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Geometric Means of Positive Operators

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ABSTRACT. Based on Ricatti equation $XA^{-1}X = B$ for two (positive invertible) operators A and B which has the geometric mean $A\sharp B$ as its solution, we consider a cubic equation

$$X(A\sharp B)^{-1}X(A\sharp B)^{-1}X = C$$

for A, B and C. The solution $X = (A\sharp B)\sharp_{\frac{1}{3}}C$ is a candidate of the geometric mean of the three operators. However, this solution is not invariant under permutation unlike the geometric mean of two operators. To supply the lack of the property, we adopt a limiting process due to Ando-Li-Mathias. We define reasonable geometric means of k operators for all integers $k \geq 2$ by induction. For three positive operators, in particular, we define the weighted geometric mean as an extension of that of two operators.

1. Introduction

The quadratic equation

$$XA^{-1}X = B$$

for (positive invertible) operator A and B on a Hilbert space H is said the Ricatti equation [17], which has a unique solution

$$A\sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$$

and this operator is defined as the geometric mean of A and B [2]. As its extension, the weighted geometric (or α-power) mean $A\sharp_\alpha B$ for $0 \leq \alpha \leq 1$ is defined [15] by

$$A\sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}}. \tag{1.1}$$

The geometric means for more than two operators have been already defined by many authors [1], [3], [8], [14], etc. We here want to introduce a new geometric mean, extending the notion of the Ricatti equation. Consider the equation

$$X(A\sharp B)^{-1}X(A\sharp B)^{-1}X = C$$

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for operator A, B and C. Then we easily have a unique solution which is given by

$$X = (A\sharp B)\sharp_{\frac{1}{3}} C. \tag{1.2}$$

If A, B, C commute with each other, then the above operator X is reduced to $(ABC)^{1/3}$, so that it seems a candidate of the geometric mean. But the operator lacks the property P3 below, i.e., permutation invariance for $k = 3$ to be a reasonable geometric mean. The following properties were postulated for a geomtric mean $G(A_1, A_2, \cdots, A_k) = G_k(A_1, A_2, \cdots, A_k)$ of k operators in [3]:

P1 Consistency with scalars. If $A_1, A_2, \cdots, A_k$ commute then

$$G(A_1, A_2, \cdots, A_k) = (A_1 A_2 \cdots A_k)^{\frac{1}{k}}.$$

P1′ This implies $G(\overbrace{A, \cdots, A}^{k}) = A$.

P2 Joint homogeneity.
$G(a_1A_1, a_2A_2, \cdots, a_kA_k) = (a_1a_2 \cdots a_k)^{\frac{1}{k}} G(A_1, A_2, \cdots, A_k)$ for $a_i \geq 0$ with $i = 1, \cdots, k$.

P2′ This implies $G(aA_1, aA_2, \cdots, aA_k) = aG(A_1, A_2, \cdots, A_k)$ $(a \geq 0)$.

P3 Permutation invariance. For any permutation $\pi(A_1, A_2, \cdots, A_k)$ of $(A_1, A_2, \cdots, A_k)$, $G(A_1, A_2, \cdots, A_k) = G(\pi(A_1, A_2, \cdots, A_k))$.

P4 Monotonicity. The map $(A_1, A_2, \cdots, A_n) \mapsto G(A_1, A_2, \cdots, A_n)$ is monotone, i.e., if $A_i \geq B_i$ for $i = 1, \cdots, k$, then $G(A_1, A_2, \cdots, A_k) \geq G(B_1, B_2, \cdots, B_k)$.

P5 Continuity from above. If $\{A_1^{(n)}\}, \{A_2^{(n)}\}, \cdots, \{A_k^{(n)}\}$ are monotonic decreasing sequences converging to $A_1, A_2, \cdots, A_k$, respectively, then $\{G(A_1^{(n)}, A_2^{(n)}, \cdots, A_k^{(n)})\}$ converges to $G(A_1, A_2, \cdots, A_k)$.

P6 Congruence invariance. For any invertible S,

$$G(S^*A_1S, S^*A_2S, \cdots, S^*A_kS) = S^*G(A_1, A_2, \cdots, A_k)S.$$

P7 Joint concavity. The map $(A_1, A_2, \cdots, A_k) \mapsto G(A_1, A_2, \cdots, A_k)$ is jointly concave:

$$G(\lambda A_1 + (1-\lambda)A_1', \lambda A_2 + (1-\lambda)A_2', \cdots, \lambda A_k + (1-\lambda)A_k')$$

$$\geq \lambda G(A_1, A_2, \cdots, A_k) + (1-\lambda)G(A_1', A_2', \cdots, A_k') \quad (0 < \lambda < 1).$$

P8 Self-duality. $G(A_1, A_2, \cdots, A_k)^* = G(A_1, A_2, \cdots, A_k)$. The dual $G(A_1, A_2, \cdots, A_k)^*$ is defined by

$$G(A_1, A_2, \cdots, A_k)^* = G(A_1^{-1}, A_2^{-1}, \cdots, A_k^{-1})^{-1}.$$

P9 (*In case $A_1, A_2, \cdots, A_k$ are matrices.*) Determinant identity.

$$\det G(A_1, A_2, \cdots, A_k) = (\det A_1 \cdot \det A_2 \cdot \cdots \cdot \det A_k)^{\frac{1}{k}}.$$

P10 The arithmetic-geometric-harmonic mean inequaility.

$$\frac{A_1 + A_2 + \cdots + A_k}{k} \geq G(A_1, A_2, \cdots, A_k) \geq \Big(\frac{A_1^{-1} + A_2^{-1} + \cdots + A_k^{-1}}{k}\Big)^{-1}.$$

Mentioned as before, we can show that the operator defined by (1.2) does not satisfy the basic property P3. Supplying this fact, we employ the iteration technique due to Ando-Li-Mathias [3]: We define the three sequences $\{A_n\}, \{B_n\}$ and $\{C_n\}$, for example, by $A_1 = A,\ B_1 = B,\ C_1 = C$,

$$\begin{aligned} A_{n+1} &= (B_n \sharp C_n)\sharp_{\frac{1}{3}} A_n = A_n \sharp_{\frac{2}{3}} (B_n \sharp C_n) \quad \text{(see WG3)},\\ B_{n+1} &= B_n \sharp_{\frac{2}{3}} (C_n \sharp A_n) \quad \text{and}\\ C_{n+1} &= C_n \sharp_{\frac{2}{3}} (A_n \sharp B_n) \quad \text{for } n \geq 1. \end{aligned}$$

Then we see that $\{A_n\}, \{B_n\}$ and $\{C_n\}$ converge and have a common limit. Now we may define the limit as the desired geometric mean of the operators A, B and C, which posseses all properties P1-P10. In this paper we define a geometric mean, somewhat different from that presented in [3] of k (≥ 3) operators which satisfies the above properties P1-P10. For three operators we define a weighted geometric mean of operators, which really extends that of two operators.

All operators are assumed *positive invertible* if stated otherwise.

2. Definition of geometric mean of more than two operators

Before we define a geometric mean of k operators, we want to state some useful facts for our discussion. First we introduce the metric $d(A, B)$, called Thompson metric on the positive cone Ω of all (positive invertible) operators defined ([18], [4], [5], [7]) by

$$d(A, B) = \max\{\log M(A/B),\ \log M(B/A)\} \quad \text{for } A, B \in \Omega,$$

where

$$M(A/B) = \inf\{\lambda > 0 : A \leq \lambda B\} = \| B^{-1/2} A B^{-1/2} \| .$$

We remark that Ω is complete with respect to the corresponding metric topology [7]. As a basic inequality between weighted geometric means of two operators, the

following holds [4]:
WG0 $d(A_1\sharp_\alpha A_2, B_1\sharp_\alpha B_2) \le (1-\alpha)d(A_1,B_1) + \alpha d(A_2,B_2)$
for $A_1, A_2, B_1, B_2 \in \Omega$ and $\alpha \in (0,1)$. Next for convenience sake, we, parallel to P1-P10, state basic properties of the weighted geometric mean $A\sharp_\alpha B$ defined by (1.1) [2], [11]:

WG1 $A\sharp_\alpha A = A$.

WG2 $(aA)\sharp_\alpha(bB) = a^{1-\alpha}b^\alpha A\sharp_\alpha B$.

WG3 $A\sharp_\alpha B = B\sharp_{1-\alpha} A$.

WG4 $A\sharp_\alpha B$ is monotone, i.e., if $A \ge C$ and $B \ge D$, then $A\sharp_\alpha B \ge C\sharp_\alpha D$.

WG5 The map $(A,B) \mapsto A\sharp_\alpha B$ is continuous from above.

WG6 $A\sharp_\alpha B$ is invariant with respect to congruence, i.e.,
$S^*(A\sharp_\alpha B)S = S^*AS\sharp_\alpha S^*BS$ for any invertible operator S.

WG7 The map $(A,B) \mapsto A\sharp_\alpha B$ is jointly concave.

WG8 $A\sharp_\alpha B$ is self-dual, i.e., $A\sharp_\alpha B = (A^{-1}\sharp_\alpha B^{-1})^{-1}$.

WG9 (*In case A and B are matrices,*) $\det(A\sharp_\alpha B) = (\det A)^{1-\alpha}(\det B)^\alpha$.

WG10 The weighted arithmetic-geometric-harmonic mean inequality holds:

$$(1-\alpha)A + \alpha B \ge A\sharp_\alpha B \ge ((1-\alpha)A^{-1} + \alpha B^{-1})^{-1}.$$

The following fact [12, (11)] is also useful, so we add it to the above properties:

WG11 $A\sharp_\alpha(A\sharp_\beta B) = A\sharp_{\alpha\beta} B$.

Now we want to define our geometric means for all integers $k \ge 2$.

Definition 2.1. (1) First for $k = 2$, define $G(A_1, A_2) = A_1\sharp A_2$ (the usual geometric mean) for two operators A_1 and A_2. Then $G(A_1, A_2)$ satisfies all properties P1-P10, and moreover, from WG0, between two geometric means $G(A_1, A_2)$ and $G(B_1, B_2)$ the following inequality holds:

$$d(G(A_1,A_2), G(B_1,B_2)) \le \frac{1}{2}(d(A_1,B_1) + d(A_2,B_2)). \tag{2.1}$$

To define geometric means for $k \ge 3$ by induction, we assume that for k operators $A_1, \cdots, A_k$ we have obtained a geometric mean $G(A_1, \cdots, A_k)$ such that the mean satisfies all properties P1-P10, and the following inequality, for another k-tuple of operators $B_1, \cdots, B_k$, holds:

$$d(G(A_1, \cdots, A_k),\ G(B_1, \cdots, B_k)) \leq \frac{1}{k} \sum_{i=1}^{k} d(A_i, B_i). \tag{2.2}$$

(2) Then we shall define a geometric mean $G(A_1, \cdots, A_{k+1})$ of $(k+1)$ operators $A_1, \cdots, A_{k+1}$ as the common limit of the sequences $\{A_i^{(r)}\}_{r=1}^{\infty}$ $(i = 1, \cdots, k+1)$ defined by

$$A_i^{(1)} = A_i \ \text{ and } \ A_i^{(r+1)} = A_i^{(r)} \sharp_{\frac{k-1}{k}} G((A_j^{(r)})_{j \neq i}) \text{ for } r \geq 1. \tag{2.3}$$

Here

$$G((A_j^{(r)})_{j \neq i}) = G(A_1^{(r)}, \cdots, A_{i-1}^{(r)}, A_{i+1}^{(r)}, \cdots, A_{k+1}^{(r)}).$$

Now we have to show:

Theorem 2.2. *For a fixed $k \geq 2$, assume $G(A_1, \cdots, A_k)$ is defined and satisfies P1-P10 and* (2.2). *Then the $(k+1)$ sequences $\{A_i^{(r)}\}_{r=1}^{\infty}$ $(i = 1, \cdots, k+1)$ defined by* (2.3) *are convergent in the Thompson metric and have a common limit. The limit defined as the geometric mean $G(A_1, \cdots, A_{k+1})$ satisfies all properties P1-P10 and the inequality* (2.2) *for $(k+1)$ operators $A_1, \cdots, A_{k+1}$.*

Proof. First for $d(A_i^{(r+1)}, A_i^{(r)})$ $(i = 1, \cdots, k+1)$, we have

$$d(A_i^{(r+1)}, A_i^{(r)}) \leq \frac{1}{k+1} \sum_{j=1, j \neq i}^{k+1} d(A_i^{(r)}, A_j^{(r)}). \tag{2.4}$$

In fact, from P1', P3 and (2.2)

$$\begin{aligned}
d(A_i^{(r+1)}, A_i^{(r)}) &= d(A_i^{(r)} \sharp_{\frac{k}{k+1}} G((A_j^{(r)})_{j \neq i}), A_i^{(r)}) \\
&= d(A_i^{(r)} \sharp_{\frac{k}{k+1}} G(A_1^{(r)}, \cdots, A_{i-1}^{(r)}, A_{i+1}^{(r)}, \cdots, A_{k+1}^{(r)}), A_i^{(r)} \sharp_{\frac{k}{k+1}} G(A_i^{(r)}, \cdots, A_i^{(r)})) \\
&\leq \frac{1}{k+1} d(A_i^{(r)}, A_i^{(r)}) + \frac{k}{k+1} \cdot \frac{1}{k} \Big(d(A_1^{(r)}, A_i^{(r)}) + d(A_2^{(r)}, A_i^{(r)}) \\
&\qquad + \cdots + d(A_{i-1}^{(r)}, A_i^{(r)}) + d(A_{i+1}^{(r)}, A_i^{(r)}) + \cdots + d(A_{k+1}^{(r)}, A_i^{(r)}) \Big) \\
&= \frac{1}{k+1} \sum_{j=1, j \neq i}^{k+1} d(A_j^{(r)}, A_i^{(r)}) = \frac{1}{k+1} \sum_{j=1, j \neq i}^{k+1} d(A_i^{(r)}, A_j^{(r)}).
\end{aligned}$$

For $d(A_i^{(r+1)}, A_j^{(r+1)})$ $(i \neq j)$, say, for $i < j$, we have

$$\begin{aligned}
&d(A_i^{(r+1)}, A_j^{(r+1)}) = d(A_i^{(r)} \sharp_{\frac{k}{k+1}} G((A_l^{(r)})_{l\neq i}), A_j^{(r)} \sharp_{\frac{k}{k+1}} G((A_l^{(r)})_{l\neq j})) \\
&= d(A_i^{(r)} \sharp_{\frac{k}{k+1}} G(A_j^{(r)}, A_1^{(r)}, \cdots, A_{i-1}^{(r)}, A_{i+1}^{(r)}, \cdots, A_{j-1}^{(r)}, A_{j+1}^{(r)}, \cdots, A_{k+1}^{(r)}), \\
&\qquad A_j^{(r)} \sharp_{\frac{k}{k+1}} G(A_i^{(r)}, A_1^{(r)}, \cdots, A_{i-1}^{(r)}, A_{i+1}^{(r)}, \cdots, A_{j-1}^{(r)}, A_{j+1}^{(r)}, \cdots, A_{k+1}^{(r)})) \\
&\leq \frac{1}{k+1} d(A_i^{(r)}, A_j^{(r)}) + \frac{k}{k+1} \cdot \frac{1}{k} d(A_j^{(r)}, A_i^{(r)}) = \frac{2}{k+1} d(A_i^{(r)}, A_j^{(r)}).
\end{aligned}$$

Hence we have

$$\begin{aligned}
(2.5) \qquad & d(A_i^{(r+1)}, A_j^{(r+1)}) \leq \frac{2}{k+1} d(A_i^{(r)}, A_j^{(r)}) \\
&\leq \left(\frac{2}{k+1}\right)^2 d(A_i^{(r-1)}, A_j^{(r-1)}) \leq \cdots \leq \left(\frac{2}{k+1}\right)^r d(A_i, A_j).
\end{aligned}$$

From (2.4) and the above inequality, we then have

$$d(A_i^{(r+1)}, A_i^{(r)}) \leq \frac{1}{k+1} \sum_{j=1, j\neq i}^{k+1} \left(\frac{2}{k+1}\right)^{r-1} d(A_i, A_j) = \left(\frac{2}{k+1}\right)^{r-1} K_i,$$

where $K_i = \frac{1}{k+1} \sum_{j=1, j\neq i}^{k+1} d(A_i, A_j)$. Hence for any r, s such that $r \geq s$

$$\begin{aligned}
&d(A_i^{(r+1)}, A_i^{(s)}) \leq \sum_{l=s}^{r} d(A_i^{(l+1)}, A_i^{(l)}) \\
&\leq \sum_{l=s}^{r} \left(\frac{2}{k+1}\right)^{l-1} K_i \leq \frac{(k+1)K_i}{k-1} \left(\frac{2}{k+1}\right)^{s-1} \to 0 \text{ (as } s \to \infty).
\end{aligned}$$

This implies that $\{A_i^{(r)}\}$ (for all $i = 1, \ldots, k+1$) are convergent, and then from (2.5) their limits are identical. For the properties P1-P10 of $G(A_1, \cdots, A_{k+1})$, it is not difficult to show them. For example, to see P3, let

$$(B_1, \cdots, B_{k+1}) = \pi(A_1, \cdots, A_{k+1}) \ (= (A_{\pi(1)}, \cdots, A_{\pi(k+1)}))$$

be a permutation of $(A_1, \cdots, A_{k+1})$. Put for all $i = 1, \cdots, k+1$

$$B_i^{(1)} = B_i = A_{\pi(i)} \text{ and } B_i^{(r+1)} = B_i^{(r)} \sharp_{\frac{k}{k+1}} G((B_j^{(r)})_{j\neq i}) \text{ for } r \geq 1.$$

Then we can see that

$$B_i^{(r)} = A_{\pi(i)}^{(r)} \text{ for } r \geq 1, \quad (A_{\pi(i)}^{(r)} \text{ is defined before by (2.3)}),$$

so that $(B_1^{(r)}, \cdots, B_{k+1}^{(r)})$ is a rearrangement of $(A_1, \cdots, A_{k+1})$. Hence all sequences $\{B_i^{(r)}\}_{r=1}^{\infty}$ are convergent and have the common limit $G(A_1, \cdots, A_{k+1})$. This implies the desired $G(\pi(A_1, \cdots, A_{k+1})) = G(A_1, \cdots, A_{k+1})$.

For the inequality (2.2) for two tuples $(A_1, \cdots, A_{k+1})$ and $(B_1, \cdots, B_{k+1})$ of operators, let $\{A_i^{(r)}\}$ $(i = 1, \cdots, k+1)$ be the sequences defined by (2.3) and $\{B_i^{(r)}\}$ $(i = 1, \cdots, k+1)$ be those defined by $B_i^{(1)} = B_1$ and

$$B_i^{(r+1)} = B_i^{(r)} \sharp_{\frac{k}{k+1}} G((B_j^{(r)})_{j \neq i}) \quad \text{for } r \geq 1.$$

Then by the assumption (2.2), we have, for all $i = 1, \cdots, k+1$

$$\begin{aligned} d(A_i^{(r+1)}, B_i^{(r+1)}) &= d(A_i^{(r)} \sharp_{\frac{k}{k+1}} G((A_j^{(r)})_{j \neq i}), B_i^{(r)} \sharp_{\frac{k}{k+1}} G((B_j^{(r)})_{j \neq i})) \\ &\leq \frac{1}{k+1} d(A_i^{(r)}, B_i^{(r)}) + \frac{1}{k+1} \sum_{j=1, j \neq i}^{k+1} d(A_j^{(r)}, B_j^{(r)}) = \frac{1}{k+1} \sum_{j=1}^{k+1} d(A_j^{(r)}, B_j^{(r)}). \end{aligned}$$

Thus we have

$$\sum_{i=1}^{k+1} d(A_i^{(r+1)}, B_i^{(r+1)}) \leq \sum_{i=1}^{k+1} d(A_i^{(r)}, B_i^{(r)}) \leq \cdots \leq \sum_{i=1}^{k+1} d(A_i, B_i).$$

Taking the limit as $r \to \infty$, we obtain

$$(k+1) d(G(A_1, \cdots, A_{k+1}), G(B_1, \cdots, B_{k+1})) \leq \sum_{i=1}^{k+1} d(A_i, B_i),$$

which is the desired. □

Other definitions of geometric means

1. Ando-Li-Mathias [3] defined a geometric mean $G_A = G_A(A_1, \cdots, A_k)$ having all properties P1-P10 as follows: For $k = 2$, put $G_A = A_1 \sharp A_2$. Assuming G_A for $k (\geq 2)$ operators, by induction, define $G_A(A_1, \cdots, A_{k+1})$ as the common limit of the $(k+1)$ sequences $\{A_i^{(r)}\}_{r=1}^{\infty}$ $(i = 1, \cdots, k+1)$ such that $A_i^{(1)} = A_i$ and

$$A_i^{(r+1)} = G_A((A_j^{(r)})_{j \neq i}) \quad \text{for } r \geq 1.$$

2. Kosaki [14] defined a geometric mean $G_K = G_K(A_1, \cdots, A_k)$ as follows: First define

$$\text{(2.6)} \quad {G_K}^+ = G_K^+(A_1, \cdots, A_k) :$$
$$= \frac{1}{(\Gamma(1/k))^k} \int_{\Lambda_k} \left\{ \sum_{j=1}^k \lambda_j A_j^{-1} \right\}^{-1} \left\{ \sum_{j=1}^k \lambda_j \right\} \left\{ \prod_{j=1}^k \lambda_j^{1/k-1} \right\} d\lambda_1 \cdots d\lambda_k,$$

where $\Lambda_k = \left\{(\lambda_1,\cdots,\lambda_k); \lambda_j \geq 0\ (j=1,\cdots,k),\ \sum_{j=1}^k \lambda_j \leq 1\right\}$. Then define $G_K^- = (G_K^+)^*$, the dual of G_K^+ (see P8) and $G_K = G_K^+ \sharp G_K^-$. The function G_K^+ does not have the property P8-self-duality (nor P9), but G_K has P8 by the modification [3].

3. Anderson-Morley-Trapp [1] defined a geometric mean G_{amt} of $k(\geq 2)$ operators based on symmetric functions. For example, for three operators A, B and C, define $A_1^{(1)} = A, A_2^{(1)} = B, A_3^{(1)} = C,$ and for $r \geq 1$

$$\begin{aligned}
A_1^{(r+1)} &= (A_1^{(r)} + A_2^{(r)} + A_3^{(r)})/3, \\
A_2^{(r+1)} &= \{A_1^{(r)} : (A_2^{(r)} + A_3^{(r)}) + A_2^{(r)} : (A_3^{(r)} + A_1^{(r)}) + A_3^{(r)} : (A_1^{(r)} + A_2^{(r)})\}/2, \\
A_3^{(r+1)} &= 3((A_1^{(r)} : A_2^{(r)}) : A_3^{(r)}) = 3(A_1^{(r)} : A_2^{(r)} : A_3^{(r)}).
\end{aligned}$$

Here $X : Y = (X^{-1} + Y^{-1})^{-1}$ is the parallel sum of operators X and Y. Then the sequences $\{A_1^{(r+1)}\}, \{A_2^{(r+1)}\}, \{A_3^{(r+1)}\}$ converge to a common limit, which is denoted by $G_{\mathrm{amt}}^+ = G_{\mathrm{amt}}^+(A,B,C)$. The mean G_{amt} is defined by $G_{\mathrm{amt}} = G_{\mathrm{amt}}^+ \sharp G_{\mathrm{amt}}^-$, where $G_{\mathrm{amt}}^- = (G_{\mathrm{amt}}^+)^*$, the dual of G_{amt}^+. None of $G_{\mathrm{amt}}, G_{\mathrm{amt}}^+, G_{\mathrm{amt}}^-$ satisfiy P2-joint homogeneity [3].

Denote by G the geometric mean of our definition in Theorem 2.2. Then we compare $G_A, G_K, G_{\mathrm{amt}}$ and G by the following three 2×2 matrices: Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},\quad B = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then by numerical computation cited from [3] (less than 10^{-4} discarded),

$$G_A = \begin{bmatrix} 0.9319 & 0.6636 \\ 0.6636 & 1.5456 \end{bmatrix},\ G_K = \begin{bmatrix} 0.9320 & 0.6628 \\ 0.6628 & 1.5444 \end{bmatrix},$$

$$G_{\mathrm{amt}} = \begin{bmatrix} 0.9317 & 0.6608 \\ 0.6608 & 1.5419 \end{bmatrix}.$$

For our geometric mean G (by (2.3))

$$G = \begin{bmatrix} 0.9319 & 0.6618 \\ 0.6618 & 1.5431 \end{bmatrix} (= A_1^{(r)} = A_2^{(r)} = A_3^{(r)} \quad \text{for} \quad r \geq 3).$$

3. Weighted geometric means

We show two facts with respect to weighted geometric means. First we give

Proposition 3.1. *For any operators A, B and for any positive integers $\ell, m \geq 1$*

$$G_{\ell+m} = G_{\ell+m}(\overbrace{A,\cdots,A}^{\ell}, \overbrace{B,\cdots,B}^{m}) = A \sharp_{\frac{m}{\ell+m}} B. \tag{3.1}$$

Proof. First for $m = 1$, we obtain, by induction with respect to ℓ,

$$G_{\ell+1}(\overbrace{A, \cdots, A}^{\ell}, B) = A \sharp_{\frac{1}{\ell+1}} B. \tag{3.2}$$

In fact, for $\ell = 1$, (3.2) is obvious. Assuming (3.2) (for ℓ), we want to prove

$$G_{\ell+2}(\overbrace{A, \cdots, A}^{\ell+1}, B) = A \sharp_{\frac{1}{\ell+2}} B.$$

Put $A_i^{(1)} = A$ for $i = 1, \cdots, \ell+1$ and $A_{\ell+2}^{(1)} = B$, then

$$\begin{aligned} A_1^{(2)} &= A \sharp_{\frac{\ell+1}{\ell+2}} G_{\ell+1}(A_2^{(1)}, \cdots, A_{\ell+1}^{(1)}, A_{\ell+2}^{(1)}) \\ &= A \sharp_{\frac{\ell+1}{\ell+2}} G_{\ell+1}(\overbrace{A, \cdots, A}^{\ell}, B) = A \sharp_{\frac{\ell+1}{\ell+2}} (A \sharp_{\frac{1}{\ell+1}} B) = A \sharp_{\frac{1}{\ell+2}} B. \end{aligned}$$

Similarly, we have $A_2^{(2)} = \cdots = A_{\ell+1}^{(2)} = A \sharp_{\frac{1}{\ell+2}} B$. For $A_{\ell+2}^{(2)}$, we have

$$A_{\ell+2}^{(2)} = A_{\ell+2}^{(1)} \sharp_{\frac{\ell+1}{\ell+2}} G_{\ell+1}(A_2^{(1)}, \cdots, A_{\ell+1}^{(1)}) = B \sharp_{\frac{\ell+1}{\ell+2}} G(A, \cdots, A) = B \sharp_{\frac{\ell+1}{\ell+2}} A = A \sharp_{\frac{1}{\ell+2}} B.$$

Hence for $r \geq 3$, we have $A_i^{(r)} = A \sharp_{\frac{1}{\ell+2}} B$ for $i = 1, \cdots, \ell+2$, so that $G_{\ell+2} = A \sharp_{\frac{1}{\ell+2}} B$ as the limit of $A_i^{(r)}$ $(r \to \infty)$, which is desired.

Now we have obtained (3.1) for $m = 1$ and all $\ell \geq 1$. Assuming (3.1) for a fixed $m \geq 1$ and all $\ell \geq 1$, we then have to show, for all $\ell \geq 1$,

$$G_{\ell+m+1}(\overbrace{A, \cdots, A}^{\ell}, \overbrace{B, \cdots, B}^{m+1}) = A \sharp_{\frac{m+1}{\ell+m+1}} B. \tag{3.3}$$

For $\ell = 1$, we obtain

$$G_{m+2}(A, \overbrace{B, \cdots, B}^{m+1}) = G_{m+2}(\overbrace{B, \cdots, B}^{m+1}, A) = B \sharp_{\frac{1}{m+2}} A = A \sharp_{\frac{m+1}{m+2}} B.$$

Now assuming (3.3) (for ℓ), we want to prove

$$G_{\ell+m+2}(\overbrace{A, \cdots, A}^{\ell+1}, \overbrace{B, \cdots, B}^{m+1}) = A \sharp_{\frac{m+1}{\ell+m+2}} B.$$

Put $A_i^{(1)} = A$ for $i = 1, \cdots, \ell+1$ and $A_i^{(1)} = B$ for $i = \ell+2, \cdots, \ell+m+2$. Then for $i = 1, \cdots, \ell+1$

$$A_i^{(2)} = A \sharp_{\frac{\ell+m+1}{\ell+m+2}} G_{\ell+m+1}(\overbrace{A, \cdots, A}^{\ell}, \overbrace{B, \cdots, B}^{m+1}) = A \sharp_{\frac{\ell+m+1}{\ell+m+2}} (A \sharp_{\frac{m+1}{\ell+m+1}} B) = A \sharp_{\frac{m+1}{\ell+m+2}} B.$$

For $i = \ell+2, \cdots, \ell+m+2$

$$\begin{aligned} A_i^{(2)} &= B\sharp_{\frac{\ell+m+1}{\ell+m+2}} G_{\ell+m+1}(\overbrace{A, \cdots, A}^{\ell+1}, \overbrace{B, \cdots, B}^{m}) \\ &= B\sharp_{\frac{\ell+m+1}{\ell+m+2}}(A\sharp_{\frac{m}{\ell+m+1}} B) = B\sharp_{\frac{\ell+m+1}{\ell+m+2}}(B\sharp_{\frac{\ell+1}{\ell+m+1}} A) = B\sharp_{\frac{\ell+1}{\ell+m+2}} A = A\sharp_{\frac{m+1}{\ell+m+2}} B. \end{aligned}$$

Hence for $r \geq 2$ we have $A_i^{(r)} = A\sharp_{\frac{m+1}{\ell+m+2}} B$ for all $i = 1, \cdots, \ell+m+2$, which is desired. □

Next we define a weighted geometric mean of three positive operators. Let α, β, γ be real numbers such that

$$\alpha, \beta, \gamma \geq 0 \text{ and } \alpha + \beta + \gamma = 1. \tag{3.4}$$

Then for operators A, B, C and real numbers α, β, γ satisfying (3.4), we define the sequences $\{A_n\}, \{B_n\}$ and $\{C_n\}$ as follows:

$$A_1 = A, \quad B_1 = B, \quad C_1 = C \text{ and for } n \geq 1$$

$$\begin{cases} A_{n+1} = A_n\sharp_{\alpha_1}(B_n\sharp_{\alpha_2} C_n), \\ B_{n+1} = B_n\sharp_{\beta_1}(C_n\sharp_{\beta_2} A_n), \\ C_{n+1} = C_n\sharp_{\gamma_1}(A_n\sharp_{\gamma_2} B_n). \end{cases} \tag{3.5}$$

Here the above constants are:

$$\alpha_1 = 1-\alpha, \ \alpha_2 = 1 - \frac{\beta}{\alpha_1}, \ \beta_1 = 1-\beta, \ \beta_2 = 1 - \frac{\gamma}{\beta_1}, \ \gamma_1 = 1-\gamma, \ \gamma_2 = 1 - \frac{\alpha}{\gamma_1}$$

with a convention $\frac{y}{1-x} = 0$ for $x = 1$. These are the solutions for the equations

$$\begin{cases} \alpha = 1-\alpha_1 = \beta_1\beta_2 = \gamma_1(1-\gamma_2), \\ \beta = \alpha_1(1-\alpha_2) = 1-\beta_1 = \gamma_1\gamma_2, \\ \gamma = \alpha_1\alpha_2 = \beta_1(1-\beta_2) = 1-\gamma_1. \end{cases} \tag{3.6}$$

which are obtained by observing the exponents for commuting operators:

$$A_{n+1} = A_n^{1-\alpha_1} B_n^{\alpha_1(1-\alpha_2)} C_n^{\alpha_1\alpha_2},$$

$$B_{n+1} = B_n^{1-\beta_1} C_n^{\beta_1(1-\beta_2)} A_n^{\beta_1\beta_2},$$

$$C_{n+1} = C_n^{1-\gamma_1} A_n^{\gamma_1(1-\gamma_2)} B_n^{\gamma_1\gamma_2}.$$

Now we obtain a weighted geometric mean as the common limit of $\{A_n\}, \{B_n\}$ and $\{C_n\}$:

Theorem 3.2. *Let $\{A_n\}, \{B_n\}$ and $\{C_n\}$ be the sequences given by* (3.5) *for operators A, B, C and real numbers $\alpha_i,\ \beta_i,\ \gamma_i,\ \alpha,\ \beta,\ \gamma$ satisfying* (3.6)*. Then the sequences converge and have a common limit, which we denote by $G(A, B, C; \alpha, \beta, \gamma)$. Moreover, the limit is permutation invariant, that is,*

$$G(A, B, C; \alpha, \beta, \gamma) = G(\pi(A, B, C); \pi(\alpha, \beta, \gamma))$$

for any permutations $\pi(A, B, C),\ \pi(\alpha, \beta, \gamma)$ of $(A, B, C),\ (\alpha, \beta, \gamma)$, respectively.

Lemma 3.3. *Let $\{A_n\}, \{B_n\}$ and $\{C_n\}$ be the sequences given by* (3.5) *and $\alpha,\ \beta,\ \gamma,\ \alpha_i,\ \beta_i,\ \gamma_i\ (i = 1, 2)$ be real numbers satisfying* (3.6)*. Then*

$$\begin{cases} d(A_{n+1}, B_{n+1}) \le 2\min\{\alpha, \beta\} d(A_n, B_n) \le (2M)^n d(A, B), \\ d(B_{n+1}, C_{n+1}) \le 2\min\{\beta, \gamma\} d(B_n, C_n) \le (2M)^n d(B, C), \\ d(C_{n+1}, A_{n+1}) \le 2\min\{\gamma, \alpha\} d(C_n, A_n) \le (2M)^n d(C, A), \end{cases} \tag{3.7}$$

where

$$M = \max\{\min\{\alpha, \beta\}, \min\{\beta, \gamma\}, \min\{\gamma, \alpha\}\}, \tag{3.8}$$

or, is the second number of $\alpha,\ \beta,\ \gamma$ in size.

Proof. By the definition (3.5), we have

$$\begin{aligned} &d(A_{n+1}, B_{n+1}) = d(A_n \sharp_{\alpha_1} (B_n \sharp_{\alpha_2} C_n), B_n \sharp_{\beta_1} (C_n \sharp_{\beta_2} A_n)) \\ &\le d(A_n \sharp_{\alpha_1} (B_n \sharp_{\alpha_2} C_n), A_n \sharp_{\beta_1} (C_n \sharp_{\beta_2} A_n)) + d(A_n \sharp_{\beta_1} (C_n \sharp_{\beta_2} A_n), B_n \sharp_{\beta_1} (C_n \sharp_{\beta_2} A_n)) \\ &(:= I + II). \end{aligned}$$

Note that from WG3 and WG11,

$$A_n \sharp_{\beta_1} (C_n \sharp_{\beta_2} A_n) = A_n \sharp_{\beta_1} (A_n \sharp_{1-\beta_2} C_n) = A_n \sharp_{\beta_1(1-\beta_2)} C_n = A_n \sharp_{\alpha_1} (A_n \sharp_{\alpha_2} C_n).$$

Hence we have

$$\begin{aligned} &I := d(A_n \sharp_{\alpha_1} (B_n \sharp_{\alpha_2} C_n), A_n \sharp_{\beta_1} (C_n \sharp_{\beta_2} A_n)) = d(A_n \sharp_{\alpha_1} (B_n \sharp_{\alpha_2} C_n), A_n \sharp_{\alpha_1} (A_n \sharp_{\alpha_2} C_n)) \\ &\le (1 - \alpha_1) d(A_n, A_n) + \alpha_1 d(B_n \sharp_{\alpha_2} C_n, A_n \sharp_{\alpha_2} C_n) \le \alpha_1 (1 - \alpha_2) d(B_n, A_n) = \beta d(A_n, B_n). \\ &II := d(A_n \sharp_{\beta_1} (C_n \sharp_{\beta_2} A_n), B_n \sharp_{\beta_1} (C_n \sharp_{\beta_2} A_n)) \\ &\le (1 - \beta_1) d(A_n, B_n) + \beta_1 d(C_n \sharp_{\beta_2} A_n, C_n \sharp_{\beta_2} A_n) = \beta d(A_n, B_n). \end{aligned}$$

Hence

$$d(A_{n+1}, B_{n+1}) \le I + II \le 2\beta d(A_n, B_n). \tag{3.9}$$

Again

$$\begin{aligned} &d(A_{n+1}, B_{n+1}) = d(A_n \sharp_{\alpha_1} (B_n \sharp_{\alpha_2} C_n), B_n \sharp_{\beta_1} (C_n \sharp_{\beta_2} A_n)) \\ &\le d(A_n \sharp_{\alpha_1} (B_n \sharp_{\alpha_2} C_n), B_n \sharp_{\alpha_1} (B_n \sharp_{\alpha_2} C_n)) + d(B_n \sharp_{\alpha_1} (B_n \sharp_{\alpha_2} C_n), B_n \sharp_{\beta_1} (C_n \sharp_{\beta_2} A_n)) \\ &(:= III + IV). \end{aligned}$$

Note that

$$B_n\sharp_{\alpha_1}(B_n\sharp_{\alpha_2}C_n) = B_n\sharp_{\alpha_1\alpha_2}C_n = B_n\sharp_{\beta_1}(C_n\sharp_{\beta_2}B_n).$$

So that

$$\begin{aligned}
&III := d(A_n\sharp_{\alpha_1}(B_n\sharp_{\alpha_2}C_n), B_n\sharp_{\alpha_1}(B_n\sharp_{\alpha_2}C_n))\\
&\le (1-\alpha_1)d(A_n, B_n) + \alpha_1 d(B_n\sharp_{\alpha_2}C_n, B_n\sharp_{\alpha_2}C_n) = \alpha d(A_n, B_n).\\
&IV := d(B_n\sharp_{\alpha_1}(B_n\sharp_{\alpha_2}C_n), B_n\sharp_{\beta_1}(C_n\sharp_{\beta_2}A_n))\\
&= d(B_n\sharp_{\beta_1}(C_n\sharp_{\beta_2}B_n), B_n\sharp_{\beta_1}(C_n\sharp_{\beta_2}A_n))\\
&\le (1-\beta_1)d(B_n, B_n) + \beta_1((1-\beta_2)d(C_n, C_n) + \beta_2 d(B_n, A_n)) = \alpha d(A_n, B_n).
\end{aligned}$$

From the above inequalities we have

$$d(A_{n+1}, B_{n+1}) \le III + IV \le 2\alpha d(A_n, B_n). \tag{3.10}$$

Now from (3.9) and (3.10), we have $d(A_{n+1}, B_{n+1}) \le 2\min\{\alpha, \beta\} d(A_n, B_n)$, which implies the first inequalities of (3.7).

Similarly we can show the second and the third inequalities. □

Lemma 3.4. *Let $\{A_n\}, \{B_n\}$ and $\{C_n\}$ be sequences given by (3.5) and let $\alpha,\ \beta,\ \gamma,$ $\alpha_i, \beta_i, \gamma_i$ $(i = 1, 2)$ be real numbers satisfying (3.6). Then*

(3.11)

$$\begin{cases}
d(A_{n+1}, A_n) \le \beta d(A_n, B_n) + \gamma d(C_n, A_n) \le (2M)^{n-1}\{\beta d(A, B) + \gamma d(C, A)\},\\
d(B_{n+1}, B_n) \le \gamma d(B_n, C_n) + \alpha d(A_n, B_n) \le (2M)^{n-1}\{\gamma d(B, C) + \alpha d(A, B)\},\\
d(C_{n+1}, C_n) \le \alpha d(C_n, A_n) + \beta d(B_n, C_n) \le (2M)^{n-1}\{\alpha d(C, A) + \beta d(B, C)\},
\end{cases}$$

where M is defined in (3.8).

Proof. Using WG0, we have

$$\begin{aligned}
d(A_{n+1}, A_n) &= d(A_n\sharp_{\alpha_1}(B_n\sharp_{\alpha_2}C_n), A_n\sharp_{\alpha_1}(A_n\sharp_{\alpha_2}A_n))\\
&\le (1-\alpha_1)d(A_n, A_n) + \alpha_1 d(B_n\sharp_{\alpha_2}C_n, A_n\sharp_{\alpha_2}A_n)\\
&\le \alpha_1(1-\alpha_2)d(A_n, B_n) + \alpha_1\alpha_2 d(C_n, A_n)\\
&= \beta d(A_n, B_n) + \gamma d(C_n, A_n) \le (2M)^{n-1}\{\beta d(A, B) + \gamma d(C, A)\}.
\end{aligned}$$

In the same manner, we can obtain other inequalities in (3.11). □

Proof of Theorem 3.2. We may only consider the case that all of α, β and γ are nonzero or smaller than 1, so that we can assume $M < 1/2$. Then from Lemmas 3.3 and 3.4 we can show that the sequences $\{A_n\}, \{B_n\}$ and $\{C_n\}$ converge and have a common limit by using the similar argument as in the proof of Theorem 2.2. For the property of permutation invariance of the limit $G(A, B, C; \alpha, \beta, \gamma)$, we can also show the fact almost similarly as in the proof of Theorem 2.2. □

The following result implies that $G(A, B, C; \alpha, \beta, \gamma)$ is really an extension of the weighted mean of two operators:

Proposition 3.5. *Let A and B be positive operators, and let α, β, γ be real numbers satisfying* (3.4). *Then*

$$G(A, A, B; \alpha, \beta, \gamma) = A\sharp_\gamma B. \tag{3.12}$$

Proof. In (3.5), replace $A_1 = B_1 = A$ and $C_1 = B$, then

$$A_2 = A\sharp_{\alpha_1}(A\sharp_{\alpha_2}B) = A\sharp_{\alpha_1\alpha_2}B = A\sharp_\gamma B.$$

Similarly, we can obtain $B_2 = C_2 = A\sharp_\gamma B$, so that

$$A_n = B_n = C_n = A\sharp_\gamma B \quad \text{for } n \geq 2.$$

This implies the desired identity (3.12).

Kosaki [14] presented the following definition of a weighted geometric mean $\tilde{G}_K = \tilde{G}_K^+ \sharp \tilde{G}_K^-$. Here $\tilde{G}_K^+ = \tilde{G}_K^+(A_1, \cdots, A_k; \alpha_1, \cdots, \alpha_k)$ is defined as the extended form of (2.6):

$$\tilde{G}_K^+ = \frac{1}{\Pi_{j=1}^k \Gamma(\alpha_j)} \int_{\Lambda_k} \left\{ \sum_{j=1}^k \lambda_j A_j^{-1} \right\}^{-1} \left\{ \sum_{j=1}^k \lambda_j \right\} \left\{ \prod_{j=1}^k \lambda_j^{1/\alpha_j - 1} \right\} d\lambda_1 \cdots d\lambda_k,$$

and $\tilde{G}_K^- = (\tilde{G}_K^+)^*$, the dual of $\tilde{G}_K^+$.

For numerical computation of $\tilde{G}_K$ and $\tilde{G}$, the weighted geometric mean by our definition, take three matrices

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}.$$

Then for $\alpha = 1/2, \beta = 1/3, \gamma = 1/6$ we have, by the Simpson's formula for the integral on the interval $[0, 1]$ divided into 2×10^4 segments,

$$\tilde{G}_K = \begin{bmatrix} 1.6119 & 0.9374 \\ 0.9374 & 1.1655 \end{bmatrix} (= \tilde{G}_{K,1}).$$

From another computation by using Gauss-Legendre quadrature with 150 nodes, we have

$$\tilde{G}_K = \begin{bmatrix} 1.6118 & 0.9375 \\ 0.9375 & 1.1648 \end{bmatrix} (= \tilde{G}_{K,2}).$$

Hence it seems that the true $\tilde{G}_k$ is a matrix with an error at most 10^{-3} for each component of $\tilde{G}_{k,1}$ or $\tilde{G}_{k,2}$. For $\tilde{G}$ by our definition we have

$$\tilde{G} = \begin{bmatrix} 1.6185 & 0.9375 \\ 0.9375 & 1.1608 \end{bmatrix} (= A_1^{(r)} = A_2^{(r)} = A_3^{(r)} \quad \text{for} \quad r \geq 3).$$

4. Reverse inequality

Recently Kantorovich type reverse inequalities of the arithmetic-geometric, the arithmetic-harmonic, or the arithmetic-geometric-harmonic ones for two or more than two operators were presented in [7], [9], [19]. The following fact was shown in [9]:

Lemma 4.1 ([9, Theorem 9]). *Let $A_1, A_2, \cdots, A_n$ be operators such that $0 < mI \leq A_i \leq MI$ for $i = 1, 2, \cdots, n$ for some scalars m and M with $0 < m < M$. (The letter I stands the identity operator.) Then*

$$\frac{A_1 + \cdots + A_n}{n} \leq \frac{(M+m)^2}{4Mm} \left(\frac{A_1^{-1} + \cdots + A_n^{-1}}{n} \right)^{-1}.$$

From the above lemma and the property P10 of our geometric mean, we immediately obtain the following result:

Proposition 4.2. *Let $A_1, A_2, \cdots, A_n$ be operators such that $0 < mI \leq A_i \leq MI$ for $i = 1, 2, \cdots, n$ for some scalars m and M with $0 < m < M$. Then*

$$\frac{A_1 + A_2 + \cdots + A_n}{n} \leq \frac{(M+m)^2}{4Mm} G(A_1, A_2, \cdots, A_n).$$

For the weighted version of the arithmetic-geometric-harmonic mean inequality we can show for operators A, B, C and real numbers $\alpha,\ \beta,\ \gamma$ with the assumption (3.4),

$$\alpha A + \beta B + \gamma C \geq G(A, B, C; \alpha, \beta, \gamma) \geq (\alpha A^{-1} + \beta B^{-1} + \gamma C^{-1})^{-1}.$$

As a reverse version of the arithmetic-geometric mean inequality, we can obtain

Proposition 4.3. *Let $mI \leq A,\ B,\ C \leq MI$ for some scalars m and M with $0 < m < M$. Then with the assumption (3.4) for real numbers $\alpha,\ \beta,\ \gamma$,*

$$\alpha A + \beta B + \gamma C \leq \frac{(M+m)^2}{4Mm} G(A, B, C; \alpha, \beta, \gamma).$$

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