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Meromorphic Functions Sharing a Small Function with Their Derivatives

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Abstract. In this paper, we investigate uniqueness problems of meromorphic functions that share a small function with one of their derivatives, and give some results to improve some previous results.

1. Introduction and results

In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We shall use the standard notations in Nevanlinna value distribution theory of meromorphic functions such as $T(r, f)$, $N(r, f)$, $m(r, f)$ (see e.g., [5], [8]). For any nonconstant meromorphic function f, we denote by $S(r, f)$ any quantity satisfying

$$\lim_{r\to\infty} \frac{S(r, f)}{T(r, f)} = 0,$$

possibly outside of a set of finite linear measure in R_+. A meromorphic function $a(z)$ is said to be a small function of f, provided $T(r, a) = S(r, f)$.

We say that two meromorphic functions f and g share a small function a IM (ignoring multiplicities) when $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that f and g share a CM (counting multiplicities).

The uniqueness theory of entire and meromorphic functions has grown up to an extensive subfield of the value distribution theory, see e.g. the monograph [8] by Yang and Yi. A widely studied subtopic of the uniqueness theory has been to considering shared value problems relative to a meromorphic function f and its derivative $f^{(k)}$. Some of the basic papers in this direction are due to Rubel and Yang [7], Gundersen [3], Mues and Steinmetz [6] and Yang [9].

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Recently, L. Z. Yang and the present author [10] considered value sharing relative to a power of a meromorphic function $F = f^n$ and its derivative F', proving the following 2 theorems.

Theorem A. *Let f be a nonconstant entire function, $n \geq 7$ be an integer. Denote $F = f^n$. If F and F' share 1 CM, then $F = F'$, and f assumes the form*

$$f(z) = ce^{\frac{1}{n}z},$$

where c is a nonzero constant.

Theorem B. *Let f be a nonconstant meromorphic function and $n \geq 12$ be an integer. Denote $F = f^n$. If F and F' share 1 CM, then $F = F'$, and f assumes the form*

$$f(z) = ce^{\frac{1}{n}z},$$

where c is a nonzero constant.

In this paper, we improve Theorem A and B by obtaining the following results.

Theorem 1.1. *Let f be a nonconstant entire function, n, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and $n > k + 4$, then $f^n = (f^n)^{(k)}$, and f assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^k = 1$.

Theorem 1.2. *Let f be a nonconstant meromorphic function, k, $n(\geq k)$ be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and*

$$(n - k - 1)(n - k - 4) > 3k + 6, \tag{1.1}$$

then $f^n = (f^n)^{(k)}$, and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^k = 1$.

Corollary 1.3. *Let f be a nonconstant entire function and $n \geq 6$ be an integer. Denote $F = f^n$. If F and F' share 1 CM, then $F = F'$, and f assumes the form*

$$f(z) = ce^{\frac{1}{n}z},$$

where c is a nonzero constant.

Corollary 1.4. *Let f be a nonconstant meromorphic function and $n \geq 7$ be an* *integer. Denote $F = f^n$. If F and F' share 1 CM, then $F = F'$, and f assumes the form*

$$f(z) = ce^{\frac{1}{n}z},$$

where c is a nonzero constant.

Remark. Obviously, Corollary 1.3 and Corollary 1.4 improve Theorem A and Theorem B respectively.

For any $a \in \mathbf{C}\bigcup\{\infty\}$, we denote by $E_{l)}(a, f)$ the set of a-points of f with the multiplicity $m \leq l$, counting multiplicities.

Obviously, if $E_{l)}(a, f) = E_{l)}(a, g)$ and $l = \infty$, then f and g share a CM. It is natural to ask what happens if $F - a$ and $F' - a$ share 0 CM is replaced by $E_{l)}(0, F - a) = E_{l)}(0, F' - a)$ in Theorem A and B? Corresponding to this question, we obtain the following results.

Theorem 1.5. *Let f be a nonconstant entire function, n, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $E_{3)}(0, f^n - a) = E_{3)}(0, (f^n)^{(k)} - a)$ and $n > k + 4$, then $f^n = (f^n)^{(k)}$, and f assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^k = 1$.

From Theorem 1.5, we can easily get Theorem 1.1.

Theorem 1.6. *Let f be a nonconstant meromorphic function, n, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $E_{3)}(0, f^n - a) = E_{3)}(0, (f^n)^{(k)} - a)$ and*

$$n \geq \begin{cases} 8 & if\ k = 1, \\ 10 & if\ k = 2, \\ \left[\dfrac{3k}{2}\right] + 8 & if\ k \geq 3, \end{cases} \tag{1.2}$$

then $f^n = (f^n)^{(k)}$, and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^k = 1$.

2. Some lemmas

Let F and G be two non-constant meromorphic functions. We denote by $N_E^{1)}(r, \frac{1}{F-1})$ the counting function of common simple 1-points of F and G.

Lemma 2.1([11], Lemma 3)**.** *Let*

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right), \tag{2.1}$$

where F and G are two nonconstant meromorphic functions. If $H \not\equiv 0$, then

$$N_E^{1)}\left(r, \frac{1}{F-1}\right) \le N(r,H) + S(r,F) + S(r,G). \tag{2.2}$$

Let p be a positive integer and $a \in \mathbf{C}\bigcup\{\infty\}$. We denote by $N_{p)}\left(r, \frac{1}{f-a}\right)$ the counting function of the zeros of $f-a$ with the multiplicities less than or equal to p, and by $N_{(p+1}\left(r, \frac{1}{f-a}\right)$ the counting function of the zeros of $f-a$ with the multiplicities larger than p. And we use $\overline{N}_{p)}\left(r, \frac{1}{f-a}\right)$ and $\overline{N}_{(p+1}\left(r, \frac{1}{f-a}\right)$ to denote the corresponding reduced counting functions (ignoring multiplicities). However, $N_p\left(r, \frac{1}{f-a}\right)$ denotes the counting function of the zeros of $f-a$ where m-fold zeros are counted m times if $m \le p$ and p times if $m > p$.

Lemma 2.2([12], Lemma 3)**.** *Suppose that f is a nonconstant meromorphic function and k, p are positive integers. Then*

$$N_p\left(r, 1/f^{(k)}\right) \le T(r, f^{(k)}) - T(r,f) + N_{p+k}\left(r, 1/f\right) + S(r,f), \tag{2.3}$$

$$N_p\left(r, 1/f^{(k)}\right) \le k\overline{N}(r,f) + N_{p+k}\left(r, 1/f\right) + S(r,f). \tag{2.4}$$

Lemma 2.3. *Suppose that f is a nonconstant meromorphic function and a is a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. Let*

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right), \tag{2.5}$$

where $F = \dfrac{f^n}{a}$, $G = \dfrac{(f^n)^{(k)}}{a}$ and n, k are positive integers. If $V = 0$ and $n \ge 2$, then $F = G$.

Proof. From $V = 0$, we get

$$1 - \frac{1}{F} = B - \frac{B}{G}, \tag{2.6}$$

where B is a non-zero constant. We discuss the following two cases.

Case 1. Suppose that the counting function of poles of f is not $S(r,f)$. Then there exists a z_0 which is not a zero or pole of a such that $\dfrac{1}{f(z_0)} = 0$, thus $\dfrac{1}{F(z_0)} = \dfrac{1}{G(z_0)} = 0$. We get $B = 1$ from (2.6).

Case 2. Suppose that the counting function of poles of f is $S(r,f)$. If $B \ne 1$, then

$N\left(r, \frac{1}{F-\frac{1}{1-B}}\right) = S(r,f)$. From the second fundamental theorem, we have

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F-\frac{1}{1-B}}\right) + S(r,F) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + S(r,f), \end{aligned}$$

which is a contradiction since $n \geq 2$. Therefore $B = 1$. Thus $F = G$, completing the proof of Lemma 2.3. □

Lemma 2.4. *Let V be given by (2.5) and suppose that $V \not\equiv 0$. Then the poles of f are the zeros of V, and*

$$(n-1)\overline{N}(r,f) \leq N(r,V) + S(r,f).$$

Proof. We get from (2.5) that

$$V = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}.$$

Suppose that z_0 is a pole of f with the multiplicity p such that $a(z_0) \neq 0$ and $a(z_0) \neq \infty$. Then z_0 is a zero of $\dfrac{F'}{F(F-1)}$ with the multiplicity $np-1$ and a zero of $\dfrac{G'}{G(G-1)}$ with the multiplicity $np+k-1$. So z_0 is zero of V with the multiplicity at least $n-1$. Noting that $m(r,V) = S(r,f)$, we have

$$(n-1)\overline{N}(r,f) \leq N\left(r, \frac{1}{V}\right) + S(r,f) \leq T(r,V) + S(r,f) \leq N(r,V) + S(r,f).$$

□

Lemma 2.5. *Let H be given by (2.1), where F and G are given by Lemma 2.3. If $H \equiv 0$ and $n > k+2$, then $F = G$, and f assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^k = 1$.

Proof. By integration, we get from (2.1) that

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{2.7}$$

where $A(\neq 0)$ and B are constants. From (2.7) we have

$$N(r,F) = N(r,G) = N(r,f) = S(r,f), \tag{2.8}$$

and

$$F=\frac{(B+1)G+(A-B-1)}{BG+(A-B)},\quad G=\frac{(B-A)F+(A-B-1)}{BF-(B+1)}. \tag{2.9}$$

We discuss the following three cases.
Case 1. Suppose that $B\neq 0,-1$. From (2.9) we have $\overline{N}\left(r,1/\left(F-\frac{B+1}{B}\right)\right)=\overline{N}(r,G)$. From (2.8) and the second fundamental theorem, we have

$$\begin{aligned} nT(r,f) &\leq T(r,F)+S(r,f)\\ &\leq \overline{N}(r,F)+\overline{N}(r,1/F)+\overline{N}\left(r,\frac{1}{F-\frac{B+1}{B}}\right)+S(r,f)\\ &\leq \overline{N}(r,1/f)+\overline{N}(r,F)+\overline{N}(r,G)+S(r,f)\\ &\leq T(r,f)+S(r,f), \end{aligned}$$

which contradicts the assumption $n\geq 2$.
Case 2. Suppose that $B=0$. From (2.9) we have

$$F=\frac{G+(A-1)}{A},\quad G=AF-(A-1). \tag{2.10}$$

If $A\neq 1$, from (2.10) we obtain $\overline{N}\left(r,1/\left(F-\frac{A-1}{A}\right)\right)=\overline{N}(r,1/G)$. By (2.4), (2.8) and the second fundamental theorem, we have

$$\begin{aligned} nT(r,f) &\leq T(r,F)+S(r,f)\\ &\leq \overline{N}(r,F)+\overline{N}(r,1/F)+\overline{N}\left(r,\frac{1}{F-\frac{A-1}{A}}\right)+S(r,f)\\ &\leq \overline{N}(r,1/f)+\overline{N}(r,F)+\overline{N}\left(r,1/G\right)+S(r,f)\\ &\leq \overline{N}(r,1/f)+N_1\left(r,1/G\right)+S(r,f)\\ &\leq (k+2)\overline{N}(r,1/f)+S(r,f)\\ &\leq (k+2)T(r,f)+S(r,f), \end{aligned}$$

which contradicts the assumption $n>k+2$. Thus $A=1$. From (2.10) we have $F=G$, then

$$f^n=(f^n)^{(k)}. \tag{2.11}$$

We claim that 0 is a Picard exceptional value of f. In fact, if z_0 is a zero of f with the multiplicity p, then z_0 is a zero of f^n with the multiplicity np and a zero of $(f^n)^{(k)}$ with the multiplicity $np-k$, which is impossible from (2.11). Then from (2.11), we have

$$f(z)=ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^k = 1$.
Case 3. Suppose that $B = -1$. From (2.9) we have

$$F = \frac{A}{-G + (A+1)}, \quad G = \frac{(A+1)F - A}{F}. \tag{2.12}$$

If $A \neq -1$, we obtain from (2.12) that $\overline{N}\left(r, 1/\left(F - \frac{A}{A+1}\right)\right) = \overline{N}(r, 1/G)$. By the same reasoning discussed in Case 2, we obtain a contradiction. Hence $A = -1$. From (2.12), we get $F \cdot G = 1$, that is

$$f^n \cdot (f^n)^{(k)} = a^2.$$

From above equation, we have

$$N\left(r, \frac{1}{f}\right) + N(r, f) = S(r, f),$$

and so $T(r, \frac{f^{(k)}}{f}) = S(r, f)$. From above two equations, we obtain

$$2T\left(r, \frac{f^n}{a}\right) = T\left(r, \frac{f^{2n}}{a^2}\right) = T\left(r, \frac{a^2}{f^{2n}}\right) + O(1) = T\left(r, \frac{(f^n)^{(k)}}{f^n}\right) + O(1) = S(r, f).$$

So $T(r, f) = S(r, f)$, which is impossible. This completes the proof of Lemma 2.5. □

3. Proofs of results

Proof of Theorem 1.6. Let

$$F = \frac{f^n}{a}, \quad G = \frac{(f^n)^{(k)}}{a}. \tag{3.1}$$

From the conditions of Theorem 1.6, we know that $E_{3)}(1, F) = E_{3)}(1, G)$ possibly except at the zeros and poles of $a(z)$. From (3.1), we have

$$T(r, F) = n\big(T(r, f)\big) + S(r, f), \tag{3.2}$$

$$\overline{N}(r, F) = \overline{N}(r, G) + S(r, f) = \overline{N}(r, f) + S(r, f). \tag{3.3}$$

Let H be defined by (2.1). Suppose that $H \not\equiv 0$. By Lemma 2.1 we know that (2.2) holds. From (2.1) and (3.3), we have

$$\begin{aligned} N(r, H) \leq\ & \overline{N}_{(2}\left(r, \frac{1}{F}\right) + \overline{N}_{(2}\left(r, \frac{1}{G}\right) + \overline{N}(r, G) \\ +\ & \overline{N}_{(4}\left(r, \frac{1}{F-1}\right) + \overline{N}_{(4}\left(r, \frac{1}{G-1}\right) + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right), \end{aligned} \tag{3.4}$$

where $N_0(r, \frac{1}{F'})$ denotes the counting function corresponding to the zeros of F' which are not the zeros of F and $F-1$, and correspondingly for G'. From the second fundamental theorem, we have

$$\begin{aligned} T(r,F)+T(r,G) &\le \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,G) \\ &+ \overline{N}\left(r,\frac{1}{G-1}\right)-N_0\left(r,\frac{1}{F'}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,f). \end{aligned} \tag{3.5}$$

Noting that $E_{3)}(1,F)=E_{3)}(1,G)$, we have

$$\begin{aligned} \overline{N}\left(r,\frac{1}{F-1}\right) &+ \overline{N}\left(r,\frac{1}{G-1}\right) \\ &= 2N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{G-1}\right). \end{aligned}$$

Combining with (2.2) and (3.4), we obtain

$$\begin{aligned} \overline{N}\left(r,\frac{1}{F-1}\right) &+ \overline{N}\left(r,\frac{1}{G-1}\right) \\ &\le \overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}(r,G)+\overline{N}_{(4}\left(r,\frac{1}{F-1}\right) \\ &+ \overline{N}_{(4}\left(r,\frac{1}{G-1}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{F-1}\right) \\ &+ \overline{N}_{(2}\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f). \end{aligned} \tag{3.6}$$

It is easy to see that

$$\begin{aligned} \frac{1}{2}N_E^{1)}\left(r,\frac{1}{F-1}\right) &+ \overline{N}_{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(4}\left(r,\frac{1}{F-1}\right) \\ &\le \frac{1}{2}N\left(r,\frac{1}{F-1}\right)\le\frac{1}{2}T(r,F)+O(1), \end{aligned} \tag{3.7}$$

$$\begin{aligned} \frac{1}{2}N_E^{1)}\left(r,\frac{1}{G-1}\right) &+ \overline{N}_{(2}\left(r,\frac{1}{G-1}\right)+\overline{N}_{(4}\left(r,\frac{1}{G-1}\right) \\ &\le \frac{1}{2}N\left(r,\frac{1}{G-1}\right)\le\frac{1}{2}T(r,G)+O(1). \end{aligned} \tag{3.8}$$

From (3.5) to (3.8) and (3.3), we have

$$\frac{1}{2}T(r,F)+\frac{1}{2}T(r,G)\le N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+3\overline{N}(r,f)+S(r,f).$$

Then

$$T(r,F)+T(r,G)\le 2N_2\left(r,\frac{1}{F}\right)+2N_2\left(r,\frac{1}{G}\right)+6\overline{N}(r,f)+S(r,f). \tag{3.9}$$

From (3.1), (3.9) and by using Lemma 2.2, we have

$$\begin{aligned}2T(r,F) &\le 2N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_{2+k}\left(r,\frac{1}{f^n}\right)+6\overline{N}(r,f)+S(r,f)\\ &\le 2N_2\left(r,\frac{1}{F}\right)+2N_{2+k}\left(r,\frac{1}{f^n}\right)+(6+k)\overline{N}(r,f)+S(r,f).\end{aligned}$$

Then

$$\begin{aligned}T(r,F) &\le N_2\left(r,\frac{1}{F}\right)+N_{2+k}\left(r,\frac{1}{f^n}\right)+(3+\frac{k}{2})\overline{N}(r,f)+S(r,f)\\ &\le (k+4)\overline{N}\left(r,\frac{1}{f}\right)+(3+\frac{k}{2})\overline{N}(r,f)+S(r,f).\end{aligned}$$

From (3.2) and above inequality, we get

$$nT(r,f)\le (k+4)\overline{N}\left(r,\frac{1}{f}\right)+(3+\frac{k}{2})\overline{N}(r,f)+S(r,f). \tag{3.10}$$

We now divide the discussion in two cases:
Case 1. Suppose first that $k\ge 3$. We can get a contradiction from (1.2) and (3.10).
Case 2. Suppose next that $k\le 2$. Let V be given by (2.5). If $V=0$, we get $F=G$ from Lemma 2.3. From the proof of Lemma 2.5, we obtain the conclusions of Theorem 1.6. Next, we suppose that $V\ne 0$. Since $E_{3)}(1,F)=E_{3)}(1,G)$, by Lemma 2.4 and (2.5), we obtain

$$\begin{aligned}(n-1)\overline{N}(r,f) &\le N(r,V)+S(r,f)\\ &\le \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}_{(4}\left(r,\frac{1}{F-1}\right)\\ &\quad +\overline{N}_{(4}\left(r,\frac{1}{G-1}\right)+S(r,f).\end{aligned} \tag{3.11}$$

Observe that

$$\begin{aligned}\overline{N}_{(4}\left(r,\frac{1}{F-1}\right) &\le \frac{1}{3}N\left(r,\frac{F}{F'}\right)\le \frac{1}{3}N\left(r,\frac{F'}{F}\right)+S(r,f)\\ &\le \frac{1}{3}\overline{N}(r,1/F)+\frac{1}{3}\overline{N}(r,F)+S(r,f),\end{aligned}$$

$$\begin{aligned}\overline{N}_{(4}\left(r,\frac{1}{G-1}\right) &\le \frac{1}{3}N\left(r,\frac{G}{G'}\right)\le \frac{1}{3}N\left(r,\frac{G'}{G}\right)+S(r,f)\\ &\le \frac{1}{3}\overline{N}(r,1/G)+\frac{1}{3}\overline{N}(r,G)+S(r,f).\end{aligned}$$

From (3.11) and (2.4), we have

$$\begin{aligned}(n-1)\overline{N}(r,f) &\le \frac{4}{3}\overline{N}(r,1/F)+\frac{4}{3}\overline{N}(r,1/G)+\frac{2}{3}\overline{N}(r,F)+S(r,f)\\ &\le \frac{4}{3}\overline{N}(r,1/f)+\frac{4}{3}N_1(r,1/G)+\frac{2}{3}\overline{N}(r,f)+S(r,f)\\ &\le \frac{4}{3}\overline{N}\left(r,1/f\right)+\frac{4}{3}\left((k+1)\overline{N}\left(r,1/f\right)+k\overline{N}(r,f)\right)+\frac{2}{3}\overline{N}(r,f)+S(r,f)\\ &= \frac{4(k+2)}{3}\overline{N}(r,1/f)+\frac{2(2k+1)}{3}\overline{N}(r,f)+S(r,f),\end{aligned}$$

and so

$$\left(n-1-\frac{2(2k+1)}{3}\right)\overline{N}(r,f)\le\frac{4(k+2)}{3}\overline{N}(r,1/f)+S(r,f).$$

From (1.2), we can easily get $n-1-\frac{2(2k+1)}{3}>0$. From (3.10) and above inequality, we have

$$\begin{aligned}nT(r,f) &\le \left(k+4+\frac{(2k+12)(k+2)}{3n-4k-5}\right)\overline{N}(r,1/f)+S(r,f)\\ &\le \left(k+4+\frac{(2k+12)(k+2)}{3n-4k-5}\right)T(r,f)+S(r,f),\end{aligned}$$

which contradicts the assumption (1.2) of Theorem 1.6. Thus, $H=0$. From (1.2), we have $n>k+2$. By Lemma 2.5, we get the conclusions of Theorem 1.6. This completes the proof of Theorem 1.6. □

Proof of Theorem 1.5. The proof of Theorem 1.6 applies, since f is an entire function, we get from (3.10)

$$nT(r,f)\le(k+4)\overline{N}\left(r,1/f\right)+S(r,f),$$

which contradicts the assumption $n>k+4$. Hence $H=0$. By the same reasoning as in the proof of Theorem 1.6, we obtain the results of Theorem 1.5, and we complete the proof of Theorem 1.5. □

Proof of Theorem 1.2. The proof of Theorem 1.6 applies. Since f^n-a and $(f^n)^{(k)}-a$ share the value 0 CM, then F and G share 1 CM except possibly at the zeros and poles of $a(z)$. We obtain

$$N\left(r,\frac{1}{F-1}\right)=N\left(r,\frac{1}{G-1}\right)+S(r,f),$$

and

(3.12)

$$N(r,H)\le\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f).$$

So

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right) &\leq N_E^{1)}\left(r,\frac{1}{F-1}\right)+N\left(r,\frac{1}{F-1}\right)\\ &\leq N_E^{1)}\left(r,\frac{1}{F-1}\right)+T\left(r,F\right)+O(1).\end{aligned}\tag{3.13}$$

From (2.2), (3.5), (3.12) and (3.13), we have

$$T(r,G)\leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+3\overline{N}(r,f)+S(r,f).\tag{3.14}$$

By Lemma 2.2 and (3.14), we get

$$T(r,F)\leq N_2\left(r,\frac{1}{F}\right)+N_{2+k}\left(r,\frac{1}{f^n}\right)+3\overline{N}(r,f)+S(r,f).\tag{3.15}$$

Let V be given by (2.5). If $V=0$, we get $F=G$ by Lemma 2.3. From Case 2 in the proof of Lemma 2.5, we obtain the conclusions of Theorem 1.2. Next, we suppose that $V\not\equiv 0$. Since F and G share 1 CM except at the zeros and poles of $a(z)$, by Lemma 2.4 and Lemma 2.2, we obtain

$$\begin{aligned}(n-1)\overline{N}(r,f) &\leq N(r,V)+S(r,f)\\ &\leq \overline{N}\left(r,1/F\right)+\overline{N}\left(r,1/G\right)+S(r,f)\\ &\leq \overline{N}\left(r,1/f\right)+(k+1)\overline{N}\left(r,1/f\right)+k\overline{N}(r,f)+S(r,f),\end{aligned}$$

that is

$$(n-k-1)\overline{N}(r,f)\leq (k+2)\overline{N}\left(r,1/f\right)+S(r,f).\tag{3.16}$$

Since $n\geq k$, we get from (1.1)

$$n>\frac{2k+5+\sqrt{12k+33}}{2}>k+4.\tag{3.17}$$

Combining with (3.15) and (3.16), we obtain

$$nT(r,f)\leq\left(k+4+\frac{3k+6}{n-k-1}\right)\overline{N}\left(r,1/f\right)+S(r,f),$$

which contradicts the assumption (1.1) of Theorem 1.2. Thus, $H=0$. By Lemma 2.5 and (3.17), we obtain the conclusions of Theorem 1.2. □

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