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Integral Transforms of Square Integrable Functionals on Yeh-Wiener Space

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Abstract. We give a necessary and sufficient condition that a square integrable functional $F(x)$ on Yeh-Wiener space has an integral transform $\hat{\mathcal{F}}_{\alpha,\beta}F(x)$ which is also square integrable. This extends the result by Kim and Skoug for functional $F(x)$ in $L_2(C_0[0,T])$.

1. Introduction and definitions

Let $C(Q)$ denote Yeh-Wiener space; that is the space of all real-valued continuous functions $x(s,t)$ on $Q = [0,S]\times[0,T]$ with $x(s,0) = x(0,t) = 0$ for all $0 \le s \le S$ and $0 \le t \le T$. Yeh [14] defined a Gaussian measure m_Y on $C(Q)$ (later modified in [16]) such that as a stochastic process $\{x(s,t) : (s,t) \in Q\}$ has mean $E[x(s,t)] = 0$ and covariance $E[x(s,t)x(u,v)] = \min\{s,u\}\min\{t,v\}$.

Let $\mathcal{M}$ denote the class of all Yeh-Wiener measurable subsets of $C(Q)$ and we denote the Yeh-Wiener integral of a Yeh-Wiener integrable functional F by

$$\int_{C(Q)} F(x)\, m_Y(dx).$$

Let $L_2(C(Q))$ be the space of all real or complex valued functionals F satisfying

$$\int_{C(Q)} |F(x)|^2\, m_Y(dx) < \infty.$$

Let $K(Q)$ be the space of complex valued continuous functions defined on Q and satisfying $x(s,0) = x(0,t) = 0$ for all $0 \le s \le S$ and $0 \le t \le T$. Let α and β be nonzero complex numbers. Next we state the definitions of the integral transform $\mathcal{F}_{\alpha,\beta}F$ introduced in [12] and studied in [6],[9],[10] and [11].

Definition 1.1. Let F be a functional defined on $K(Q)$. Then the integral transform $\mathcal{F}_{\alpha,\beta}F$ of F is defined by

$$\text{(1.1)} \qquad \mathcal{F}_{\alpha,\beta}F(y) = \int_{C(Q)} F(\alpha x + \beta y)\, m_Y(dx), \quad y \in K(Q)$$

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if it exists.

Remark 1.2. (1) When $\alpha = 1$ and $\beta = i$, $\mathcal{F}_{\alpha,\beta}F$ is a Yeh-Wiener space version of the Fourier-Wiener transform introduced by Cameron in [2] and used by Cameron and Martin in [3].
(2) When $\alpha = \sqrt{2}$ and $\beta = i$, $\mathcal{F}_{\alpha,\beta}F$ is a Yeh-Wiener space version of the modified Fourier-Wiener transform introduced by Cameron and Martin in [4].
(3) Equation (1.1) implies that

$$\mathcal{F}_{\alpha,\beta\beta'}F(y) = \mathcal{F}_{\alpha,\beta}F(\beta' y), \quad y \in K(Q) \tag{1.2}$$

for any nonzero complex number β'.
(4) For a detailed survey of previous work on integral transform, Fourier-Wiener transform and Fourier-Feynman transform [5], see [13].

Recently Kim and Skoug [11] established a necessary and sufficient condition that a functional $F(x)$ in $L_2(C_0[0,T])$ has an integral transform $\mathcal{F}_{\alpha,\beta}F(x)$ which also belong to $L_2(C_0[0,T])$. In this paper we extend this result for square integrable functionals on Yeh-Wiener space, that is, we give a necessary and sufficient condition that a functional $F(x)$ in $L_2(C(Q))$ has an integral transform $\hat{\mathcal{F}}_{\alpha,\beta}F(x)$, which will be defined in Section 3, also belonging to $L_2(C(Q))$.

Now we introduce a concept of the function of bounded variation of two variables. The concept of bounded variation for a function of two variables is surprisingly complex. In this paper we will use the definition used by Hardy and Krause [1],[8] which we now review.

Let $R = [a,b] \times [c,d]$ and let P be a partition of R given by

$$a = s_0 < s_1 < \cdots < s_n = b, \quad c = t_0 < t_1 < \cdots < t_m = d.$$

A function $f(s,t)$ is said to be of bounded variation on R in the sense of Hardy and Krause provided the following three conditions hold.

(1) There is a constant B such that

$$\sum_{i=1}^{n}\sum_{j=1}^{m} |f(s_i,t_j) - f(s_i,t_{j-1}) - f(s_{i-1},t_j) + f(s_{i-1},t_{j-1})| \le B \tag{1.3}$$

for all partitions P.

(2) For each $t \in [c,d]$, $f(\cdot,t)$ is a function of bounded variation on $[a,b]$.

(3) For each $s \in [a,b]$, $f(s,\cdot)$ is a function of bounded variation on $[c,d]$.

The total variation $\mathrm{Var}(f,R)$ of f over R is defined to be the supremum of the sums in (1.3) over all partitions P of R. $\mathrm{Var}(f(\cdot,t),[a,b])$ and $\mathrm{Var}(f(s,\cdot),[c,d])$ will denote the total variation of $f(\cdot,t)$ on $[a,b]$ and $f(s,\cdot)$ on $[c,d]$, respectively, as functions of single variable.

The definition of bounded variation by Hardy and Krause has the important property that if g is continuous on R and f is of bounded variation on R then the

Riemann-Stieltjes integrals $\int_R g(s,t)\,df(s,t)$ and $\int_R f(s,t)\,dg(s,t)$ both exist and satisfy an integration by parts formula [7].

Let $\{\theta_1, \theta_2, \cdots, \theta_n\}$ be an orthonormal set of real-valued functions in $L_2(C(Q))$. Furthermore assume that each θ_j is of bounded variation in the sense of Hardy and Krause on Q. Then for each $y \in K(Q)$ and $j = 1, 2, \cdots$, the Riemann-Stieltjes integral $\langle \theta_j, y\rangle \equiv \int_Q \theta_j(s,t)\,dy(s,t)$ exists. We finish this section by introducing a well-known Yeh-Wiener integration formula for functionals $f(\langle \vec{\theta}, x\rangle) \equiv f(\langle \theta_1, x\rangle, \cdots, \langle \theta_n, x\rangle)$:

$$\int_{C(Q)} f(\langle \vec{\theta}, x\rangle)\, m_Y(dx) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp\Big\{-\frac{1}{2}\|\vec{u}\|^2\Big\}\, d\vec{u}, \tag{1.4}$$

where $\|\vec{u}\|^2 = \sum_{j=1}^n u_j^2$ and $d\vec{u} = du_1 \cdots du_n$.

2. Integral transforms of the Fourier-Hermite functionals

For $n = 0, 1, 2, \cdots$, let $H_n(u)$ denote the Hermite polynomial

$$H_n(u) = (-1)^n (n!)^{-1/2} e^{u^2/2} \frac{d^n}{du^n}(e^{-u^2/2}).$$

Then, as is well known, the set

$$\{(2\pi)^{-1/4} H_n(u) e^{-u^2/4} : n = 0, 1, 2, \cdots\} \tag{2.1}$$

is a complete orthonormal(CON) set on $\mathbb{R}$.

Let $\{\theta_p(s,t) : p = 1, 2, \cdots\}$ be a CON set of functions of bounded variation on Q. Define

$$\Phi_{n,p}(y) = H_n(\langle \theta_p, y\rangle), \quad n = 0, 1, 2, \cdots, p = 1, 2, \cdots,$$

and

$$\Psi_{n_1, \cdots, n_p}(y) = \Psi_{n_1, \cdots, n_p, 0, \cdots, 0}(y) = \Phi_{n_1,1}(y) \cdots \Phi_{n_p,p}(y) \tag{2.2}$$

for $y \in K(Q)$. The functionals in (2.2) are called the Fourier-Hermite functionals on Yeh-Wiener space.

In [15], Yeh showed that the Fourier-Hermite functionals form a CON set in $L_2(C(Q))$. That is to say that every functional $F(x)$ in $L_2(C(Q))$ has a Fourier-Hermite development which converges in the $L_2(C(Q))$ sense to $F(x)$; namely that

$$F(x) = \operatorname*{l.i.m.}_{N \to \infty} \sum_{n_1, \cdots, n_N = 0}^{N} A^F_{n_1, \cdots, n_N} \Psi_{n_1, \cdots, n_N}(x), \tag{2.3}$$

where $A^F_{n_1, \cdots, n_N}$ is the Fourier-Hermite coefficient

$$A^F_{n_1, \cdots, n_N} = \int_{C(Q)} F(x) \Psi_{n_1, \cdots, n_N}(x)\, m_Y(dx). \tag{2.4}$$

Throughout this paper, in order to insure that various integrals exist, we will assume that $\beta = a + bi$ is a nonzero complex number satisfying the inequality

$$\operatorname{Re}(1-\beta^2) = 1 + b^2 - a^2 > 0. \tag{2.5}$$

Note that $\operatorname{Re}(1-\beta^2) = 1 + b^2 - a^2 > 0$ if and only if the point $(a, b) \in \mathbb{R}^2$ lies in the open region, determined by the hyperbola $a^2 - b^2 = 1$, containing the b-axis. Hence for all $|\beta| \le 1$, $\beta \ne \pm 1$, $\operatorname{Re}(1-\beta^2) > 0$. Next we define

$$\alpha \equiv \sqrt{1-\beta^2}, \quad -\pi/4 < \arg(\alpha) < \pi/4 \tag{2.6}$$

and note that $\alpha^2 + \beta^2 = 1$ and $\operatorname{Re}(\alpha^2) = \operatorname{Re}(1-\beta^2) > 0$.

The following lemma is introduced in [11] and will be needed to find the integral transform of the Fourier-Hermite functionals on Yeh-Wiener space.

Lemma 2.1. *Let β be a nonzero complex number satisfying inequality* (2.5) *and let α be defined by equation* (2.6). *Let r be a complex number. Then for $n = 0, 1, 2, \cdots$,*

$$\int_{\mathbb{R}} H_n(u) \exp\Big\{-\frac{1}{2\alpha^2}(u - r\beta)^2\Big\}\, du = \sqrt{2\pi}\alpha\beta^n H_n(r). \tag{2.7}$$

Next, using Lemma 2.1, we obtain a formula for the integral transform of the Fourier-Hermite functionals given by equation (2.2).

Theorem 2.2. *Let α and β be as in Lemma* 2.1. *Then for each $y \in K(Q)$,*

$$\mathcal{F}_{\alpha,\beta}\Psi_{n_1,\cdots,n_p}(y) = \beta^{n_1+\cdots+n_p}\Psi_{n_1,\cdots,n_p}(y). \tag{2.8}$$

Proof. For $j = 1, 2, \cdots$, let $r_j = \langle\theta_j, y\rangle$ which we know exists for all $y \in K(Q)$ since θ_j is of bounded variation on Q. Then for every $y \in K(Q)$, by the Yeh-Wiener integration formula (1.4),

$$\begin{aligned}\mathcal{F}_{\alpha,\beta}\Psi_{n_1,\cdots,n_p}(y) &= \int_{C(Q)} \Psi_{n_1,\cdots,n_p}(\alpha x + \beta y)\, m_Y(dx) \\ &= \prod_{j=1}^{p}\Big[(2\pi)^{-1/2}\int_{\mathbb{R}} H_{n_j}(\alpha u_j + \beta r_j)e^{-u_j^2/2}\, du_j\Big].\end{aligned}$$

Note that for all positive α and all $\beta \in \mathbb{C}$,

$$\int_{\mathbb{R}} H_n(\alpha u + \beta r)e^{-u^2/2}\, du = \frac{1}{\alpha}\int_{\mathbb{R}} H_n(u)e^{-(u-r\beta)^2/2\alpha^2}\, du.$$

But each side of the above expression is an analytic function of α throughout the region $\{\alpha \in \mathbb{C} : \operatorname{Re}(\alpha^2) > 0\}$. Hence by the uniqueness theorem for analytic functions, the above equality holds for all α with $\operatorname{Re}(\alpha^2) > 0$ and all $\beta \in \mathbb{C}$ and so

$$\mathcal{F}_{\alpha,\beta}\Psi_{n_1,\cdots,n_p}(y) = \prod_{j=1}^{p}\Big[(2\pi\alpha^2)^{-1/2}\int_{\mathbb{R}} H_{n_j}(u_j)e^{-(u_j-r_j\beta)^2/2\alpha^2}\, du_j\Big].$$

Then using Lemma 2.1, we obtain equation (2.8), the desired result. □

Our first corollary follows immediately from equation (2.8) and the fact that $\|\Psi_{n_1,\cdots,n_p}\|_2 = 1$.

Corollary 2.3. *Let α and β be as in Lemma 2.1. Then*

$$\|\mathcal{F}_{\alpha,\beta}\Psi_{n_1,\cdots,n_p}\|_2 = |\beta|^{n_1+\cdots+n_p}. \tag{2.9}$$

By (1.2) and Theorem 2.2, we obtain the following corollary.

Corollary 2.4. *Let α and β be as in Lemma 2.1 and let γ be any nonzero complex number. Then for each $y \in K(Q)$,*

$$\mathcal{F}_{\alpha,\gamma}\Psi_{n_1,\cdots,n_p}(y) = \beta^{n_1+\cdots+n_p}\Psi_{n_1,\cdots,n_p}\Big(\frac{\gamma y}{\beta}\Big). \tag{2.10}$$

3. Integral transforms of functionals belonging to $L_2(C(Q))$

For $F \in L_2(C(Q))$ let (2.3) denote the Fourier-Hermite expression of $F(x)$ with the Fourier-Hermite coefficients $A^F_{n_1,\cdots,n_N}$ given by equation (2.4). For $N = 1, 2, \cdots$, let

$$F_N(x) = \sum_{n_1,\cdots,n_N=0}^{N} A^F_{n_1,\cdots,n_N}\Psi_{n_1,\cdots,n_N}(x). \tag{3.1}$$

Then by Theorem 2.2, we know that for each $N = 1, 2, \cdots$, the integral transform $\mathcal{F}_{\alpha,\beta}F_N$ exists for all α and β as in Lemma 2.1, and $\mathcal{F}_{\alpha,\beta}F_N$ is an element of $L_2(C(Q))$ such that for each $y \in K(Q)$,

$$\mathcal{F}_{\alpha,\beta}F_N(y) = \sum_{n_1,\cdots,n_N=0}^{N} A^F_{n_1,\cdots,n_N}\beta^{n_1+\cdots+n_N}\Psi_{n_1,\cdots,n_N}(y). \tag{3.2}$$

Furthermore,

$$\|\mathcal{F}_{\alpha,\beta}F_N\|_2^2 = \sum_{n_1,\cdots,n_N=0}^{N} |A^F_{n_1,\cdots,n_N}\beta^{n_1+\cdots+n_N}|^2. \tag{3.3}$$

Definition 3.1. Let $F \in L_2(C(Q))$ be given by (2.3). Then for each pair of nonzero complex numbers α and β, we define *the integral transform* $\hat{\mathcal{F}}_{\alpha,\beta}F$ of F to be

$$\hat{\mathcal{F}}_{\alpha,\beta}F(x) = \underset{N\to\infty}{\mathrm{l.i.m.}}\,\mathcal{F}_{\alpha,\beta}F_N(x), \quad x \in C(Q) \tag{3.4}$$

if it exists; that is to say if

$$\lim_{N\to\infty}\int_{C(Q)} |\hat{\mathcal{F}}_{\alpha,\beta}F(x) - \mathcal{F}_{\alpha,\beta}F_N(x)|^2\, m_Y(dx) = 0. \tag{3.5}$$

Suppose that F is a functional defined on $K(Q)$ and has the integral transform $\mathcal{F}_{\alpha,\beta}F(y)$ for $y \in K(Q)$ in the sense of Definition 1.1. Further assume that F, as a function of $x \in C(Q)$, belongs to $L_2(C(Q))$ and has the integral transform $\hat{\mathcal{F}}_{\alpha,\beta}F$ for $x \in C(Q)$ in the sense of Definition 3.1. The following example shows that the two integral transforms $\mathcal{F}_{\alpha,\beta}F(x)$ and $\hat{\mathcal{F}}_{\alpha,\beta}F(x)$ for $x \in C(Q)$ need not coincide.

Example 3.2. Let F be a functional defined on $K(Q)$ by

$$F(y) = \begin{cases} 0, & \text{if } y \in C(Q) \\ 1, & \text{if } y \in K(Q) \setminus C(Q). \end{cases}$$

Then F belongs to $L_2(C(Q))$ and for $x \in C(Q)$, we have

$$\mathcal{F}_{\sqrt{2},i}F(x) = \int_{C(Q)} F(\sqrt{2}z + ix)\, m_Y(dz) = 1.$$

On the other hand, for any nonnegative integers $n_1, \cdots, n_N$,

$$A^F_{n_1,\cdots,n_N} = \int_{C(Q)} F(x)\Psi_{n_1,\cdots,n_N}(x)\, m_Y(dx) = 0$$

and so $F_N(y) = 0$ for $y \in K(Q)$ and for all $N = 1, 2, \cdots$. Now

$$\mathcal{F}_{\sqrt{2},i}F_N(x) = \int_{C(Q)} F_N(\sqrt{2}z + ix)\, m_Y(dz) = 0, \quad x \in C(Q)$$

and so

$$\hat{\mathcal{F}}_{\sqrt{2},i}F(x) = \underset{N\to\infty}{\text{l.i.m.}}\, \mathcal{F}_{\sqrt{2},i}F_N(x) = 0, \quad x \in C(Q).$$

Hence we conclude that

$$\mathcal{F}_{\sqrt{2},i}F(x) \neq \hat{\mathcal{F}}_{\sqrt{2},i}F(x)$$

for $x \in C(Q)$.

Theorem 3.3. *Let $F \in L_2(C(Q))$ be given by (2.3). Let α and β be nonzero complex numbers and let c be a nonzero real number. Then*

$$\hat{\mathcal{F}}_{\alpha,c\beta}F(x) = \hat{\mathcal{F}}_{\alpha,\beta}F(cx) \tag{3.6}$$

for $x \in C(Q)$.

Proof. By (1.2) for each $N = 1, 2, \cdots$,

$$\mathcal{F}_{\alpha,c\beta}F_N(x) = \mathcal{F}_{\alpha,\beta}F_N(cx)$$

and so

$$\hat{\mathcal{F}}_{\alpha,c\beta}F(x) = \underset{N\to\infty}{\text{l.i.m.}}\, \mathcal{F}_{\alpha,c\beta}F_N(x) = \underset{N\to\infty}{\text{l.i.m.}}\, \mathcal{F}_{\alpha,\beta}F_N(cx) = \hat{\mathcal{F}}_{\alpha,\beta}F(cx)$$

as desired. □

The following lemma gives us a relationship between the Fourier-Hermite coefficients of $\hat{\mathcal{F}}_{\alpha,\beta}F$ and F.

Lemma 3.4. *Let $F \in L_2(C(Q))$ be given by (2.3) with Fourier-Hermite coefficients given by (2.4). Let α and β be as in Lemma 2.1 and assume that $\hat{\mathcal{F}}_{\alpha,\beta}F$ exists and is in $L_2(C(Q))$. Then*

$$A^{\hat{\mathcal{F}}_{\alpha,\beta}F}_{n_1,\cdots,n_N} = A^F_{n_1,\cdots,n_N}\beta^{n_1+\cdots+n_N} \tag{3.7}$$

for each $N = 1, 2, \cdots$.

Proof. Fix $N = 1, 2, \cdots$. For any given $\epsilon > 0$, take a natural number M satisfying $\|\hat{\mathcal{F}}_{\alpha,\beta}F - \mathcal{F}_{\alpha,\beta}F_M\|_2 < \epsilon$ and $M \geq N$. Then we have

$$\begin{aligned}
&|A^{\hat{\mathcal{F}}_{\alpha,\beta}F}_{n_1,\cdots,n_N} - A^F_{n_1,\cdots,n_N}\beta^{n_1+\cdots+n_N}| \\
=&\Big|\int_{C(Q)} \hat{\mathcal{F}}_{\alpha,\beta}F(x)\Psi_{n_1,\cdots,n_N}(x)\, m_Y(dx) - A^F_{n_1,\cdots,n_N}\beta^{n_1+\cdots+n_N}\Big| \\
\leq&\Big|\int_{C(Q)} [\hat{\mathcal{F}}_{\alpha,\beta}F(x) - \mathcal{F}_{\alpha,\beta}F_M(x)]\Psi_{n_1,\cdots,n_N}(x)\, m_Y(dx)\Big| \\
&+\Big|\int_{C(Q)} \mathcal{F}_{\alpha,\beta}F_M(x)\Psi_{n_1,\cdots,n_N}(x)\, m_Y(dx) - A^F_{n_1,\cdots,n_N}\beta^{n_1+\cdots+n_N}\Big|.
\end{aligned}$$

But by the Hölder inequality and the fact that $\{\Psi_{n_1,\cdots,n_N}\}$ is an orthonormal set,

$$\Big|\int_{C(Q)} [\hat{\mathcal{F}}_{\alpha,\beta}F(x) - \mathcal{F}_{\alpha,\beta}F_M(x)]\Psi_{n_1,\cdots,n_N}(x)\, m_Y(dx)\Big| \leq \|\hat{\mathcal{F}}_{\alpha,\beta}F - \mathcal{F}_{\alpha,\beta}F_M\|_2 < \epsilon$$

and from (3.2) we know that

$$\int_{C(Q)} \mathcal{F}_{\alpha,\beta}F_M(x)\Psi_{n_1,\cdots,n_N}(x)\, m_Y(dx) = A^F_{n_1,\cdots,n_N}\beta^{n_1+\cdots+n_N}.$$

Hence

$$|A^{\hat{\mathcal{F}}_{\alpha,\beta}F}_{n_1,\cdots,n_N} - A^F_{n_1,\cdots,n_N}\beta^{n_1+\cdots+n_N}| < \epsilon$$

which establishes equation (3.7). □

The following theorem is our main result. It gives a necessary and sufficient condition that a functional F in $L_2(C(Q))$ has an integral transform $\hat{\mathcal{F}}_{\alpha,\beta}F$ belonging to $L_2(C(Q))$.

Theorem 3.5. Let $F \in L_2(C(Q))$ be given by (2.3) with Fourier-Hermite coefficients given by (2.4). Let α and β be as in Lemma 2.1. Then $\hat{\mathcal{F}}_{\alpha,\beta}F$ exists and is an element of $L_2(C(Q))$ if and only if

$$\lim_{N\to\infty} \sum_{n_1,\cdots,n_N=0}^{N} |A^F_{n_1,\cdots,n_N}\beta^{n_1+\cdots+n_N}|^2 < \infty. \tag{3.8}$$

Furthermore if (3.8) holds, then the Fourier-Hermite expression of $\hat{\mathcal{F}}_{\alpha,\beta}F$ is given by

$$\hat{\mathcal{F}}_{\alpha,\beta}F(x) = \underset{N\to\infty}{\text{l.i.m.}} \sum_{n_1,\cdots,n_N=0}^{N} A^F_{n_1,\cdots,n_N}\beta^{n_1+\cdots+n_N}\Psi_{n_1,\cdots,n_N}(x) \tag{3.9}$$

for $x \in C(Q)$.

Proof. Assume that $\hat{\mathcal{F}}_{\alpha,\beta}F$ exists and is an element of $L_2(C(Q))$. For any given $\epsilon > 0$, we have $\|\hat{\mathcal{F}}_{\alpha,\beta}F - \mathcal{F}_{\alpha,\beta}F_N\|_2 < \epsilon$ for sufficiently large N, and so

$$\Big(\sum_{n_1,\cdots,n_N=0}^{N} |A^F_{n_1,\cdots,n_N}\beta^{n_1+\cdots+n_N}|^2\Big)^{1/2} = \|\mathcal{F}_{\alpha,\beta}F_N\|_2 \le \|\hat{\mathcal{F}}_{\alpha,\beta}F\|_2 + \epsilon.$$

Hence we have

$$\lim_{N\to\infty} \sum_{n_1,\cdots,n_N=0}^{N} |A^F_{n_1,\cdots,n_N}\beta^{n_1+\cdots+n_N}|^2 \le \|\hat{\mathcal{F}}_{\alpha,\beta}F\|_2^2 < \infty.$$

To prove the converse, suppose that (3.8) holds. Let $M > N$, let

$$I_M = \{(n_1,\cdots,n_M) : n_1,\cdots,n_M = 0,1,\cdots,M\},$$

and let

$$I_N = \{(n_1,\cdots,n_M) : n_1,\cdots,n_N = 0,1,\cdots,N \text{ and } n_{N+1} = \cdots = n_M = 0\}.$$

Then

$$\begin{aligned}
&\|\mathcal{F}_{\alpha,\beta}F_M - \mathcal{F}_{\alpha,\beta}F_N\|_2^2 \\
=&\Big\|\sum_{I_M-I_N} A^F_{n_1,\cdots,n_M}\beta^{n_1+\cdots+n_M}\Psi_{n_1,\cdots,n_M}\Big\|_2^2 \\
=&\sum_{I_M-I_N} |A^F_{n_1,\cdots,n_M}\beta^{n_1+\cdots+n_M}|^2 \\
=&\sum_{n_1,\cdots,n_M=0}^{M} |A^F_{n_1,\cdots,n_M}\beta^{n_1+\cdots+n_M}|^2 - \sum_{n_1,\cdots,n_N=0}^{N} |A^F_{n_1,\cdots,n_N}\beta^{n_1+\cdots+n_N}|^2
\end{aligned}$$

which goes to 0 as $M, N \to \infty$. Hence $\{\mathcal{F}_{\alpha,\beta}F_N\}$ is a Cauchy sequence in $L_2(C(Q))$ and since $L_2(C(Q))$ is complete,

$$\hat{\mathcal{F}}_{\alpha,\beta}F(x) = \underset{N\to\infty}{\text{l.i.m.}}\, \mathcal{F}_{\alpha,\beta}F_N(x), \quad x \in C(Q)$$

exists and is an element of $L_2(C(Q))$ and is given by (3.9). □

Our first corollary follows immediately from Theorem 3.5.

Corollary 3.6. *Let F, α and β be as in Theorem* 3.5. *Furthermore assume that* $|\beta| \le 1$. *Then $\hat{\mathcal{F}}_{\alpha,\beta}F$ exists, belongs to $L_2(C(Q))$, and*

$$\|\hat{\mathcal{F}}_{\alpha,\beta}F\|_2^2 = \lim_{N\to\infty} \sum_{n_1,\cdots,n_N=0}^{N} |A^F_{n_1,\cdots,n_N}\beta^{n_1+\cdots+n_N}|^2 \le \lim_{N\to\infty} \sum_{n_1,\cdots,n_N=0}^{N} |A^F_{n_1,\cdots,n_N}|^2 = \|F\|_2^2. \tag{3.10}$$

In addition,

$$\|\hat{\mathcal{F}}_{\alpha,\beta}F\|_2 = \|F\|_2 \tag{3.11}$$

if and only if $|\beta| = 1$.

The following corollary is immediate from Theorems 3.3 and 3.5.

Corollary 3.7. *Let F, α and β be as in Theorem* 3.5 *and let c be a nonzero real number. Then*

$$\hat{\mathcal{F}}_{\alpha,c\beta}F(x) = \underset{N\to\infty}{\mathrm{l.i.m.}} \sum_{n_1,\cdots,n_N=0}^{N} A^F_{n_1,\cdots,n_N}\beta^{n_1+\cdots+n_N}\Psi_{n_1,\cdots,n_N}(cx) \tag{3.12}$$

for $x \in C(Q)$.

Next choosing $\alpha = \sqrt{2}$ and $\beta = i$, we obtain a Yeh-Wiener space version of the main theorem of [4].

Corollary 3.8. *Every functional $F(x) \in L_2(C(Q))$ has a Fourier-Wiener transform $G(x) \in L_2(C(Q))$. The functional $G(x)$ has $F(-x)$ as its transform and F and G satisfies Plancherel's relation*

$$\int_{C(Q)} |F(x)|^2\, m_Y(dx) = \int_{C(Q)} |G(x)|^2\, m_Y(dx). \tag{3.13}$$

Proof. Using Corollary 3.6 and Theorem 3.5, we obtain that $G(x) \in L_2(C(Q))$ is given by

$$G(x) = \underset{N\to\infty}{\mathrm{l.i.m.}} \sum_{n_1,\cdots,n_N=0}^{N} A^F_{n_1,\cdots,n_N} i^{n_1+\cdots+n_N}\Psi_{n_1,\cdots,n_N}(x),$$

and that

$$\hat{\mathcal{F}}_{\sqrt{2},i}G(x) = \underset{N\to\infty}{\mathrm{l.i.m.}} \sum_{n_1,\cdots,n_N=0}^{N} A^F_{n_1,\cdots,n_N}(-1)^{n_1+\cdots+n_N}\Psi_{n_1,\cdots,n_N}(x).$$

But since the Hermite polynomial H_n is an even function if n is even and an odd function if n is odd, it is easy to see that

$$(-1)^{n_1+\cdots+n_N}\Psi_{n_1,\cdots,n_N}(x)=\Psi_{n_1,\cdots,n_N}(-x)$$

and so $\hat{\mathcal{F}}_{\sqrt{2},i}G(x) = F(-x)$. Finally equation (3.13) follows immediately from (3.11). □

Recall that throughout this paper we have assumed that $\beta = a + bi$ was a nonzero complex number satisfying inequality (2.5); namely that $\mathrm{Re}(1-\beta^2) > 0$. Furthermore, in Corollary 3.6 we showed that if β also satisfies the inequality $|\beta| \le 1$, then $\hat{\mathcal{F}}_{\alpha,\beta}F$ exists as an element of $L_2(C(Q))$ for all $F \in L_2(C(Q))$ with α given by (2.6). In Example 10 of [11], Kim and Skoug showed that for any complex number β with $|\beta| > 1$ and $\mathrm{Re}(1-\beta^2) > 0$, there exists a functional $F \in L_2(C_0[0,T])$ such that $\mathcal{F}_{\alpha,\beta}F$, $\hat{\mathcal{F}}_{\alpha,\beta}F$ in our notation, doesn't exist as an element of $L_2(C_0[0,T])$. Using the same idea as in Example 10 of [11], we can construct a functional $F \in L_2(C(Q))$ such that $\hat{\mathcal{F}}_{\alpha,\beta}F$ doesn't exist as an element of $L_2(C(Q))$ when β is a complex number with $|\beta| > 1$ and $\mathrm{Re}(1-\beta^2) > 0$.

Our final results involves the inverse transform of $\hat{\mathcal{F}}_{\alpha,\beta}$. In order to insure the existence of the inverse transform of $\hat{\mathcal{F}}_{\alpha,\beta}$ we need to put an additional assumption on $\beta = a + bi$; namely that

$$\mathrm{Re}\Big(1-\frac{1}{\beta^2}\Big) > 0. \tag{3.14}$$

Now $\mathrm{Re}(1-1/\beta^2) > 0$ if and only if $(a^2+b^2)^2-(a^2-b^2) > 0$. But the graph of $(a^2+b^2)^2-(a^2-b^2) = 0$ is the lemniscate $r^2 = \cos(2\theta)$. Hence $\mathrm{Re}(1-1/\beta^2) > 0$ if and only if the point $(a,b) \in \mathbb{R}^2$ lies outside the lemniscate $(a^2+b^2)^2-(a^2-b^2) = 0$.

Theorem 3.9. *Let F, α and β be as in Theorem 3.5 and assume that (3.8) holds. Furthermore assume that β satisfies inequality (3.14). Then for $\alpha' \equiv \sqrt{1-1/\beta^2}$ and $\beta' = \pm 1/\beta$, we have that*

$$\hat{\mathcal{F}}_{\alpha',\beta'}\hat{\mathcal{F}}_{\alpha,\beta}F(x) = F(\beta\beta' x), \quad x \in C(Q). \tag{3.15}$$

That is to say,

$$\hat{\mathcal{F}}_{\alpha',1/\beta}\hat{\mathcal{F}}_{\alpha,\beta}F(x) = F(x), \quad x \in C(Q) \tag{3.16}$$

and

$$\hat{\mathcal{F}}_{\alpha',-1/\beta}\hat{\mathcal{F}}_{\alpha,\beta}F(x) = F(-x), \quad x \in C(Q). \tag{3.17}$$

Proof. Since $\hat{\mathcal{F}}_{\alpha,\beta}F$ exists, the Fourier-Hermite expression of it is given by

$$\hat{\mathcal{F}}_{\alpha,\beta}F(x) = \underset{N\to\infty}{\mathrm{l.i.m.}}\sum_{n_1,\cdots,n_N=0}^{N} A^F_{n_1,\cdots,n_N}\beta^{n_1+\cdots+n_N}\Psi_{n_1,\cdots,n_N}(x)$$

for $x \in C(Q)$. Now since $\beta\beta' = \pm 1$, we have

$$\lim_{N\to\infty} \sum_{n_1,\cdots,n_N=0}^{N} |A^F_{n_1,\cdots,n_N} \beta^{n_1+\cdots+n_N} (\beta')^{n_1+\cdots+n_N}|^2 = \lim_{N\to\infty} \sum_{n_1,\cdots,n_N=0}^{N} |A^F_{n_1,\cdots,n_N}|^2$$
$$= \|F\|_2^2 < \infty.$$

Hence by Theorem 3.5, $\hat{\mathcal{F}}_{\alpha',\beta'}\hat{\mathcal{F}}_{\alpha,\beta}F$ exists and is given by

$$\begin{aligned}\hat{\mathcal{F}}_{\alpha',\beta'}\hat{\mathcal{F}}_{\alpha,\beta}F(x) &= \underset{N\to\infty}{\text{l. i. m.}} \sum_{n_1,\cdots,n_N=0}^{N} A^F_{n_1,\cdots,n_N} (\beta\beta')^{n_1+\cdots+n_N} \Psi_{n_1,\cdots,n_N}(x) \\ &= \underset{N\to\infty}{\text{l. i. m.}} \sum_{n_1,\cdots,n_N=0}^{N} A^F_{n_1,\cdots,n_N} \Psi_{n_1,\cdots,n_N}(\beta\beta' x) \\ &= F(\beta\beta' x),\end{aligned}$$

for $x \in C(Q)$, where the second equality holds since $\beta\beta' = 1$ or -1, and this completes the proof of Theorem 3.9. □

The following corollary is immediate from Theorems 3.3 and 3.9.

Corollary 3.10. *Let $F, \alpha, \beta, \alpha'$ and β' be as in Theorem 3.9. Let c and c' be nonzero real numbers. Then*

$$\hat{\mathcal{F}}_{\alpha',c'\beta'}\hat{\mathcal{F}}_{\alpha,c\beta}F(x) = F(cc'\beta\beta' x) \tag{3.18}$$

for $x \in C(Q)$.

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