$ provided $RP(x), LP(x)$ exist in R.

It is clear that $f(0) = 0$. A pseudo-rank function f with the property $f(x) > 0$, for $x \neq 0$ is called a *rank function* on R.

Proposition 1. *Let R be a *-ring and f be a pseudo-rank function on R.*

(1) *If $x_1, \cdots, x_n \in R$ are mutually orthogonal then $f(x_1 + \cdots + x_n) = \sum_{i=1}^{n} f(x_i)$.*

(2) *If the involution in R is proper and $x \leq y$ then $f(x) \leq f(y)$.*

(3) *If the involution in R is proper and* $x_1, \cdots, x_n$ *and* $y_1, \cdots, y_k$ *are sets of orthogonal elements in R such that* $x_1 + \cdots + x_n \leq y_1 + \cdots + y_k$*, then* $\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{k} f(y_i)$.

(4) *If* e, g *are projections in R, such that* $e \sim g$ *then* $f(e) = f(g)$.

(5) *If* $e_1, \cdots, e_n$ *and* $g_1, \cdots, g_k$ *are sets of orthogonal projections in R such that* $e_1 + \cdots + e_n \precsim g_1 + \cdots + g_k$*, then* $\sum_{i=1}^{n} f(e_i) \leq \sum_{i=1}^{k} f(g_i)$.

Proof. (1) Follows from the definition of a pseudo-rank function.

(2) $x \leq y$ iff $x^*x = x^*y$ and $xx^* = xy^*$. By the definition of a pseudo-rank function $f(x) = f(x^*x) = f(x^*y) \leq f(y)$.

(3) Using (2) and the definition of a pseudo-rank function we have $\sum_{i=1}^{n} f(x_i) = f(x_1 + \cdots + x_n) \leq f(y_1 + \cdots + y_k) = \sum_{i=1}^{k} f(y_i)$.

(4) $e \sim g$ implies $e = w^*w$, $g = ww^*$ for some partial isometry $w \in R$. Then $w = ww^*w = gw = we$ and so $f(e) = f(w^*w) \leq f(w) = f(gw) \leq f(g)$. Similarly $f(g) \leq f(e)$.

(5) Follows from (4)and (3). □

It is known that for a projection e in a Rickart *-ring R, eRe is a Rickart *-ring, see; Berberian [2] p.15.

Lemma 1. *Let* f *be a pseudo-rank function on a Rickart *-ring R. Let* $e \in R$ *be a nonzero projection such that* $f(e) \neq 0$.

(1) *The function* $Q(x) = f(x)/f(e)$ *defines a pseudo-rank function on the Rickart *-ring eRe.*

(2) *If e is a central projection in R then the function* $Q(x) = f(ex)/f(e)$ *defines a pseudo-rank function on R.*

(3) *If e is a central projection such that* $f(e) = 1$ *then* $f(ex) = f(x)$ *for all* $x \in R$.

Proof. (1) $x \in eRe$ implies $x = ex = xe$. Hence $f(x) = f(ex) \leq f(e)$ shows that $Q(x) \leq 1$. Thus Q maps eRe into $[0, 1]$. By Corollary p.15 from Berberian [2], for $x \in eRe$, $RP(x)$, $LP(x)$ are same whether calculated in R or in eRe. Hence the remaining properties for Q to be a pseudo-rank function can be easily verified.

(2) Since $f(ex) \leq f(e)$ for all $x \in R$, Q maps R into $[0, 1]$.
(i). Clearly $Q(1) = 1$.
(ii). For $x, y \in R$, $f(exy) \leq f(ex), f(ey)$ and so $Q(xy) \leq Q(x), Q(y)$.
(iii). Suppose $x \perp y$ in R. Since, e is a central projection it follows that $ex \perp ey$. Hence $Q(x + y) = f(ex + ey)/f(e) = Q(x) + Q(y)$.
(iv). Since $(ex)^* = ex^*$, we get $Q(x) = Q(x^*)$.

To show that $Q(x) = Q(RP(x))$, we first show that $eRP(x) = RP(ex)$. From $x = xRP(x)$ we get $ex = exRP(x)$. Hence $ex[1 - eRP(x)] = 0$ and so $RP(ex)[1 - eRP(x)] = 0$. Thus $RP(ex) = RP(ex)eRP(x)$.

On the other hand, $ex = exRP(ex)$ implies $x[eRP(ex) - e] = 0$ and so $RP(x)[eRP(ex) - e] = 0$ i.e. $RP(x)eRP(ex) = eRP(x)$. Since e is a central projection it follows from the Lemma p.137 from Berberian [2] that $RP(ex) \leq RP(x)$. Thus $RP(x)$ and $RP(ex)$ commute with each other. This shows that $RP(ex) = eRP(x)$.
Thus $Q(x) = f(ex)/f(e) = f(RP(ex))/f(e) = f(eRP(x))/f(e) = Q(RP(x))$. Similarly we get $Q(x) = Q(LP(x))$.

(3) We have $f(x) = f(xe + x(1-e)) = f(xe) + f(x(1-e))$. Also, $1 = f(1) = f(e+(1-e)) = f(e)+f(1-e)$ implies $f(1-e) = 0$. Now $f(x(1-e)) \leq f(1-e) = 0$ leads to $f(x) = f(xe)$. □

Lemma 2. *Let $\{e_1, \cdots, e_n\}$ be a set of orthogonal projections in a Rickart *-ring R. Suppose I, J are nonempty subsets of $\{1, \cdots, n\}$. Let α_i and β_j be nonzero real numbers. For each $i \in I$ and $j \in J$, let P_i and Q_j be pseudo-rank functions on e_iRe_i and e_jRe_j respectively. If $\sum_{i\in I} \alpha_i P_i(e_ixe_i) = \sum_{j\in J} \beta_j Q_j(e_jxe_j)$ for every $x \in R$, then $I = J$, $\alpha_i = \beta_i$ and $P_i = Q_i$ for each i.*

Proof. Let $t \in J$. Then using $e_je_t = 0$ for $j \neq t$, $Q_j(0) = 0$ and $Q_j(e_j) = 1$ we get $\sum_{i\in I} \alpha_i P_i(e_ixe_t) = \sum_{j\in J} \beta_j Q_j(e_jxe_t) = \beta_t \neq 0$. Hence $P_s(e_se_t) \neq 0$ for some $s \in I$. This implies $e_se_t \neq 0$ and so $s = t$, i.e. $t \in I$. Thus $J \subseteq I$. Similarly we get $I \subseteq J$. Given $s \in I$, $y \in e_sRe_s$ then using $e_ie_s = 0$ for $i \neq s$ we get, $\alpha_s P_s(x) = \sum_{i\in I} \alpha_i P_i(e_ixe_i) = \sum_{j\in J} \beta_j Q_j(e_jxe_j) = \beta_s Q_s(x)$. In particular $\alpha_s = \alpha_s P_s(e_s) = \beta_s Q_s(e_s) = \beta_s$. Consequently $P_s(x) = Q_s(x)$ for every $x \in e_sRe_s$. Thus $P_s = Q_s$. □

Lemma 3. *Let $\{e_1, \cdots, e_n\}$ be a set of orthogonal central projections in a Rickart *-ring R. Suppose I, J are nonempty subsets of $\{1, \cdots, n\}$. Let α_i and β_j be nonzero real numbers. For each $i \in I$ and $j \in J$, let P_i and Q_j be pseudo-rank functions on R such that $P_i(e_i) = 1$ and $Q_j(e_j) = 1$. If $\sum_{i\in I} \alpha_i P_i(e_ixe_i) = \sum_{j\in J} \beta_j Q_j(e_jxe_j)$ for every $x \in R$, then $I = J$, $\alpha_i = \beta_i$ and $P_i = Q_i$ for each $i \in I$.*

Proof. Let $t \in J$. Then using Lemma 1(3) we get $Q_j(e_t) = Q_j(e_je_t) = 0$. Hence $\sum_{i\in I} \alpha_i P_i(e_t) = \sum_{j\in J} \beta_j Q_j(e_t) = \beta_t Q_t(e_t) = \beta_t \neq 0$. Therefore $P_s(e_t) \neq 0$ for some $s \in I$. This implies $e_se_t \neq 0$ and so $s = t$, i.e. $t \in I$. Thus $J \subseteq I$. Similarly we get $I \subseteq J$.
Given $s \in I$, $x \in R$ we have by using Lemma 1(3), $\alpha_s P_s(x) = \alpha_s P_s(e_sx) = \sum_{i\in I} \alpha_i P_i(e_ie_sx) = \sum_{j\in J} \beta_j Q_j(e_je_sx) = \beta_s Q_s(e_sx) = \beta_s Q_s(x)$. In particular $\alpha_s = \alpha_s P_s(e_s) = \beta_s Q_s(e_s) = \beta_s$. Consequently $P_s(x) = Q_s(x)$ for every $x \in R$. Thus $P_s = Q_s$. □

Lemma 4. *Let $e_1, \cdots, e_n$ be orthogonal central projections in a Rickart *-ring R. Let f be a pseudo-rank function on R such that $\sum_{i=1}^{n} f(e_i) = 1$ and $f(e_i) \neq 0$ for all i.*

(*a*) *There exist unique pseudo-rank functions $P_i, 1 \leq i \leq n$, on e_iR such that*

$f(x) = \sum_{i=1}^{n} f(e_i)P_i(e_ix)$ *for all* $x \in R$.

(b) *There exist unique pseudo-rank functions* $Q_i, 1 \leq i \leq n$, *on* R *such that* $Q_i(e_i) = 1$ *and* $f(x) = \sum_{i=1}^{n} f(e_i)Q_i(x)$ *for all* $x \in R$.

Proof. Since $e_1, \cdots, e_n$ are orthogonal central projections in R, $f(e_1 + \cdots + e_n) = \sum_{i=1}^{n} f(e_i) = 1$. Hence by Lemma 1(3), $f(x) = f(e_1x + \cdots + e_nx) = \sum_{i=1}^{n} f(e_ix)$.

(a) For each i, $1 \leq i \leq n$ by Lemma 1(1), $P_i(x) = f(x)/f(e_i)$ defines a pseudo-rank function P_i on e_iR. Given any $x \in R$, we then have $f(e_ix) = f(e_i)P_i(e_ix)$ for each i. Hence $f(x) = \sum_{i=1}^{n} f(e_i)P(e_ix)$. Uniqueness follows from Lemma 2.

(b) For each i, by Lemma 1(2), $Q_i(x) = f(e_ix)/f(e_i)$ defines a pseudo-rank function Q_i on R. We note that $Q_i(e_i) = 1$. Given any $x \in R$, we then have $f(e_ix) = f(e_i)Q_i(x)$ for all i. Hence $f(x) = \sum_{i=1}^{n} f(e_i)Q_i(x)$. Uniqueness follows from Lemma 3. □

Theorem 1. *Let* R *be a Rickart *-ring and* $e_1, \cdots, e_n$ *be orthogonal central projections in* R *such that* $e_1 + \cdots + e_n = 1$.

(a) *Suppose* I *is a nonempty subset of* $\{1, \cdots, n\}$. *Let* α_i *be positive real numbers such that* $\sum_{i \in I} \alpha_i = 1$. *For each* $i \in I$, *let* P_i *be a pseudo-rank function on* e_iR. *Then* $f(x) = \sum_{i \in I} \alpha_i P_i(e_ix)$ *is a pseudo-rank function on* R.

(b) *Let* α_i, $1 \leq i \leq n$ *be nonnegative real numbers such that* $\sum_{i \in I} \alpha_i = 1$. *For each* $i \in I$, *let* P_i *be a pseudo-rank function on* e_iR. *Then* $f(x) = \sum_{i=1}^{n} \alpha_i P_i(e_ix)$ *is a pseudo-rank function on* R.

(c) *Every pseudo-rank function on* R *may be uniquely obtained as in* (a). *Moreover, if there exists at least one pseudo-rank function on each* e_iR, *then every pseudo-rank function on* R *may be obtained as in* (b).

Proof. (a) and (b) follow from the definition of a pseudo-rank function.
(c) Let f be a pseudo-rank function on R. Let I be the set of those $i \in \{1, \cdots, n\}$ for which $f(e_i) \neq 0$. Put $\alpha_i = f(e_i)$ for all $i \in I$. Then $\sum_{i \in I} \alpha_i = \sum_{i \in I} f(e_i) = f(e_1 + \cdots + e_n) = f(1) = 1$. By Lemma 4(a) there exist pseudo-rank functions P_i on e_iR for each $i \in I$ such that $f(x) = \sum_{i \in I} \alpha_i P_i(e_ix)$ for all $x \in R$. Thus f has a representation as in (a).
Suppose that there exists at least one pseudo-rank function on each e_iR. Put $\alpha_i = f(e_i)$ for all $i = 1, \cdots, n$. For $i \in \{1, \cdots, n\} - I$, let P_i be any pseudo-rank function on e_iR. Then $f(x) = \sum_{i \in I} \alpha_i P_i(e_ix) = \sum_{i=1}^{n} \alpha_i P_i(e_ix)$ for all $x \in R$, which represents f as in (b). □

The proof of the following lemma follows from the definition of a pseudo-rank function.

Lemma 5. *Let* R_1, R_2 *be two Rickart *-rings,* $f : R_1 \to R_2$ *be a *-homomorphism satisfying the condition* $f(RP(x)) = RP(f(x))$. *If* g *is a pseudo-rank function on* R_2, *then* $g \circ f$ *is a pseudo-rank function on* R_1.

An ideal I of a Rickart *-ring is called a *strict ideal*, if $x \in I$ implies $RP(x) \in I$, see; Berberian [2] p. 141.

Lemma 6. *Let f be a pseudo-rank function on a Rickart *-ring R. The set $A = \{x \in R : f(x) = 0\}$ is a proper strict ideal of R.*

Proof. Since $f(0) = 0$, A is nonempty. Also, $f(x) = f(x^*) = f(RP(x))$ shows that if $x \in A$, then x^*, $RP(x) \in A$. Clearly, for $x \in A$ and $y \in R$; $f(xy) \leq f(x)$ implies $xy \in A$. Similarly $yx \in A$. Let $x, y \in A$. Then $f(x+y) = f(RP(x+y))$. By Lemma on p. 137 from Berberian [2], $RP(x+y) \leq RP(x) \vee RP(y)$. For convenience write $RP(x) = e$ and $RP(y) = g$. Then $e \vee g = g + RP[e(1-g)]$ with $g \perp RP[e(1-g)]$. We have $f[g+RP(e(1-g))] = f(g)+f[RP(e(1-g))] = f(RP(y))+f[e(1-g)]$. We have $f(g) = f(RP(y)) = f(y) = 0$ and $f[e(1-g)] \leq f(e) = f(RP(x)) = f(x) = 0$. Hence $f(x+y) = f(RP(x+y)) \leq f(e \vee g) = 0$. Thus $x+y \in A$ and so A is a strict ideal of R. Since $f(1) = 1$, A is a proper strict ideal. □

The following result is from Berberian [2] (Exercise 1, p. 144).

Lemma 7. *Let I be a strict ideal of a Rickart *-ring R. Equip R/I with the natural *-ring structure and write $x \to \overline{x}$ for the canonical mapping $R \to R/I$.*

(1) *R/I is a Rickart *-ring.*

(2) *$RP(\overline{x}) = \overline{(RP(x))}$, $LP(\overline{x}) = \overline{(LP(x))}$, for all $x \in R$; in particular, every projection in R/I has the form $\overline{e}$ with e a projection in R.*

(3) *For all projections $e, f \in R$, $\overline{e \vee f} = \overline{e} \vee \overline{f}$ and $\overline{e \wedge f} = \overline{e} \wedge \overline{f}$.*

(4) *If u, v are orthogonal projections in R/I and if $v = \overline{f}$, f a projection in R, then there exists a projection $e \in R$ such that $u = \overline{e}$ and e is orthogonal to f.*

Lemma 8. *Let R be a Rickart *-ring in which every projection is central. Let I be a strict ideal of R. Let $u, v \in R/I$ and $v = \overline{b}$ for some $b \in R$.*

(1) *If $u \leq v$, then there exists $a \in R$ such that $u = \overline{a}$ and $a \leq b$.*

(2) *If $u \perp v$, then there exists $a \in R$ such that $u = \overline{a}$ and $a \perp b$.*

Proof. Let $u = \overline{x}$ for some $x \in R$.

(1) We note that $u \leq v$ implies $u^*u = u^*v = v^*u$ and $uu^* = uv^* = vu^*$ in R/I. Then in R/I

$$\text{(a)} \qquad \overline{x^*x} = \overline{x^*b} = \overline{b^*x} \text{ and } \overline{xx^*} = \overline{xb^*} = \overline{bx^*}.$$

Put $a = bRP(x)$. Since all projections are central, $RP(x) = RP(x^*)$. We have $a^*a = a^*b = b^*a$ and $aa^* = ba^* = ab^*$. Thus $a \leq b$ in R. Moreover, $x^*a = x^*b$ and $ax^* = bx^*$. Hence in R/I,

$$\text{(b)} \qquad \overline{x^*x} = \overline{x^*b} = \overline{x^*a} \text{ and } \overline{xx^*} = \overline{bx^*} = \overline{ax^*}.$$

Thus, in R/I, $\overline{x} \leq \overline{a}$. Further we have, $\overline{xx^*} = \overline{ax^*} = \overline{xa^*}$. Hence $\overline{x}\,[\,\overline{x^*} - \overline{a^*}\,]\,\overline{a} = \overline{0}$. This implies $RP(\overline{x})\,[\,\overline{x^*} - \overline{a^*}\,]\,\overline{a} = \overline{0}$. Thus $\overline{RP(x)}\,\overline{x^*}\,\overline{a} = \overline{RP(x)a^*a}$. Using $RP(x)x^* = x^*$ and $RP(x)a^* = a^*$, we get $\overline{x^*a} = \overline{a^*a}$. Therefore, $(\overline{a} - \overline{x})^*(\overline{a} - \overline{x}) = \overline{0}$. Since R/I is a Rickart *-ring, its involution is proper and so we get $\overline{a} - \overline{x} = \overline{0}$. Thus $\overline{a} = \overline{x}$.

(2) Suppose $u \perp v$ in R/I. Then $u^*v = 0$ implies $\overline{x^*}\overline{b} = \overline{0}$ and so $\overline{x^*RP(b)} = \overline{0}$, consequently, $\overline{xRP(b)} = \overline{0}$, $\overline{x} = \overline{x[1 - RP(b)]}$. Put $a = x[1 - RP(b)]$. As all projections are central, we get $a^*b = ab^* = 0$. Hence $a \perp b$ in R. Also we have $\overline{a} = \overline{x}$. $\square$

Lemma 9. *Let f be a pseudo rank function on a Rickart *-ring R, in which all projections are central. Let I be a strict ideal of R, such that $I \subseteq ker(f)$. Then there exists a pseudo rank function g on R/I such that $g \circ \phi = f$. Further g is a rank function iff $I = ker(f)$.*

Proof. Let ϕ be the canonical *-homomorphism from R to R/I. Suppose $x, y \in R$ and $\phi(x) = \phi(y)$. Then $\phi(x-y) = 0$ and so $x-y \in I \subseteq ker(f)$ implies $f(x-y) = 0$. We have $x = (x - y) + y$. By the Lemma on p. 137 from Berberian [2] $RP(x) \leq RP(x - y) \vee RP(y)$. Put $RP(x - y) = e$ and $RP(y) = g$. We have $e \vee g = e + RP[g(1 - e)]$ with $e \perp RP[g(1 - e)]$. Hence

$$f(x) = f(RP(x)) \leq f(e) + f[RP(g(1-e))] = f(x-y) + f[g(1-e)] \leq f(g) = f(y).$$

Similarly, we get $f(y) \leq f(x)$. Thus $f(x) = f(y)$. Define a map $g : R/I \to [0, 1]$, by $g(\overline{x}) = f(x)$. In view of the above para g is well defined. We have $g(\overline{1}) = 1$, $g(\overline{xy}) \leq g(\overline{x}), g(\overline{y})$. Let $\overline{x}, \overline{y} \in R/I$ and $\overline{x} \perp \overline{y}$. Then by Lemma 8 there exist $a, b \in R$ such that $a \perp b$, $\overline{a} = \overline{x}$ and $\overline{b} = \overline{y}$. We have

$$g[\overline{x} + \overline{y}] = g[\overline{a + b}] = f[a + b] = f(a) + f(b) = g(\overline{x}) + g(\overline{y}).$$

Thus g is a pseudo rank function. We have for $x \in R$, $g \circ \phi(x) = g(0) = f(0) = 0 = f(x)$ if $x \in I$. If $x \notin I$, then $g \circ \phi(x) = g(x) = f(x)$. Thus $g \circ \phi = f$.
Clearly, g is unique.
We note that if $x \neq 0$ in R/I then $g(x) > 0$ iff $x \notin I$ iff $x \notin ker(f)$. Thus $I = ker(f)$. Conversely, if $I = ker(f)$, then $g(x) > 0$ for $x \neq 0$. Thus g is a rank function iff $I = ker(f)$. $\square$

Lemma 10. *If f, g are pseudo-rank functions on a Rickart *-ring R such that $f(e) \leq g(e)$ for all projections $e \in R$, then $f = g$.*

Proof. If $f \neq g$, then $f(x) < g(x)$ for some $x \in R$. This implies $f(RP(x)) < g(RP(x))$. Put $e = RP(x)$. Using $f(1 - e) \leq g(1 - e)$ we get

$$1 = f(1) = f(e + (1 - e)) = f(e) + f(1 - e) < g(e) + g(1 - e) = g(1) = 1,$$

a contradiction. $\square$

We recall some terms from Birkhoff [1], p. 5. The *length* of a poset P is defined as the least upper bound of the lengths of the chains in P and it is denoted by

$l(P)$. If $l(P)$ is finite then P is said to be of *finite length*. Let P be a poset of finite length and with 0 and $a \in P$. The *height* of a, denoted by $h(a)$, is defined as the least upper bound of all chains in $[0, a]$. It is known that in a modular lattice $h(a \vee b) + h(a \wedge b) = h(a) + h(b)$. The following two results are from Janowitz [7].

Theorem 2. *Every interval $[0, x]$ of a Rickart *-ring is an orthomodular lattice.*

Lemma 11. *Let R be a Rickart *-ring, $x \in R$. The interval $[0, x]$ is orthoisomorphic to $\{e \in C(x^*x) | e \leq x^{''}\}$, where $C(x^*x)$ dentes the set of all projections $f \in R$ which commute with x^*x. Moreover, $[0, x]$ is orthoisomorphic to $[0, x^*]$.*

In the notations of Berberian [2], we have $x^{''} = RP(x)$.

Theorem 3. *Let R be a Rickart *-ring, considered as a poset, of finite length and in which each projection is central. Then the function N on R defined by $N(x) = inf\{m/n : m, n \in \mathbb{Z}, m > 0, n > 0$ and $nh(x) \leq mh(1)\}$ is a pseudo-rank function on R.*

Proof. For all $x \in R$, put $N(x) = inf\{m/n : m, n \in \mathbb{Z} > 0 \, \text{and} \, nh(x) \leq mh(1)\}$. From Lemma 11, we get $h(x) = h(x^*) = h(RP(x))$. Clearly, $h(RP(x)) \leq h(1)$ and so $h(x) \leq h(1)$. Hence $0 \leq N(x) \leq 1$. Suppose, $N(1) < 1$. Then there exist positive integers m, n, $m < n$ such that $nh(1) \leq mh(1)$ but this is not possible as $h(1)$ is a positive integer. Thus $N(1) = 1$. Let $x, y \in R$. Then $h(xy) = h(RP(xy))$. By Lemma on p. 137, from Berberian [2], $RP(xy) \leq RP(y)$. Hence $h(RP(xy)) \leq h(RP(y))$. Also we get $LP(xy) \leq LP(x)$. Since all projections are central, $LP(a) = RP(a)$ for all $a \in R$. Thus $RP(xy) \leq RP(x)$. Let m, n be any positive integers such that $nh(x) \leq mh(1)$.Then $nh(xy) \leq mh(1)$ and so $N(xy) \leq m/n$. Thus $N(xy) \leq N(x)$. Similarly, $N(xy) \leq N(y)$. Let $x, y \in R$ be such that $x \perp y$. This implies $RP(x) \perp RP(y)$ and so $RP(x) \vee RP(y) = RP(x) + RP(y)$. We have $h(x + y) = h(RP(x + y))$. By Lemma on p.137 from Berberian, [2],

$$RP(x + y) \leq RP(x) \vee RP(y) = RP(x) + RP(y).$$

As all projections are central, the lattice of projections in R is distributive. Hence $h(RP(x) \vee RP(y)) = h(RP(x)) + h(RP(y))$. Thus $h(x + y) \leq h(RP(x)) + h(RP(y)) = h(x) + h(y)$. Let $\epsilon > 0$ be given. Then there exist positive integers m, n, s, t such that $nh(x) \leq mh(1)$ and $th(y) \leq sh(1)$ and $m/n < N(x) + \epsilon/2$ and $s/t < N(y) + \epsilon/2$. Then $nth(x + y) = nth(x) + nth(y) \leq mth(1) + nsh(1) = (mt + ns)h(1)$. Hence

$$N(x + y) \leq (m/n) + (s/t) < N(x) + N(y) + \epsilon.$$

Thus $N(x + y) \leq N(x) + N(y)$. If $N(x + y) < N(x) + N(y)$, then there exist positive integers m, n, s, t such that $N(x) > m/n$ and $N(y) > s/t$, while $N(x+y) < (m/n) + (s/t)$. Consequently, there exist positive integers a, b such that $bh(x+y) \leq ah(1)$ and $a/b < (m/n) + (s/t)$. Then $ant < (mt + ns)b$, hence $anth(x + y) \leq (mt + ns)bh(x + y) \leq a(mt + ns)h(1)$. Since $N(x) > m/n$, we have $nh(x) \nleq mh(1)$. Hence $mh(1) \leq nh(x)$. Now $anth(x) + anth(y) = anth(x + y) \leq a(mt + ns)h(1) \leq$

$anth(x) + ansh(y)$. Hence $anth(y) \leq ansh(1)$. But then $N(y) \leq ans/ant = s/t$, which is false. Thus $N(x + y) = N(x) + N(y)$.
Therefore N is a pseudo-rank function on R. □

4. Properties of the set of pseudo-rank functions

For a Rickart *-ring R, we denote the set of all pseudo-rank functions on R by $\mathbb{P}(R)$ (this set might be empty). We consider it as a subset of the real vector space $\mathbb{R}^R = \{f \mid f : R \to \mathbb{R}\}$ equipped with the product topology.
We note that $\mathbb{R}^R$ is a Housdorff topological vector space. The topology on $\mathbb{R}^R$ can be described in terms of convergence of nets. Given a net $\{f_i\}$ in $\mathbb{R}^R$ and some $f \in \mathbb{R}^R$, we have $f_i \to f$ if and only if $f_i(x) \to f(x)$ for all $x \in R$. A partial order can be defined on $\mathbb{R}^R$ by $f \leq g$ iff for each $x \in R$, $f(x) \leq g(x)$.
We recall some terms. Let E be a real vector space. A *convex combination* of points $x_1, \cdots, x_n \in E$ is any linear combination of the form $\alpha_1 x_1 + \cdots + \alpha_n x_n$ where $\alpha_i \in \mathbb{R}$ and $\alpha_i \geq 0$, $\sum_{i=1}^n \alpha_i = 1$. A *convex subset* of E is any subset $K \subseteq E$ such that for $0 \leq \alpha \leq 1$ and any $x, y \in K$, $\alpha x + (1 - \alpha)y \in K$. A *convex cone* in E is a subset $C \subseteq E$ such that $0 \in C$ and $\alpha x + \beta y \in C$ for all $x, y \in C$ and nonnegative real numbers α and β. A convex cone C is called *strict* if $C \cap (-C) = 0$. A subset A of E is called an *affine subspace* if it is closed under linear combinations of the form $\sum_{i=1}^n \alpha_i x_i$ where $x_i \in A$ and $\sum \alpha_i = 1$. A *hyperplane* in E is an affine subspace of the form $V + x$ such that V is a linear subspace of E of codimension 1. A *base* for a strict cone in C is a convex $K \subseteq E$ such that K is contained in a hyperplane not containing the origin and $C = \{\alpha x : x \in K \text{and} \, \alpha \geq 0\}$.

Proposition 2. *For a Rickart *-ring R, $\mathbb{P}(R)$ is a compact convex subset of $\mathbb{R}^R$.*

Proof. Clearly $\mathbb{P}(R)$ is a convex set.
We note that $\mathbb{P}(R)$ is contained in $W = [0, 1]^R$ which is compact by Tichonoff's theorem. Thus it is sufficient to show that $\mathbb{P}(R)$ is closed in W. Let N_i be a net in $\mathbb{P}(R)$ which converges to some $N \in W$. Since $N_i(1) \to N(1)$ we have $N(1) = 1$. For $x \in R$ we have $N_i(x) \to N(x)$ and $N_i(x^*) \to N(x^*)$, $N_i(x) = N_i(x^*)$ imply $N(x) = N(x^*)$. Also $N_i(xy) \leq N_i(x)$ for all i implies $N(xy) \leq N(x)$. Similarly $N(xy) \leq N(y)$ and if $x \perp y$ then $N(x+y) = N(x) + N(y)$ and $N(x) = N(RP(x)) = N(LP(x))$. Thus $N \in \mathbb{P}(R)$ so $\mathbb{P}(R)$ is closed in W. □

A convex subset F of a convex set K is called a *face* of K if for $x, y \in K$ and $0 < \alpha < 1$, $\alpha x + (1 - \alpha)y \in F$ implies $x, y \in F$.

Lemma 12. *Let R be a Rickart *-ring and $X \subseteq R$. Then the set $F = \{N \in \mathbb{P}(R) \mid N(x) = 0$ for all $x \in X\}$ is a closed face of $\mathbb{P}(R)$.*

Proof. Let N_i be a net in F converging to some $N \in \mathbb{P}(R)$. Clearly for all $x \in R$, $N_i(x) = 0$ for each i and so $N(x) = 0$. Thus $N \in F$. i. e. F is a closed subset of $\mathbb{P}(R)$. If $0 < \alpha < 1$, then for any $P, Q \in F$, $[\alpha P + (1 - \alpha)Q](x) = 0$. Thus F is convex. Suppose that for some α, $0 < \alpha < 1$ and for some $P, Q \in \mathbb{P}(R)$, $\alpha P + (1 - \alpha)Q = N \in F$. For all $x \in X$, we have $P(x) \leq \alpha^{-1}([\alpha P + (1 - \alpha)Q](x)) = \alpha^{-1}N(x) = 0$.

Thus $P(x) = 0$, i. e. $P \in F$. Similarly $Q \in F$. Therefore, F is a face of $\mathbb{P}(R)$. $\square$

A *-ring R with unity is called *factorial* if 0 and 1 are the only central projections in R, see; Berberian [2], p.36.

Lemma 13. *Let R be a Rickart *-ring. Let $f \in \mathbb{P}(R)$ be such that $ker(f) = \{0\}$. If f is an extreme point of $\mathbb{P}(R)$ then R is factorial.*

Proof. Let $z \in R$ be a nonzero central projection in R and $z \neq 1$. Then $1 - z$ is also a central projection. We have $f(z) > 0$, $f(1 - z) > 0$ and $f(z) + f(1 - z) = 1$. By Lemma 4(b) there exist pseudo-rank functions g_1 and g_2 on R such that $g_1(z) = 1$, $g_2(1 - z) = 1$ and $f(y) = f(z)g_1(y) + f(1 - z)g_2(y)$ for all $y \in R$. Since $g_1(z) = 1$ implies $g_1(1 - z) = 0$ we get $g_1 \neq g_2$. Thus f is a convex combination of distinct pseudo-rank functions in $\mathbb{P}(R)$. This contradicts the assumption that f is an extreme point. $\square$

A *-ring is said to satisfy the *general comparability*, (GC), if for any pair of projections $e, f \in R$ there exists a central projection h such that $he \precsim hf$ and $h(1 - f) \precsim h(1 - e)$, see; Berberian [2], p.77.

Lemma 14. *Let R be a Rickart *-ring with the generalized comparability and $f \in \mathbb{P}(R)$. If f is an extreme point of $\mathbb{P}(R)$ then $R/ker(f)$ is factorial.*

Proof. Let $K = ker(f)$ and $\phi : R \to R/K$ be the natural homomorphism. By Lemma 9 there exists a rank function g on R/K such that $g \circ \phi = f$. Suppose there exists a nontrivial central projection $e \in R/K$. We have $g(e) > 0$, $g(1 - e) > 0$ and $g(e) + g(1 - e) = 1$. By Lemma 4(b) there exist pseudo-rank functions g_1 and g_2 on R/K such that $g_1(e) = 1$, $g_2(1 - e) = 1$ and $g = g(u)g_1 + g(1 - u)g_2$. Since $g_1(e) = 1$ implies $g_1(1 - e) = 0$, we get $g_1 \neq g_2$. By Proposition 5, p.141 from Berberian [2] there exists a central projection $h \in R$ such that $\overline{h} = e$. Then $g_1 \circ \phi(h) = g_1(e) = 1$ and $g_2 \circ \phi(h) = g_2(e) = 0$ show that $g_1 \circ \phi \neq g_2 \circ \phi$. Thus $f = g(u)[g_1 \circ \phi] + g(1 - u)[g_2 \circ \phi]$ is a convex combination of distinct pseudo-rank functions in $\mathbb{P}(R)$. This contradicts the assumption that f is an extreme point. $\square$

Theorem 4. *Let R be a Rickart *-ring with the generalized comparability. Let $P \in \mathbb{P}(R)$. Consider the following statements.*

(1) *P is an extreme point of $\mathbb{P}(R)$.*

(2) *$B(R) \cap ker(P)$ is a maximal ideal of $B(R)$ where $B(R)$ is the Boolean algebra of all central projections in R.*

(3) *$ker(P)$ is a prime strict ideal of R.*

(4) *The set of strict ideals of the Rickart *-ring $R/ker(P)$ is linearly ordered.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

Proof. (1) $\Rightarrow$ (2) Let $ker(P) = K$. Since K is a proper strict ideal, $B(R) \cap K$ is a proper ideal of $B(R)$. If $B(R) \cap K$ is not maximal, then there exists an ideal J of $B(R)$ such that $B(R) \cap K \subseteq J$. Let $e \in J$ but $e \notin B(R) \cap K$. Clearly,

$1-e \notin B(R)\cap K$. Then $\overline{e}$ is a nontrivial central projection in R/K. This contradicts Lemma 14.
(2) $\Leftrightarrow$ (3) and (3) $\Leftrightarrow$ (4) follow from Proposition 1.2 of Thakare and Nimbhorkar [13]. □

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