

## A Class of Starlike Functions Defined by the Dziok-Srivastava Operator

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ABSTRACT. A comprehensive class of starlike univalent functions defined by Dziok-Srivastava operator is introduced. Necessary and sufficient coefficient bounds are given for functions in this class to be starlike. Further distortion bounds, extreme points and results on partial sums are investigated.

### 1. Introduction

Let  $A$  denote the class of functions of the form

$$(1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc  $U = \{z : |z| < 1\}$ . Denote by  $S$  the subclass of  $A$  consisting functions normalized by  $f(0) = 0 = f'(0) - 1$  which are univalent in  $U$  and  $ST$  and  $CV$  the subclasses of  $S$  that are respectively, starlike and convex. Also denote by  $V$ , the class of analytic functions with varying arguments introduced by Silverman [13] consisting of functions  $f$  of the form (1) in  $S$  for which there exists a real number  $\eta$  such that

$$(2) \quad \theta_n + (n - 1)\eta = \pi(\text{mod } 2\pi), \text{ where } \arg(a_n) = \theta_n \text{ for all } n \geq 2.$$

Goodman [4], [5] introduced and defined the following subclasses of  $CV$  and  $ST$ .

**Definition 1.** A function  $f(z)$  is *uniformly convex* (*uniformly starlike*) in  $U$  if  $f(z)$  is in  $CV$  ( $ST$ ) and has the property that for every circular arc  $\zeta$  contained in  $U$ , with center  $\xi$  also in  $U$ , the arc  $f(\zeta)$  is convex (starlike) with respect to  $f(\xi)$ . The

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class of uniformly convex functions is denoted by  $UCV$  and the class of uniformly starlike functions by  $UST$ .

It is well known from [7], [10] that

$$f \in UCV \Leftrightarrow \left| \frac{zf''(z)}{f'(z)} \right| \leq \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\}$$

and

$$f \in SP \Leftrightarrow \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\}.$$

Following Goodman [4], [5], Rønning [11] generalized the class  $SP$  by introducing a parameter  $\gamma$  ( $-1 \leq \gamma < 1$ ).

(i) A function  $f \in S$  is said to be in the class  $SP(\gamma)$  uniformly  $\gamma$ -starlike functions if it satisfies the condition

$$f \in SP(\gamma) \Leftrightarrow \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} - \gamma \right\}.$$

(ii) A function  $f \in S$  is said to be in the class  $SP(\gamma, k)$  uniformly  $k$ -starlike functions if it satisfies the condition

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} - \gamma \right\} > k \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad z \in U,$$

and  $k \geq 0$ .

(iii) A function  $f \in S$  is said to be in the class  $UCV(k, \gamma)$ , uniformly  $k$ -convex functions if it satisfies the condition

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} - \gamma \right\} > k \left| \frac{zf''(z)}{f'(z)} \right|, \quad z \in U,$$

and  $k \geq 0$ .

Note that

$$f \in UCV(k, \gamma) \Leftrightarrow zf' \in SP(k, \gamma).$$

For functions  $f \in S$  given by (1) and  $g(z) \in S$  given by  $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ , we define the Hadamard product (or convolution) of  $f$  and  $g$  by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U.$$

For complex parameters  $\alpha_1, \dots, \alpha_p$  and  $\beta_1, \dots, \beta_q$  ( $\beta_j \neq 0, -1, \dots; j = 1, 2, \dots, q$ ) the *generalized hypergeometric function*  ${}_pF_q(z)$  is defined by

$$(3) \quad {}_pF_q(z) \equiv {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}$$

$$(p \leq q + 1; p, q \in N_0 := N \cup \{0\}; z \in U),$$

where  $N$  denotes the set of all positive integers and  $(\nu)_n$  is the Pochhammer symbol defined by

$$(4) \quad (\nu)_n = \begin{cases} 1, & n = 0 \\ \nu(\nu + 1)(\nu + 2) \cdots (\nu + n - 1), & n \in N. \end{cases}$$

Let  $H(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q) : A \rightarrow A$  be a linear operator defined by

$$(5) \quad \begin{aligned} [(H(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q))(f)](z) &:= z {}_pF_q(\alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_2, \dots, \beta_q; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \Gamma(\alpha_1, n) a_n z^n, \end{aligned}$$

where

$$(6) \quad \Gamma(\alpha_1, n) = \left| \frac{(\alpha_1)_{n-1} \cdots (\alpha_p)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_q)_{n-1}} \frac{1}{(n-1)!} \right|.$$

Let  $p, q \in N$  and suppose that the parameters  $\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q$  are positive real numbers. For notational simplicity, we use a shorter notation  $H_q^p[\alpha_1]$  for  $H(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q)$  in the sequel. It follows from (5) that

$$(7) \quad H_0^1[1]f(z) = f(z), H_0^1[2]f(z) = zf'(z).$$

The linear operator  $H_q^p[\alpha_1]$  is called Dziok-Srivastava operator (see [3]), which contains such well known operators as the Hohlov linear operator, Saitho generalized linear operator, the Carlson-Shaffer linear operator, the Ruscheweyh derivative operator as well as its generalized versions, the Bernardi-Libera-Livingston operator, and the Srivastava-Owa fractional derivative operator. For more details on these operators see [1], [2], [3], [8], [9], [12] and [16]. Recently, Kanas and Srivastava[6], Srivastava and Mishra [17] and Vijaya and Murugusundaramoorthy [19] defined subclasses of functions of  $UCV$  and  $SP$  making use of various operators. In this paper, we make use of Dziok-Srivastava operator[3] to investigate new subclasses of  $SP$  and  $UCV$ .

For  $-1 \leq \gamma < 1$ , we let  $S_q^p([\alpha_1], \gamma)$  denote the subclass of starlike functions corresponding to the family  $UCV$  for functions  $f(z)$  of the form (1) such that

$$(8) \quad \operatorname{Re} \left\{ \frac{z(H_q^p[\alpha_1, \beta_1]f(z))'}{H_q^p[\alpha_1, \beta_1]f(z)} - \gamma \right\} \geq \left| \frac{z(H_q^p[\alpha_1, \beta_1]f(z))'}{H_q^p[\alpha_1, \beta_1]f(z)} - 1 \right|.$$

We also let  $VS_q^p([\alpha_1], \gamma) = V \cap S_q^p([\alpha_1], \gamma)$ .

We obtain a sufficient coefficient condition for functions  $f$  given by (1) to be in the class  $S_q^p([\alpha_1], \gamma)$  and it is shown that this coefficient condition is also necessary for functions to belong to the class  $VS_q^p([\alpha_1], \gamma)$ . Distortion results and extreme points for functions in  $VS_q^p([\alpha_1], \gamma)$  are obtained. A more general subclass is defined for which we obtain similar results. Finally, we investigate partial sums for these

classes.

## 2. Main results

We obtain a sufficient condition for functions in  $S_q^p([\alpha_1], \gamma)$ .

**Theorem 1.** *Let  $f(z)$  be given by (1). If  $-1 \leq \gamma < 1$  and*

$$(9) \quad \sum_{n=2}^{\infty} (2n-1-\gamma)\Gamma(\alpha_1, n)|a_n| \leq 1-\gamma,$$

then  $f(z) \in S_q^p([\alpha_1], \gamma)$ .

*Proof.* By definition of the class  $S_q^p([\alpha_1], \gamma)$ , it suffices to show that

$$\left| \frac{z(H_q^p[\alpha_1, \beta_1]f(z))'}{H_q^p[\alpha_1, \beta_1]f(z)} - 1 \right| \leq \operatorname{Re} \left\{ \frac{z(H_q^p[\alpha_1, \beta_1]f(z))'}{H_q^p[\alpha_1, \beta_1]f(z)} - \gamma \right\}.$$

The above inequality implies

$$\begin{aligned} \left| \frac{z(H_q^p[\alpha_1, \beta_1]f(z))'}{H_q^p[\alpha_1, \beta_1]f(z)} - 1 \right| - \operatorname{Re} \left\{ \frac{z(H_q^p[\alpha_1, \beta_1]f(z))'}{H_q^p[\alpha_1, \beta_1]f(z)} - \gamma \right\} \\ \leq 2 \left| \frac{z(H_q^p[\alpha_1, \beta_1]f(z))'}{H_q^p[\alpha_1, \beta_1]f(z)} - 1 \right| \\ \leq 2 \frac{\sum_{n=2}^{\infty} (n-1)\Gamma(\alpha_1, n)|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty} \Gamma(\alpha_1, n)|a_n||z|^{n-1}}. \end{aligned}$$

Now the last expression is bounded above by  $(1-\gamma)$  if and only if

$$\sum_{n=2}^{\infty} (2n-1-\gamma)\Gamma(\alpha_1, n)|a_n| \leq 1-\gamma.$$

□

In the following theorem, we show that the condition (9) is also necessary for functions  $f \in VS_q^p([\alpha_1], \gamma)$ .

**Theorem 2.** *Let  $f(z)$  be given by (1) and satisfy (2). The function  $f$  is in  $VS_q^p([\alpha_1], \gamma)$  if and only if*

$$(10) \quad \sum_{n=2}^{\infty} (2n-1-\gamma)\Gamma(\alpha_1, n)|a_n| \leq 1-\gamma.$$

*Proof.* In view of Theorem 1 we need only show that  $f$  in  $VS_q^p([\alpha_1], \gamma)$  satisfies the coefficient inequality. If  $f \in VS_q^p([\alpha_1], \gamma)$  then by definition,

$$\left| \frac{z + \sum_{n=2}^{\infty} n\Gamma(\alpha_1, n)a_n z^n}{z + \sum_{n=2}^{\infty} \Gamma(\alpha_1, n)a_n z^n} - 1 \right| \leq \operatorname{Re} \left\{ \frac{z + \sum_{n=2}^{\infty} n\Gamma(\alpha_1, n)a_n z^n}{z + \sum_{n=2}^{\infty} \Gamma(\alpha_1, n)a_n z^n} - \gamma \right\},$$

or

$$\left| \frac{\sum_{n=2}^{\infty} (n-1)\Gamma(\alpha_1, n)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} \Gamma(\alpha_1, n)a_n z^{n-1}} \right| \leq \operatorname{Re} \left\{ \frac{(1-\gamma) + \sum_{n=2}^{\infty} (n-\gamma)\Gamma(\alpha_1, n)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} \Gamma(\alpha_1, n)a_n z^{n-1}} \right\}.$$

In view of (2), we set  $z = r^{in}$  in the above inequality to obtain

$$(11) \quad \frac{\sum_{n=2}^{\infty} (n-1)\Gamma(\alpha_1, n)|a_n|r^{n-1}}{1 - \sum_{n=2}^{\infty} \Gamma(\alpha_1, n)|a_n|r^{n-1}} \leq \frac{(1-\gamma) - \sum_{n=2}^{\infty} (n-\gamma)\Gamma(\alpha_1, n)|a_n|r^{n-1}}{1 - \sum_{n=2}^{\infty} \Gamma(\alpha_1, n)|a_n|r^{n-1}}.$$

Letting  $r \rightarrow 1^-$  in (11) yields the desired inequality (10). □

**Corollary 1.** *If  $f \in VS_q^p([\alpha_1], \gamma)$  then  $|a_n| \leq \frac{1-\gamma}{(2n-1-\gamma)\Gamma(\alpha_1, n)}$  for  $n \geq 2$ .*

*The equality holds for*

$$f(z) = z + \frac{(1-\gamma)e^{i\theta_n} z^n}{(2n-1-\gamma)\Gamma(\alpha_1, n)}$$

for  $(n = 2, 3, \dots)$  and  $z \in U$ .

Specific choices of parameters in Theorem 2 give the following necessary and sufficient coefficient conditions .

**Corollary 2.** *Let  $f(z)$  be given by (1) and satisfy (2). The function  $f$  is in  $VS_0^1([1], \gamma)$  if and only if*

$$(12) \quad \sum_{n=2}^{\infty} (2n-1-\gamma)|a_n| \leq 1-\gamma.$$

**Corollary 3.** *Let  $f(z)$  be given by (1) and satisfy (2). The function  $f$  is in  $VS_0^1([2], \gamma)$  if and only if*

$$(13) \quad \sum_{n=2}^{\infty} n(2n-1-\gamma)|a_n| \leq 1-\gamma.$$

Next we obtain the distortion bounds for functions belonging to the class  $VS_q^p([\alpha_1], \gamma)$ .

**Theorem 3.** *Let  $f(z)$  of the form (1) be in the class  $VS_q^p([\alpha_1], \gamma)$ . Then*

$$(14) \quad r - \frac{1-\gamma}{(3-\gamma)\Gamma(\alpha_1, 2)}r^2 \leq |f(z)| \leq r + \frac{1-\gamma}{(3-\gamma)\Gamma(\alpha_1, 2)}r^2$$

and

$$(15) \quad 1 - \frac{2(1-\gamma)}{(3-\gamma)\Gamma(\alpha_1, 2)}r \leq |f'(z)| \leq 1 + \frac{2(1-\gamma)}{(3-\gamma)\Gamma(\alpha_1, 2)}r.$$

*The result is sharp.*

*Proof.* Since  $f(z) \in VS_q^p([\alpha_1], \gamma)$ , we apply Theorem 2 to obtain

$$(3-\gamma)\Gamma(\alpha_1, 2) \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} (2n-1-\gamma)\Gamma(\alpha_1, n)|a_n| \leq 1-\gamma.$$

Thus

$$|f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} |a_n| \leq r + \frac{1-\gamma}{(3-\gamma)\Gamma(\alpha_1, 2)}r^2.$$

Also we have

$$|f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} |a_n| \geq r - \frac{1-\gamma}{(3-\gamma)\Gamma(\alpha_1, 2)}r^2$$

and (14) follows. In similar manner for  $f'(z)$ , the inequalities

$$|f'(z)| \leq 1 + \sum_{n=2}^{\infty} n|a_n||z|^{n-1} \leq 1 + |z| \sum_{n=2}^{\infty} na_n$$

and

$$\sum_{n=2}^{\infty} n|a_n| \leq \frac{2(1-\gamma)}{(3-\gamma)\Gamma(\alpha_1, 2)}$$

lead to (15). This completes the result.  $\square$

**Theorem 4.** *Let the function  $f(z)$  defined by (1) satisfy (2). Define  $f_1(z) = z$  and  $f_n(z) = z + \frac{1-\gamma}{(2n-1-\gamma)\Gamma(\alpha_1, n)}e^{i\theta_n}z^n$ ,  $n \geq 2$ ,  $z \in U$ . Then  $f(z) \in VS_q^p([\alpha_1], \gamma)$  if*

*and only if  $f(z)$  can be expressed in the form  $f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z)$  where  $\mu_n \geq 0$  and*

$$\sum_{n=1}^{\infty} \mu_n = 1.$$

*Proof.* If  $f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z)$  with  $\sum_{n=1}^{\infty} \mu_n = 1$  and  $\mu_n \geq 0$  then

$$\sum_{n=2}^{\infty} (2n-1-\gamma)\Gamma(\alpha_1, n) \frac{1-\gamma}{(2n-1-\gamma)\Gamma(\alpha_1, n)} \mu_n = \sum_{n=2}^{\infty} \mu_n(1-\gamma) = (1-\mu_1)(1-\gamma) \leq 1-\gamma.$$

Hence  $f(z) \in VS_q^p([\alpha_1], \gamma)$ .

Conversely, if  $f(z)$  is in the class  $VS_q^p([\alpha_1], \gamma)$ , then  $|a_n| \leq \frac{1-\gamma}{(2n-1-\gamma)\Gamma(\alpha_1, n)}$ ,

$n = 2, 3, \dots$ . We may set  $\mu_n = \frac{(2n-1-\gamma)\Gamma(\alpha_1, n)}{1-\gamma}$ ,  $n \geq 2$  and  $\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n$ .

Then  $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$ , and this completes the proof. □

### 3. The class $VS_q^p([\alpha_1], k, \gamma)$

**Definition 2.** A functions  $f(z) \in S$  of the form (1) is in the  $VS_q^p([\alpha_1], k, \gamma)$  if  $f(z)$  satisfies the analytic criteria

$$(16) \quad \operatorname{Re} \left\{ \frac{z(H_q^p[\alpha_1, \beta_1]f(z))'}{H_q^p[\alpha_1, \beta_1]f(z)} - \gamma \right\} \geq k \left| \frac{z(H_q^p[\alpha_1, \beta_1]f(z))'}{H_q^p[\alpha_1, \beta_1]f(z)} - 1 \right|,$$

where  $-1 \leq \gamma < 1$ ,  $k \geq 0$ , and  $z \in U$ .

Note that the special case  $k = 1$  reduces to  $VS_q^p([\alpha_1], \gamma)$ . Since the proofs for coefficient and distortion bounds for this family are similar to those when  $k = 1$ , we state them without proof.

**Theorem 5.** (i) *The function  $f(z)$  is in  $VS_q^p([\alpha_1], k, \gamma)$  if and only if*

$$(17) \quad \sum_{n=2}^{\infty} E_n \Gamma(\alpha_1, n) |a_n| \leq 1 - \gamma \quad \text{where } E_n = n(1+k) - (k+\gamma).$$

(ii) *If  $f(z)$  is in  $VS_q^p([\alpha_1], k, \gamma)$ , then*

$$r - \frac{1-\gamma}{E_2 \Gamma(\alpha_1, 2)} r^2 \leq |f(z)| \leq r + \frac{1-\gamma}{E_2 \Gamma(\alpha_1, 2)} r^2$$

and

$$1 - \frac{2(1-\gamma)}{E_2 \Gamma(\alpha_1, 2)} r \leq |f'(z)| \leq 1 + \frac{2(1-\gamma)}{E_2 \Gamma(\alpha_1, 2)} r.$$

**Corollary 4.** *If  $f \in V$  is in  $VS_q^p([\alpha_1], k, \gamma)$  then  $a_n \leq \frac{1-\gamma}{E_n \Gamma(\alpha_1, n)}$  for  $n \geq 2$ .*

The inequality holds for the function  $f$  given by  $f(z) = z + \frac{(1-\gamma)e^{i\theta_n}z^n}{E_n\Gamma(\alpha_1, n)}$ , for  $(n = 2, 3, \dots)$  and  $z \in U$ .

**Theorem 6.** Let the function  $f(z)$  defined by (1) be in the class  $VS_q^p([\alpha_1], k, \gamma)$  and satisfy (2). Define  $f(z) = z$  and  $f_n(z) = z + \frac{1-\gamma}{E_n\Gamma(\alpha_1, n)}e^{i\theta_n}z^n$ ,  $n \geq 2$ ,  $z \in U$ . Then  $f(z) \in VS_q^p([\alpha_1], k, \gamma)$  if and only if  $f(z)$  can be expressed in the form  $f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z)$  where  $\mu_n \geq 0$  and  $\sum_{n=1}^{\infty} \mu_n = 1$ .

#### 4. Partial sums

Following the earlier works by Silverman [14] and Silvia [15] on partial sums of analytic functions, we consider in this section partial sums of functions in the class  $VS_q^p([\alpha_1], k, \gamma)$  and obtain sharp lower bounds for the ratios of real part of  $f(z)$  to  $f_m(z)$  and  $f'(z)$  to  $f'_m(z)$ .

**Theorem 7.** Let  $f(z) \in VS_q^p([\alpha_1], k, \gamma)$  be given by (1) and define the partial sums  $f_1(z)$  and  $f_m(z)$ , by

$$(18) \quad f_1(z) = z \text{ and } f_m(z) = z + \sum_{n=2}^m a_n z^n, \quad (m \in N/1).$$

Suppose also that

$$(19) \quad \sum_{n=2}^{\infty} d_n |a_n| \leq 1,$$

where

$$(20) \quad d_n := \frac{[n(1+k) - (k+\gamma)]}{(1-\gamma)} \Gamma(\alpha_1, n).$$

Then,

$$(21) \quad \operatorname{Re} \left\{ \frac{f(z)}{f_m(z)} \right\} \geq 1 - \frac{1}{d_{m+1}} \quad z \in U, \quad m \in N$$

and

$$(22) \quad \operatorname{Re} \left\{ \frac{f_m(z)}{f(z)} \right\} \geq \frac{d_{m+1}}{1 + d_{m+1}}.$$

*Proof.* For the coefficients  $d_n$  given by (20) it is not difficult to verify that

$$(23) \quad d_{n+1} > d_n > 1.$$



Therefore we have

$$(24) \quad \sum_{n=2}^m |a_n| + d_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} d_n |a_n| \leq 1.$$

Setting

$$(25) \quad \begin{aligned} g_1(z) &= d_{m+1} \left\{ \frac{f(z)}{f_m(z)} - \left( 1 - \frac{1}{d_{m+1}} \right) \right\} \\ &= 1 + \frac{d_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^m a_n z^{n-1}}, \end{aligned}$$

it suffices to show that  $g_1(z) \geq 0$ . Applying (24), we find that

$$(26) \quad \left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq \frac{d_{m+1} \sum_{n=m+1}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^m |a_n| - d_{m+1} \sum_{n=m+1}^{\infty} |a_n|} \leq 1, \quad z \in U,$$

which readily yields the assertion (21) of Theorem 7. In order to see that

$$(27) \quad f(z) = z + \frac{z^{m+1}}{d_{m+1}}$$

gives sharp result, we observe that for  $z = re^{i\pi/m}$  that  $\frac{f(z)}{f_m(z)} = 1 + \frac{z^m}{d_{m+1}} \rightarrow 1 - \frac{1}{d_{m+1}}$  as  $z \rightarrow 1^-$ . Similarly, if we take

$$(28) \quad \begin{aligned} g_2(z) &= (1 + d_{m+1}) \left\{ \frac{f_m(z)}{f(z)} - \frac{d_{m+1}}{1 + d_{m+1}} \right\} \\ &= 1 - \frac{(1 + d_{m+1}) \sum_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \end{aligned}$$

and making use of (24), we can deduce that

$$(29) \quad \left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{(1 + d_{m+1}) \sum_{n=m+1}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^m |a_n| - (1 - d_{m+1}) \sum_{n=m+1}^{\infty} |a_n|}$$

which leads us immediately to the assertion (22) of Theorem 7.

The bound in (21) is sharp for each  $m \in N$  with the extremal function  $f(z)$  given by (27), proof of the Theorem 7, is thus complete.  $\square$

**Theorem 8.** *If  $f(z)$  of the form (1) satisfies the condition (17), then*

$$(30) \quad \operatorname{Re} \left\{ \frac{f'(z)}{f'_m(z)} \right\} \geq 1 - \frac{m+1}{d_{m+1}}.$$

*Proof.* By setting

$$\begin{aligned} g(z) &= d_{m+1} \left\{ \frac{f'(z)}{f'_m(z)} - \left( 1 - \frac{m+1}{d_{m+1}} \right) \right\} \\ &= \frac{1 + \frac{d_{m+1}}{m+1} \sum_{n=m+1}^{\infty} na_n z^{n-1} + \sum_{n=2}^{\infty} na_n z^{n-1}}{1 + \sum_{n=2}^m na_n z^{n-1}} \\ &= 1 + \frac{\frac{d_{m+1}}{m+1} \sum_{n=m+1}^{\infty} na_n z^{n-1}}{1 + \sum_{n=2}^m na_n z^{n-1}}, \end{aligned} \quad (31)$$

we get

$$(32) \quad \left| \frac{g(z) - 1}{g(z) + 1} \right| \leq \frac{\frac{d_{m+1}}{m+1} \sum_{n=m+1}^{\infty} n|a_n|}{2 - 2 \sum_{n=2}^m n|a_n| - \frac{d_{m+1}}{m+1} \sum_{n=m+1}^{\infty} n|a_n|}.$$

It suffices to show that the left hand side of the (32) is bounded above by 1. This is equivalent to

$$(33) \quad \sum_{n=2}^m n|a_n| + \frac{d_{m+1}}{m+1} \sum_{n=m+1}^{\infty} n|a_n| \leq 1.$$

But the left hand side of (33) is bounded above by  $\sum_{n=2}^m d_n |a_n|$  if

$$(34) \quad \sum_{n=2}^m (d_n - n)|a_n| + \sum_{n=m+1}^{\infty} \left( d_n - \frac{d_{m+1}}{m+1} n \right) |a_n| \geq 0,$$

and the proof is complete.

The result is sharp for the extremal function  $f(z) = z + \frac{z^{m+1}}{d_{m+1}}$ .  $\square$

**Theorem 9.** *If  $f(z)$  of the form (1) satisfies the condition (17) then*

$$(35) \quad \operatorname{Re} \left\{ \frac{f'_m(z)}{f'(z)} \right\} \geq \frac{d_{m+1}}{m+1+d_{m+1}}.$$

*Proof.* By setting

$$\begin{aligned} g(z) &= [(m+1) + d_{m+1}] \left\{ \frac{f'_m(z)}{f'(z)} - \frac{d_{m+1}}{m+1+d_{m+1}} \right\} \\ &= 1 - \frac{\left(1 + \frac{d_{m+1}}{m+1}\right) \sum_{n=m+1}^{\infty} na_n z^{n-1}}{1 + \sum_{n=2}^m na_n z^{n-1}} \end{aligned}$$

and making use of (33), we can deduce that

$$\left| \frac{g(z) - 1}{g(z) + 1} \right| \leq \frac{\left(1 + \frac{d_{m+1}}{m+1}\right) \sum_{n=m+1}^{\infty} n|a_n|}{2 - 2 \sum_{n=2}^m n|a_n| - \left(1 + \frac{d_{m+1}}{m+1}\right) \sum_{n=m+1}^{\infty} n|a_n|} \leq 1,$$

which leads us immediately to the assertion of Theorem 9.  $\square$

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