Application of Bifurcation Method to a Generalized Modified Boussinesq Equation

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ABSTRACT. Bifurcation method of dynamical systems is employed to investigate exact solitary wave solutions and kink wave solutions in the generalized modified Boussinesq equation. Under some parameter conditions, their explicit expressions are obtained. Some previous results are extended.

1. Introduction and main results

Zhang et al.[1] and Kaya[2] considered the following generalized modified Boussinesq(GMB) equation

$$(1.1) \qquad u_{tt} - \delta u_{ttxx} - (a_1 u + a_2 u^{p+1} + a_3 u^{2p+1})_{xx} = 0,$$

where $\delta \neq 0$, $p \geqslant 1$, a_1, a_2 and a_3 are arbitrary constants. This equation is used to represent a nonlinear model of longitudinal wave propagation of elastic rods [2]. When $\delta = 1$, $a_1 = 1$, $a_2 = \frac{1}{p+1}$ and $a_3 = 0$, (1.1) becomes the equation

$$(1.2) \qquad u_{tt} - u_{ttxx} - (u + \frac{1}{p+1}u^{p+1})_{xx} = 0,$$

which was studied in [3], [4]. Clarkson et al.[3] and Bogolubsky[4] gave some solitary wave solutions of (1.2) with $p = 1, 2, 4$ and studied the interaction of two solitary waves numerically. When $\delta = 1$, $a_1 = 0$, $a_2 = -\frac{1}{2}$, $a_3 = 0$ and $p = 1$, (1.1) reduces to the equation

$$(1.3) \qquad u_{tt} - u_{ttxx} + \frac{1}{2}(u^2)_{xx} = 0.$$

Parker[5] also studied the solitary wave and exact solutions of (1.3). When $\delta = 1$, $p = 1, 2$, (1.1) reduces to the equation

$$(1.4) \qquad u_{tt} - u_{ttxx} - (a_1 u + a_2 u^2 + a_3 u^3)_{xx} = 0,$$

and

$$(1.5) \qquad u_{tt} - u_{ttxx} - (a_1 u + a_2 u^3 + a_3 u^5)_{xx} = 0.$$

Zhang and Ma[6] gave some explicit solitary wave solutions of (1.4) and (1.5) by using the method of solving algebraic equations. Li and Zhang[7] gave many explicit solitary wave solutions and kink wave solutions of (1.4) and (1.5) by using the bifurcation method. Zhang et al.[1] used the judgement methods and solution formulae to obtain two bell profile solitary wave solutions

(1.6)

$$u(x,t) = \left(\frac{\frac{|a_1 - v^2|(p+2)\sqrt{p+1}}{\sqrt{(p+1)a_2^2 - a_3(a_1 - v^2)(p+2)^2}} \operatorname{sech}^2 \frac{p}{2} \sqrt{-\frac{a_1 - v^2}{\delta v^2}}(x - vt + \xi_0)}{2 + \left(-1 + \frac{a_2|\delta|\sqrt{p+1}}{\delta\sqrt{(p+1)a_2^2 - a_3(a_1 - v^2)(p+2)^2}} \right) \operatorname{sech}^2 \frac{p}{2} \sqrt{-\frac{a_1 - v^2}{\delta v^2}}(x - vt + \xi_0)} \right)^{\frac{1}{p}},$$

(1.7)

$$u(x,t) = \left(\frac{-\frac{|a_1 - v^2|(p+2)\sqrt{p+1}}{\sqrt{(p+1)a_2^2 - a_3(a_1 - v^2)(p+2)^2}} \operatorname{sech}^2 \frac{p}{2} \sqrt{-\frac{a_1 - v^2}{\delta v^2}}(x - vt + \xi_0)}{2 - \left(1 + \frac{a_2|\delta|\sqrt{p+1}}{\delta\sqrt{(p+1)a_2^2 - a_3(a_1 - v^2)(p+2)^2}} \right) \operatorname{sech}^2 \frac{p}{2} \sqrt{-\frac{a_1 - v^2}{\delta v^2}}(x - vt + \xi_0)} \right)^{\frac{1}{p}}$$

and a kink profile solitary wave solution

$$(1.8) \qquad u(x,t) = \left(-\frac{a_2(p+1)}{2a_3(p+2)} \left[1 \pm \tanh \frac{p}{2}\sqrt{-\frac{a_1 - v^2}{\delta v^2}}(x - vt + \xi_0) \right] \right)^{\frac{1}{p}},$$

where wave velocity $v^2 = a_1 - \frac{a_2^2(p+1)}{a_3(p+2)^2}$, and when $p = 1, a_2 = 0$, gave a kink wave solution

$$(1.9) \qquad u(x,t) = \pm\sqrt{-\frac{a_1 - v^2}{a_3}} \tanh \frac{1}{2}\sqrt{\frac{2(a_1 - v^2)}{\delta v^2}}(x - vt + \xi_0)$$

for (1.1).

Zhang[8] used the bifurcation method to obtain a solitary wave solution and two kink wave solutions of (1.1) when p is even, $\delta = 1$, $a_3 < 0$, $a_2 > 0$ and $\frac{a_2^2}{4a_3} < a_1 - c^2 < 0$. Kaya[2] used the Adomian decomposition method to obtain a solitary wave solution

$$(1.10) \qquad u(x,t) = S_1(1 + \tanh R(x - ct))^{\frac{1}{p}},$$

for (1.1), where $p = 1, 4,$

$$(1.11) \qquad S_1 = \frac{a_2(p+1)}{2a_3(p+2)}, \quad R = p\sqrt{\frac{-a_1^2 + \zeta}{\zeta^2 \delta}}, \quad \zeta = \sqrt{a_1 - \frac{a_2^2(p+1)}{a_3(p+2)^2}}.$$

Rafei et al.[9] used the homotopy perturbation method to obtain a approximate solitary wave solution of (1.1).

In this paper, we employ similar method used [10,11] to investigate solitary wave solutions and kink wave solutions of (1.1). To facilitate discussions, we let

$$(1.12) \qquad a_3^* = \frac{(p+1)a_2^2}{(p+2)^2(a_1 - c^2)}$$

for given p, a_1, a_2 and constant $c \neq 0$. Our main results are as follows:

1.1 Solitary wave solutions

Under some parameter conditions, we give some solitary wave solutions of (1.1) in the following Theorem 1 and Theorem 2.

Theorem 1. (1) *When p is odd, $a_2 \neq 0$ and δ, a_1, a_3 satisfy one of the following conditions*
  (a) $\delta > 0$, $a_3 > 0$ and $a_1 - c^2 < 0$,
  (b) $\delta < 0$, $a_3 < 0$ and $a_1 - c^2 > 0$,
  (1.1) has two solitary wave solutions

$$(1.13) \qquad u_1(x,t) = \left(\frac{(p+2)(a_1 - c^2)}{\alpha \cosh \beta(x - ct) - a_2} \right)^{\frac{1}{p}}$$

and

$$(1.14) \qquad u_2(x,t) = \left(\frac{(p+2)(c^2 - a_1)}{\alpha \cosh \beta(x - ct) + a_2} \right)^{\frac{1}{p}},$$

where

$$(1.15) \qquad \alpha = \sqrt{\frac{(p+1)a_2^2 + (p+2)^2(c^2 - a_1)a_3}{p+1}}, \quad \beta = \frac{p}{|c|}\sqrt{\frac{c^2 - a_1}{\delta}}.$$

(2) *When p is odd, $a_2 \neq 0$ and δ, a_1, a_3 satisfy one of the following conditions*
  (a) $\delta > 0$, $a_1 - c^2 < 0$ and $a_3^* < a_3 < 0$,
  (b) $\delta < 0$, $a_1 - c^2 > 0$ and $0 < a_3 < a_3^*$,
  (1.1) has a solitary wave solution $u_1(x,t)$.
(3) *When p is odd, $\delta a_3 > 0$, $a_2 \neq 0$ and $a_1 - c^2 = 0$, (1.1) has a solitary wave solution*

$$(1.16) \qquad u_3(x,t) = \left(\frac{4\gamma\delta c^2}{\gamma^2 p^2(x - ct)^2 - 4\mu\delta c^2} \right)^{\frac{1}{p}},$$

where

(1.17)
$$\gamma = -\frac{2a_2}{p+2}, \qquad \mu = -\frac{a_3}{p+1}.$$

Theorem 2. (1) *When p is even and δ, a_1, a_2, a_3 satisfy one of the following conditions*
   (a) $\delta > 0$, $a_1 - c^2 < 0$, $a_2 \neq 0$ and $a_3 > 0$,
   (b) $\delta > 0$, $a_1 - c^2 < 0$, $a_2 > 0$ and $a_3^* < a_3 < 0$,
   (c) $\delta < 0$, $a_1 - c^2 > 0$, $a_2 \neq 0$ and $a_3 < 0$,
   (d) $\delta < 0$, $a_1 - c^2 > 0$, $a_2 < 0$ and $0 < a_3 < a_3^*$,
   (1.1) *has two solitary wave solutions $\pm u_2(x,t)$.*
(2) *When p is even, $a_1 - c^2 = 0$ and δ, a_2, a_3 satisfy one of the following conditions*
   (a) $\delta > 0$, $a_2 < 0$ and $a_3 > 0$,
   (b) $\delta < 0$, $a_2 > 0$ and $a_3 < 0$,
   (1.1) *has two solitary wave solutions $\pm u_3(x,t)$.*

1.2   Kink wave solutions

Under some parameter conditions, we give some kink wave solutions of (1.1) in the following Theorem 3 and Theorem 4.

Theorem 3. *When p is odd, $a_2 \neq 0, a_3 = a_3^*$ and $\delta(a_1 - c^2) < 0$, (1.1) has two kink wave solutions*

(1.18)
$$u_4(x,t) = \left(\frac{(p+2)(c^2 - a_1)}{\omega e^{-\beta(x-ct)} + a_2} \right)^{\frac{1}{p}},$$

and

(1.19)
$$u_5(x,t) = \left(\frac{(p+2)(c^2 - a_1)}{\omega e^{\beta(x-ct)} + a_2} \right)^{\frac{1}{p}},$$

where ω is an arbitrary constant satisfied $\omega a_2 > 0$ and β is in (1.15).

Theorem 4. *When p is even, $a_3 = a_3^*$ and δ, a_1, a_2 satisfy one of the following conditions*
(a) $\delta > 0$, $a_2 > 0$ and $a_1 - c^2 < 0$,
(b) $\delta < 0$, $a_2 < 0$ and $a_1 - c^2 > 0$, (1.1) *has four kink wave solutions $\pm u_4(x,t)$ and $\pm u_5(x,t)$.*

1.3   Remarks

Remark 1. Some previous results becoming our special cases. For example, our results $u_1(x,t)$ and $u_2(x,t)$ become Li's results (see (4.10) and (4.9) in [7]) when $\delta = 1$, $p = 1$ and $\phi_i = 0$ ($i = 1, 3$). Our results $\pm u_2(x,t)$ become Li's results (see (5.7) in [7]) when $\delta = 1$ and $p = 2$. When $\delta = 1$, $p = 1$ and $c = 1$, Our results $u_3(x,t)$

becomes Li's results (see (4.7) in [7]). When $\delta = 1$, $p = 2$ and $c = 1$, Our results $\pm u_3(x,t)$ become Li's results (see (5.2) in [7]). When $\delta = 1$, $p = 2$, $a_2 = 4\sqrt{\frac{(a_1-c^2)}{3}}$ and $\omega = a_2$, Our results $\pm u_4(x,t)$ and $\pm u_5(x,t)$ become Li's results (see (5.9) and (5.10) in [7]).

Remark 2. Our some results obtained by using bifurcation method and Zhang's some results [1] obtained by using the judgement methods and solution formulae are the same, i.e., Zhang's results (1.6) and (1.7) are the same with our results (1.13) and (1.14). When $\omega = a_2$ and $c^2 = a_1 - \frac{a_2^2(p+1)}{a_3(p+2)^2}$, our results $u_4(x,t)$ and $u_5(x,t)$ become Zhang's result (1.8).

Remark 3. Our results include Zhang's results[8], i.e., when p is even, $\delta = 1$, $a_3 < 0$, $a_2 > 0$ and $\frac{a_2^2}{4a_3} < a_1 - c^2 < 0$, our result $u_2(x,t)$ becomes Zhang's result (see (4.6) in [8]), and our results $\pm u_4(x,t)$ and $\pm u_5(x,t)$ become Zhang's result (see (4.19) and (4.20) in [8]).

2. Preliminaries

To derive our results, we given some preliminaries in this section. Substituting $u(x,t) = \varphi(\xi)$ with $\xi = x - ct$ (c is a constant) into (1.1), we have the following ordinary differential equation

$$(2.1) \qquad c^2\varphi'' - c^2\delta\varphi^{(4)} - (a_1\varphi + a_2\varphi^{p+1} + a_3\varphi^{2p+1})'' = 0.$$

Integrating (2.1) twice and letting integral constants be zero, we get

$$(2.2) \qquad c^2\varphi - c^2\delta\varphi'' - (a_1\varphi + a_2\varphi^{p+1} + a_3\varphi^{2p+1}) = 0.$$

Via (2.2) we establish a planar system

$$(2.3) \qquad \frac{d\varphi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{(c^2 - a_1)\varphi - a_2\varphi^{p+1} - a_3\varphi^{2p+1}}{c^2\delta}.$$

Obviously, if let

$$(2.4) \qquad H(\varphi, y) = c^2\delta y^2 + \varphi^2\left(\frac{a_3}{p+1}\varphi^{2p} + \frac{2a_2}{p+2}\varphi^p + a_1 - c^2\right),$$

then system (2.3) has first integral

$$(2.5) \qquad\qquad H(\varphi, y) = h,$$

where h is integral constant.

In the following, we discuss bifurcation phase portraits of system (2.3).

2.1   Bifurcation phase portraits when p is odd

Let

$$(2.6) \qquad f(\varphi) = (c^2 - a_1)\varphi - a_2\varphi^{p+1} - a_3\varphi^{2p+1},$$

then system (2.3) reduces to

$$(2.7) \qquad \frac{d\varphi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{f(\varphi)}{c^2\delta}.$$

Thus, to study the the distribution of singular points of system (2.3), we need to investigate the zero points of $f(\varphi)$. If p is odd, $a_3 \neq 0$, and $a_2^2 - 4a_3(a_1 - c^2) \geq 0$, then we get three zero points of $f(\varphi)$ as

(2.8)

$$\varphi_0 = 0, \varphi_- = \left(\frac{-a_2 - \sqrt{a_2^2 - 4a_3(a_1 - c^2)}}{2a_3} \right)^{\frac{1}{p}}, \varphi_+ = \left(\frac{-a_2 + \sqrt{a_2^2 - 4a_3(a_1 - c^2)}}{2a_3} \right)^{\frac{1}{p}},$$

and two zero points of $f'(\varphi)$ as

$$(2.9) \qquad \tilde{\varphi}_1 = \left(\frac{-(p+1)a_2 - \sqrt{(p+1)^2 a_2^2 - 4a_3(2p+1)(a_1 - c^2)}}{2a_3(2p+1)} \right)^{\frac{1}{p}},$$

$$(2.10) \qquad \tilde{\varphi}_2 = \left(\frac{-(p+1)a_2 + \sqrt{(p+1)^2 a_2^2 - 4a_3(2p+1)(a_1 - c^2)}}{2a_3(2p+1)} \right)^{\frac{1}{p}}.$$

Suppose $(\varphi^*, 0)$ is a singular point of system (2.7), then at $(\varphi^*, 0)$ the eigenvalue of the linearized system of system (2.7) is

$$(2.11) \qquad \lambda_\pm(\varphi^*, 0) = \pm\sqrt{\frac{f'(\varphi^*)}{c^2\delta}}.$$

According to the theory of dynamical systems (e.g.,[12,13]), we obtain the following conclusions.

(1) When $\delta f'(\varphi^*) < 0$, $(\varphi^*, 0)$ is a center point.
(2) When $\delta f'(\varphi^*) > 0$, $(\varphi^*, 0)$ is a saddle point.
(3) When $f'(\varphi^*) = 0$, $(\varphi^*, 0)$ is a degenerate saddle point.

Solving equations $f'(\varphi^*) = 0$ and $H(\varphi^*, 0) = 0$ respectively, we get two bifurcation curves L_1 and L_2 as follows

$$(2.12) \qquad L_1 : a_3 = \frac{1}{4(a_1 - c^2)}a_2^2,$$

and

$$(2.13) \qquad L_2 : a_3 = \frac{p+1}{(a_1 - c^2)(p+2)^2}a_2^2.$$

From the above analysis, we obtain the bifurcation phase portraits of system (2.3) as Fig.1(A), Fig.2(C), Fig.3(E), Fig.4(G), Fig.5(I) and Fig.6(K).

2.2   Bifurcation phase portraits when p is even

When p is even, similar to above analysis, we obtain the bifurcation phase portraits of system (2.3) as Fig.1(B), Fig.2(D), Fig.3(F), Fig.4(H), Fig.5(J) and Fig.6(L).
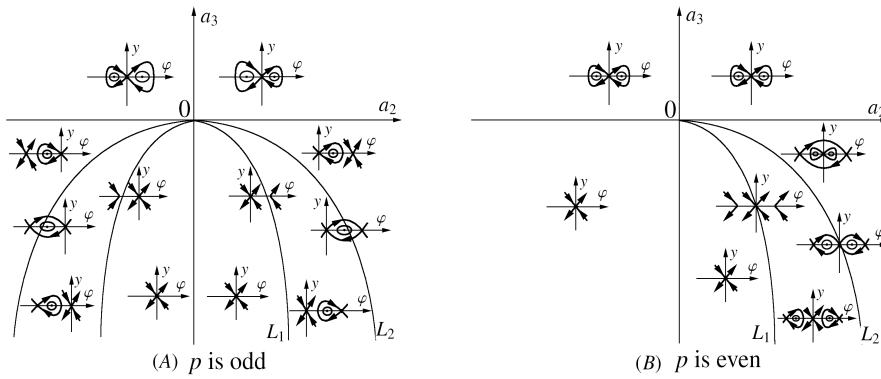

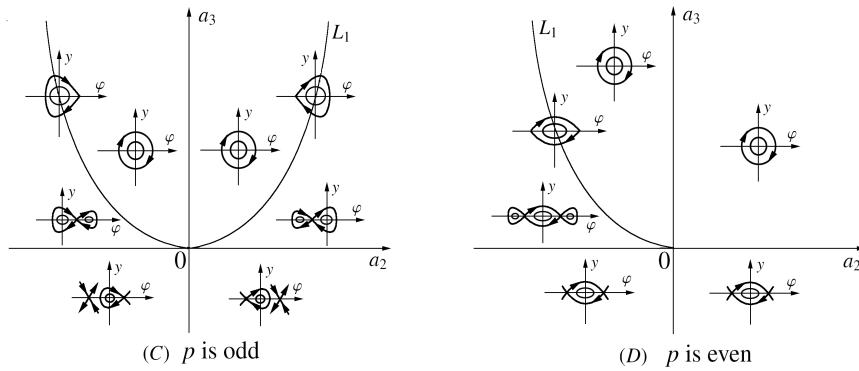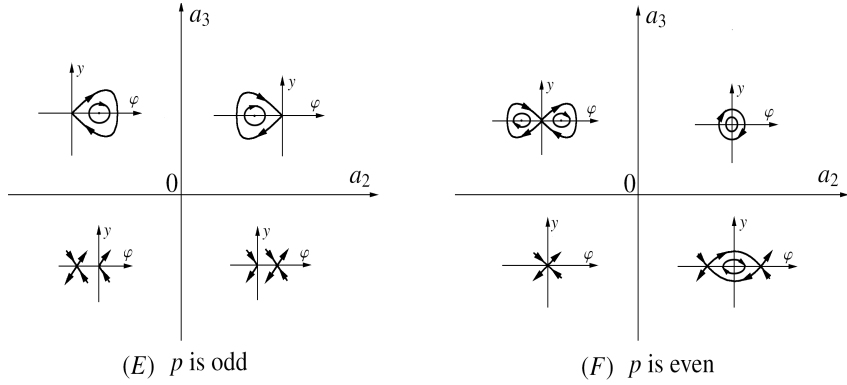Figure 1: The bifurcation phase portraits of (2.3) when $\delta > 0$ and $a_1 - c^2 < 0$.

Figure 2: The bifurcation phase portraits of (2.3) when $\delta > 0$ and $a_1 - c^2 > 0$.

3. Derive of Main Results

In this section, we derive the Theorem $i(i = 1, 2, 3, 4)$. Firstly, we derive the Theorem 1.

Figure 3: The bifurcation phase portraits of (2.3) when $\delta > 0$ and $a_1 - c^2 = 0$.
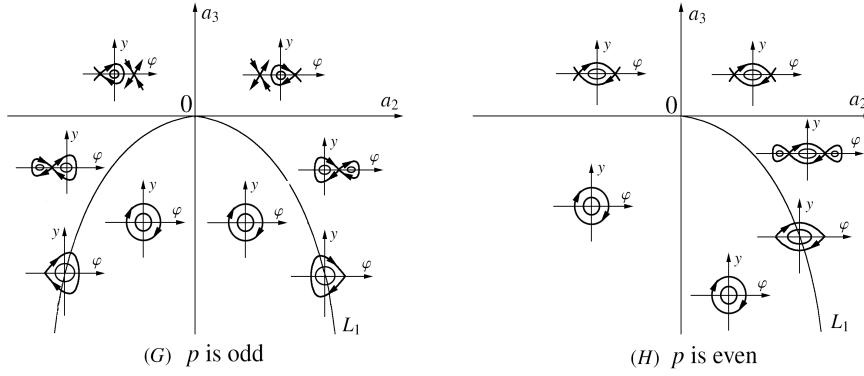

Figure 4: The bifurcation phase portraits of (2.3) when $\delta < 0$ and $a_1 - c^2 < 0$.

(1) When p is odd, $\delta > 0$, $a_2 \neq 0$, $a_3 > 0$ and $a_1 - c^2 < 0$, from Fig. 1(A) we see that system (2.3) has two homoclinic orbits (denote them as Γ_1 and Γ_2(or Γ_1^* and Γ_2^*)(see Fig.7)) which connect with singular point $(0, 0)$.
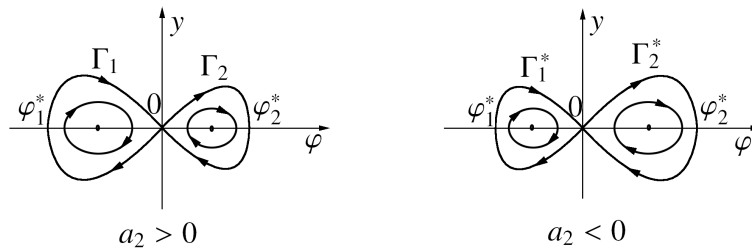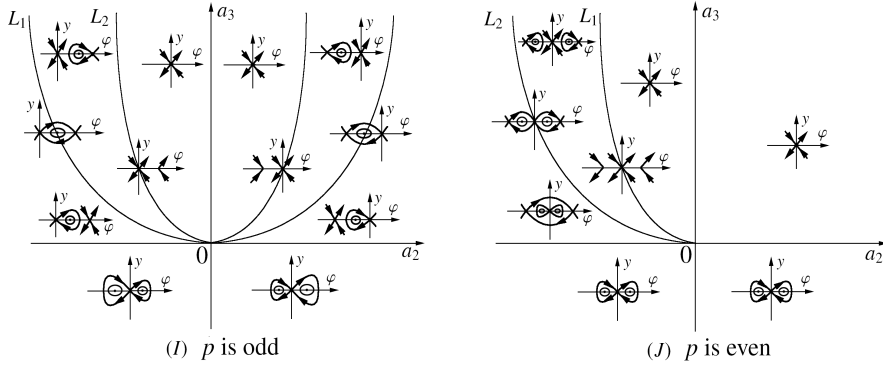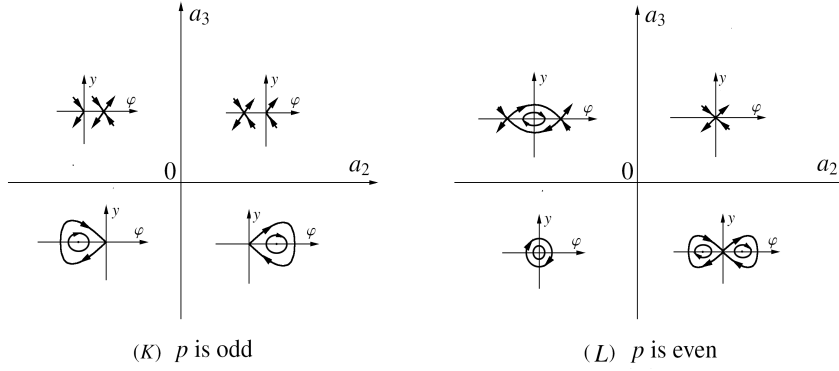

Fig. 7. Homoclinic orbits Γ_i and $\Gamma_i^*(i = 1, 2)$ when $\delta > 0$, $a_3 > 0$ and $a_1 - c^2 < 0$.

Figure 5: The bifurcation phase portraits of (2.3) when $\delta < 0$ and $a_1 - c^2 > 0$.

Figure 6: The bifurcation phase portraits of (2.3) when $\delta < 0$ and $a_1 - c^2 = 0$.

On φ-y plane, Γ_1 and Γ_2 (or Γ_1^* and Γ_2^*) are expressed by

$$(3.1) \qquad y^2 = \frac{\varphi^2(c^2 - a_1 - \frac{2a_2}{p+2}\varphi^p - \frac{a_3}{p+2}\varphi^{2p})}{c^2\delta}, \quad \varphi \in [\varphi_1^*, \varphi_2^*],$$

where

$$(3.2) \qquad \varphi_1^* = \left(\frac{-(p+1)a_2}{(p+2)a_3} - \frac{p+1}{a_3}\sqrt{\frac{a_2^2}{(p+2)^2} + \frac{a_3(c^2 - a_1)}{p+1}} \right)^{\frac{1}{p}},$$

$$(3.3) \qquad \varphi_2^* = \left(\frac{-(p+1)a_2}{(p+2)a_3} + \frac{p+1}{a_3}\sqrt{\frac{a_2^2}{(p+2)^2} + \frac{a_3(c^2 - a_1)}{p+1}} \right)^{\frac{1}{p}}.$$

Substituting (3.1) into $\frac{d\varphi}{d\xi} = y$, we have

$$(3.4) \qquad \pm\sqrt{\frac{c^2\delta}{\varphi^2(c^2 - a_1 - \frac{2a_2}{p+2}\varphi^p - \frac{a_3}{p+2}\varphi^{2p})}}\, d\varphi = d\xi.$$

Assuming $\varphi(0) = \varphi_1^*$ and integrating (3.4) along homoclinic orbit Γ_1, we get

$$(3.5) \qquad \int_{\varphi_1^*}^{\varphi} \frac{|c|\sqrt{\delta}}{s\sqrt{c^2 - a_1 - \frac{2a_2}{p+2}s^p - \frac{a_3}{p+1}s^{2p}}}\, ds = \int_{\xi}^{0} ds, \quad \xi > 0,$$

and

$$(3.6) \qquad -\int_{\varphi}^{\varphi_1^*} \frac{|c|\sqrt{\delta}}{s\sqrt{c^2 - a_1 - \frac{2a_2}{p+2}s^p - \frac{a_3}{p+1}s^{2p}}}\, ds = \int_{0}^{\xi} ds, \quad \xi \leq 0.$$

Completing above two integrals we obtain

$$(3.7) \qquad \varphi(\xi) = \left(\frac{(p+2)(a_1 - c^2)}{\alpha\cosh\beta\xi - a_2}\right)^{\frac{1}{p}},$$

where α and β are in (1.15). From $u(x,t) = \varphi(\xi)$ and $\xi = x - ct$, we obtain $u_1(x,t)$ as formula (1.13). Similarly, assuming $\varphi(0) = \varphi_2^*$ and integrating (3.4) along homoclinic orbit Γ_2, we get

$$(3.8) \qquad \int_{\varphi}^{\varphi_2^*} \frac{|c|\sqrt{\delta}}{s\sqrt{c^2 - a_1 - \frac{2a_2}{p+2}s^p - \frac{a_3}{p+1}s^{2p}}}\, ds = \int_{0}^{\xi} ds, \quad \xi < 0$$

and

$$(3.9) \qquad -\int_{\varphi_2^*}^{\varphi} \frac{|c|\sqrt{\delta}}{s\sqrt{c^2 - a_1 - \frac{2a_2}{p+2}s^p - \frac{a_3}{p+1}s^{2p}}}\, ds = \int_{\xi}^{0} ds, \quad \xi \geq 0.$$

Completing above two integrals we obtain

$$(3.10) \qquad \varphi(\xi) = \left(\frac{(p+2)(c^2 - a_1)}{\alpha\cosh\beta\xi + a_2}\right)^{\frac{1}{p}},$$

where α, β are in (1.15). From (3.10) we obtain $u_2(x,t)$ as formula (1.14). Similarly, we can derive (2), (3) of Theorem 1 and Theorem 2.

Secondly, we derive the Theorem 3. From Fig. 1(A) we see that when p is odd, $\delta > 0$, $a_2 \neq 0$, $a_3 = a_3^*$ and $a_1 - c^2 < 0$, system (2.3) has two heteroclinic orbits (denote them as $\Gamma_i(i = 3, 4)$(or $\Gamma_i^*(i = 3, 4)$)(see Fig. 8)) connecting with singular point $(0, 0)$.
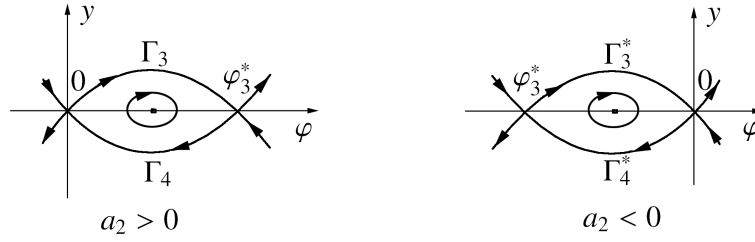
Fig. 8. Heteroclinic orbits Γ_i and $\Gamma_i^*(i = 3, 4)$ When $\delta > 0$, $a_3 = a_3^*$ and $a_1 - c^2 < 0$.

On $\varphi - y$ plane, Γ_3 and Γ_4 are expressed by

$$(3.11) \qquad y^2 = \frac{\varphi^2}{c^2\delta}\left(c^2 - a_1 - \frac{2a_2}{p+2}\varphi^p + \frac{a_2^2}{(p+2)^2(c^2-a_1)}\varphi^{2p}\right), \quad \varphi \in (0, \varphi_3^*),$$

Γ_3^* and Γ_4^* are expressed by

$$(3.12) \qquad y^2 = \frac{\varphi^2}{c^2\delta}\left(c^2 - a_1 - \frac{2a_2}{p+2}\varphi^p + \frac{a_2^2}{(p+2)^2(c^2-a_1)}\varphi^{2p}\right), \quad \varphi \in (\varphi_3^*, 0),$$

where

$$(3.13) \qquad \varphi_3^* = \left(\frac{(p+2)(c^2 - a_1)}{a_2}\right)^{\frac{1}{p}}.$$

Substituting (3.11) into $\frac{d\varphi}{dy} = y$ and integrating it along Γ_3 and Γ_4 respectively, we have

$$(3.14) \qquad \int_{\bar{\varphi}_0}^{\varphi} \frac{ds}{s\sqrt{c^2 - a_1 - \frac{2a_2}{p+2}s^p - \frac{a_2^2}{(p+2)^2(c^2-a_1)}s^{2p}}} = \int_0^{\xi} \frac{ds}{|c|\sqrt{\delta}}$$

and

$$(3.15) \qquad -\int_{\bar{\varphi}_0}^{\varphi} \frac{ds}{s\sqrt{c^2 - a_1 - \frac{2a_2}{p+2}s^p - \frac{a_2^2}{(p+2)^2(c^2-a_1)}s^{2p}}} = \int_0^{\xi} \frac{ds}{|c|\sqrt{\delta}},$$

where $\varphi(0) = \bar{\varphi}_0 \in (0, \varphi_3^*)$ is a constant. Solving (3.14) it follows that

$$(3.16) \qquad \varphi(\xi) = \left(\frac{2\sqrt{c^2 - a_1}}{\nu e^{-\beta\xi} - 2\theta}\right)^{\frac{1}{p}}.$$

Solving (3.15) we have

$$(3.17) \qquad \varphi(\xi) = \left(\frac{2\sqrt{c^2 - a_1}}{\nu e^{\beta\xi} - 2\theta}\right)^{\frac{1}{p}}.$$

In (3.16) and (3.17) ν is a integral constant, β is in (1.15) and

$$(3.18) \qquad\qquad \theta = -\frac{a_2}{(p+2)\sqrt{c^2 - a_1}}.$$

From (3.16), (3.17) we obtain $u_4(x, t)$ and $u_5(x, t)$ as (1.18) and (1.19).

Similarly, we can derive the Theorem 4. Substituting $u_i(i = 1, 2, 3, 4, 5)$ into (1.1), we find that they are its solutions indeed.

4. Conclusions

In this paper, we have employed bifurcation method in dynamical systems to study bifurcation phase portraits of (1.1)(see Fig.1-Fig.6). By using bifurcation phase portraits, its solitary wave solutions (see expressions (1.13), (1.14), (1.16)) and kink wave solutions (see expressions (1.18) and (1.19)) have been obtained.

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