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On Approximation by Post-Widder and Stancu Operators Preserving x^2

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Abstract. In the papers [5]-[7] was examined approximation of functions by the modified Szász-Mrakyan operators and other positive linear operators preserving $e_2(x) = x^2$. In this paper we introduce the Post-Widder and Stancu operators preserving x^2 in polynomial weighted spaces. We show that these operators have better approximation properties than classical Post-Widder and Stancu operators.

1. Introduction

1.1. The Post-Widder operators

$$P_n(f;x) \equiv P_n(f(t);x) := \int_0^\infty f(t)\, p_n(x,t)dt, \qquad x \in I, \;\; n \in N, \tag{1}$$

$$p_n(x,t) := \frac{(n/x)^n\, t^{n-1}}{(n-1)!} \exp\left(-\frac{nt}{x}\right), \tag{2}$$

$I = (0,\infty)$, $N = \{1, 2, \cdots\}$, were examined in many papers and monographs (e.g. [4]) for real-valued functions f bounded on I. It is known ([4], Chapter 9) that P_n are well defined also for functions $e_k(x) = x^k$, $k \in N_0 = N \cup \{0\}$, and

$$P_n(e_0;x) = 1, \qquad P_n(e_1;x) = x, \qquad P_n(e_2;x) = \frac{n+1}{n}x^2 \tag{3}$$

and generally

$$P_n\left(e_k;x\right) = \frac{n(n+1)\cdots(n+k-1)x^k}{n^k}, \qquad k \in N, \tag{4}$$

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for $x \in I$ and $n \in N$. Denoting by

$$\varphi_x(t) := t - x \quad \text{for } t \in I \text{ and a fixed } x \in I, \tag{5}$$

we have

$$P_n\left(\varphi_x^2(t); x\right) = \frac{x^2}{n} \quad \text{for } x \in I, \; n \in N. \tag{6}$$

From the results given in [4], Chapter 9, we can deduce that for every function f continuous and bounded on I there holds

$$|P_n(f; x) - f(x)| \le M\,\omega\left(f; \frac{x}{\sqrt{n}}\right), \quad x \in I, \; n \in N, \tag{7}$$

where $\omega(f; \cdot)$ is the modulus of continuity of f and $M = const. > 0$ independent on x and n.

1.2. The Stancu operators

$$L_n(f; x) \equiv L_n(f(t); x) := \int_0^\infty f(t)\, s_n(x,t) dt, \quad x \in I, \; n \in N, \tag{8}$$

$$s_n(x,t) := \frac{1}{B(nx, n+1)} \frac{t^{nx-1}}{(1+t)^{nx+n+1}}, \tag{9}$$

with the Euler beta function

$$B(a,b) := \int_0^1 t^{a-1}(1-t)^{b-1} dt \equiv \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}} dt, \quad a, b > 0,$$

were introduced in [10] for real-valued functions f bounded and locally integrable on $I = (0, \infty)$. The Stancu operators L_n are also well defined for functions $e_k(x) = x^k$, $k \in N_0$, (see [10], [1], [2]) and

$$\begin{aligned} &L_n(e_0; x) = 1, \qquad L_n(e_1; x) = x, \quad \text{for } n \in N, \\ &L_n(e_2; x) = x^2 + \frac{x(x+1)}{n-1} \quad \text{for } n \ge 2, \end{aligned} \tag{10}$$

and generally

$$L_n(e_k; x) = \frac{nx(nx+1)\cdots(nx+k-1)}{n(n-1)\cdots(n-k+1)}, \quad x \in I, \; n \ge k \ge 2. \tag{11}$$

In [10] was proved that for every function f continuous and bounded on I there holds the following inequality

$$|L_n(f; x) - f(x)| \le \left(1 + \sqrt{x(x+1)}\right) \omega\left(f; \frac{1}{\sqrt{n-1}}\right) \tag{12}$$

for $x \in I$ and $n \geq 2$, where $\omega(f;\cdot)$ is the modulus of continuity of f.

1.3. In papers [8] and [9] were examined approximation properties certain modified Post-Widder and Stancu operators for differentiable functions in polynomial weighted spaces. In [5] were investigated modified Szász-Mirakyan operators D_n^* preserving the function $e_2(x) = x^2$ and was proved that these operators have better approximation properties than classical Szász-Mirakyan operators. The similar results were given for certain positive linear operators in the papers [6] and [7].

1.4. The purpose of this note is to investigate modified Post-Widder and Stancu operators P_n^* and L_n^* preserving $e_2(x) = x^2$ in polynomial weighted spaces. These operators have better approximation properties than P_n and L_n given by (1) and (8). The definition and some properties of operators P_n^* and L_n^* will be given in Section 2. The main theorems will be given in Section 3.

1.5. First we give definition of polynomial weighted space C_r.

Similarly to [3] let $r \in N_0$,

$$w_0(x) := 1, \quad w_r(x) := (1 + x^r)^{-1} \quad \text{if} \quad r \geq 1, \ \ x \in I, \tag{13}$$

and let $C_r \equiv C_r(I)$ be the set of all real–valued functions f defined on I, for which $w_r f$ is uniformly continuous and bounded on I and the norm is given by

$$\|f\|_r \equiv \|f(\cdot)\|_r := \sup_{x\in I} w_r(x)\,|f(x)|\,. \tag{14}$$

It is obvious that if $q < r$, then $C_q \subset C_r$ and $\|f\|_r \leq \|f\|_q$ for $f \in C_q$. For $f \in C_r$, $r \in N_0$, we shall consider the modulus of continuity

$$\omega(f; C_r; t) := \sup_{0\leq h\leq t} \|\Delta_h f(\cdot)\|_r\,, \qquad t \geq 0, \tag{15}$$

where $\Delta_h f(x) = f(x+h) - f(x)$.

In this paper we shall apply the following inequalities

$$(w_r(x))^2 \leq w_{2r}(x), \qquad (w_r(x))^{-2} \leq 4(w_{2r}(x))^{-1}, \tag{16}$$

for $x \in I$ and $r \in N_0$, which immediately result from (13).

We shall denote by $M_i(r)$, $i \in N$, suitable positive constants depending only on indicated parameter r.

2. The definition and elementary properties of P_n^* and L_n^*

2.1. We introduce for $f \in C_r$, $r \in N_0$, the following modified Post-Widder operators P_n^*

$$P_n^*(f;x) := \int_0^\infty f(t)\,p_n(u_n(x),t)dt = P_n(f;u_n(x)), \quad x \in I, \ \ n \in N, \tag{17}$$

where $P_n(f)$ and p_n are given by (1) and (2) and

$$u_n(x) := \sqrt{\frac{n}{n+1}}\, x, \tag{18}$$

and modified Stancu operators

$$L_n^*(f;x) := \int_0^\infty f(t)\, s_n(v_n(x),t)dt = L_n(f;v_n(x)) \tag{19}$$

for $x \in I$ and $n \geq r \geq 2$ or $n \geq 2$ if $r = 0, 1$, where $L_n(f)$ and s_n are given by (8) and (9) and

$$v_n(x) := \frac{-1 + \sqrt{1 + 4n(n-1)x^2}}{2n}\ . \tag{20}$$

2.2. The formulas (18) and (20) imply that

$$0 < u_n(x) < x, \qquad 0 \leq v_n(x) \leq x \quad \text{for } \ x \in I, \ \ n \in N. \tag{21}$$

From (17)-(20) and (1)-(4) and (8)-(11) we immediately obtain the following

Lemma 1. *Let $e_k(x) = x^k$ for $k \in N_0$ and $x \in I$. Then for all $x \in I$ and $n \in N$ we have*

$$P_n^*(e_0;x) = 1, \quad P_n^*(e_1;x) = u_n(x), \quad P_n^*(e_2;x) = x^2 \tag{22}$$

and

$$P_n^*(e_k;x) = \frac{n(n+1)\cdots(n+k-1)u_n^k(x)}{n^k} \quad \text{if } \ k \geq 3.$$

Moreover, for $x \in I$ and $n \geq 2$ we have

$$L_n^*(e_0;x) = 1, \quad L_n^*(e_1;x) = v_n(x), \quad L_n^*(e_2;x) = x^2 \tag{23}$$

and generally

$$L_n^*(e_k;x) = \frac{nv_n(x)(nv_n(x)+1)\cdots(nv_n(x)+k-1)}{n(n-1)\cdots(n-k+1)} \ \ \text{for } \ n \geq k \geq 2.$$

The formulas (22) and (23) show that P_n^* and L_n^* preserve the functions e_0 and e_2.

Lemma 2. *For function φ_x given by (5) there hold the following analogies of (6):*

$$P_n^*(\varphi_x^2(t);x) = 2x(x - u_n(x)) \leq \frac{x^2}{n} \quad \text{for } \ x \in I, \ \ n \in N, \tag{24}$$

and

$$L_n^*(\varphi_x^2(t);x) = 2x(x - v_n(x)) \leq \frac{x(x+1)}{n-1} \ \ \text{for } \ x \in I, \ \ n \geq 2. \tag{25}$$

Proof. We shall prove only (25) because the proof of (24) is analogous. By linearity of L_n^* and (5) and (23) we have

$$\begin{aligned} L_n^*(\varphi_x^2(t);x) &= L_n^*(e_2;x) - 2xL_n^*(e_1;x) + x^2 L_n^*(e_0;x) \\ &= 2x(x - v_n(x)) \quad \text{for } x > 0, \ \ n \geq 2. \end{aligned}$$

Next, by (20) we get

$$\begin{aligned} 0 < x - v_n(x) &= \frac{2nx + 1 - \sqrt{1 + 4n(n-1)x^2}}{2n} \\ &= \frac{2x(x+1)}{2nx + 1 + \sqrt{1 + 4n(n-1)x^2}} \leq \frac{2x(x+1)}{2nx + 1 + 2(n-1)x} \\ &\leq \frac{2x(x+1)}{4(n-1)x} = \frac{x+1}{2(n-1)} \quad \text{for } x > 0, \ \ n \geq 2. \end{aligned}$$

This completes the proof of (25). □

Lemma 3. *Let $r \in N_0$ and let w_r be the weighted function given by* (13)*. Then for $n \in N$ the following inequalities*

(26) $$\|P_n^*(1/w_r)\|_r \leq 1, \qquad \|L_n^*(1/w_r)\|_r \leq 1 \quad \text{if } r = 0, 1,$$

(27) $$\|P_n^*(1/w_r)\|_r \leq 1 + (r-1)!\ , \quad \text{if } r \geq 2,$$

and

(28) $$\|L_n^*(1/w_r)\|_r \leq 1 + 2^{2r-1}(1 + r^{r-1}) \quad \text{for } n \geq r \geq 2,$$

hold. Moreover, for every $f \in C_r$ we have

(29) $$\|P_n^*(f)\|_r \leq \|f\|_r \, \|P_n^*(1/w_r)\|_r, \quad n \in N,$$

(30) $$\|L_n^*(f)\|_r \leq \|f\|_r \, \|L_n^*(1/w_r)\|_r, \quad n \geq r.$$

The formulas (17)-(20) *and inequalities* (29) *and* (30) *show that P_n^*, $n \in N$, and L_n^* with $n \geq r$ are positive linear operators acting from the space C_r to C_r, $r \in N_0$.*

Proof. Similarly to Lemma 2 we shall consider only operators L_n^*. The inequality (26) is obvious by (13), (23), (21) and (14). If $r \geq 2$, then by linearity of L_n^* and

(13), Lemma 1 and (21) we get

$$\begin{aligned} L_n^*(1/w_r;x) &= L_n^*(e_0;x)+L_n^*(e_r;x) \\ &\le 1+\frac{nx(nx+1)\cdots(nx+r-1)}{n(n-1)\cdots(n-r+1)} \\ &\le 1+\frac{n^{r-1}x(x+1/n)\cdots(x+(r-1)/n)}{(n-1)(n-2)\cdots(n-r+1)} \\ &\le 1+\frac{2^{r-1}\{(n-r+1)^{r-1}+r^{r-1}\}(x+1)^r}{(n-r+1)^{r-1}} \\ &\le 1+2^{2r-1}(1+r^{r-1})(1+x^r) \end{aligned}$$

for $x\in I$ and $n\ge r$. This inequality and (14) imply (28).

The inequality (30) immediately follows from (19) and (14). □

Applying the Hölder inequality and Lemma 2, Lemma 3 and (16), we easily obtain the following

Lemma 4. *Let $r\in N_0$ and let φ_x be given by* (5). *Then there exist $M_i(r)=const.>0$, $i=1,2$, such that for $x\in I$ and $n\in N$*

$$w_r(x)P_n^*\left(\frac{|\varphi_x(t)|}{w_r(t)};x\right)\le M_1(r)\sqrt{2x(x-u_n(x))} \tag{31}$$

and

$$w_r(x)L_n^*\left(\frac{|\varphi_x(t)|}{w_r(t)};x\right)\le M_2(r)\sqrt{2x(x-v_n(x))},\quad \textit{for}\ \ n\ge 2r. \tag{32}$$

3. Theorems

3.1. Denote by $C_r^1\equiv C_r^1(I)$, with a fixed $r\in N_0$, the set of all functions $f\in C_r$ which the first derivative belonging also to C_r.

Theorem 1. *Let $r\in N_0$. Then there exist $M_i(r)=const.>0$, $i=3,4$, such that for every $f\in C_r^1$, $x\in I$ and $n\in N$ the following inequalities*

$$w_r(x)|P_n^*(f;x)-f(x)|\le M_3(r)\,\|f'\|_r\sqrt{2x(x-u_n(x))} \tag{33}$$

and

$$w_r(x)|L_n^*(f;x)-f(x)|\le M_4(r)\,\|f'\|_r\sqrt{2x(x-v_n(x))},\quad n\ge 2r, \tag{34}$$

hold.

Proof. From (17), (18) and Lemma 1 we deduce that

$$|P_n^*(f(t);x)-f(x)|=|P_n^*(f(t)-f(x);x)|\le P_n^*\left(|\int_x^t f'(y)dy|;x\right)$$

for every $f \in C_r^1$, $x \in I$ and $n \in N$. Next by (13) and (14) we have

$$\begin{aligned} \left|\int_x^t f'(y)dy\right| &\le \|f'\|_r \left|\int_x^t \frac{dy}{w_r(y)}\right| \\ &\le \|f'\|_r \left(\frac{1}{w_r(t)} + \frac{1}{w_r(x)}\right)|t-x|, \quad x,t \in I. \end{aligned}$$

Consequently, we get

$$w_r(x)|P_n^*(f(t);x) - f(x)| \le \|f'\|_r \left\{ P_n^*\left(\frac{|\varphi_x(t)|}{w_r(t)};x\right) + P_n^*\left(\frac{|\varphi_x(t)|}{w_0(t)};x\right)\right\},$$

for $x \in I$, $n \in N$, where φ_x is defined by (5). Now using (31), we obtain the desired estimation (33).

Similarly, applying (32), we obtain (34). □

Theorem 2. *Let $r \in N_0$. Then there exist $M_i(r) = const. > 0$, $i = 5, 6$, such that for every $f \in C_r$, $x \in I$ and $n \in N$ we have*

$$w_r(x)|P_n^*(f;x) - f(x)| \le M_5(r)\,\omega(f;C_r;\sqrt{2x(x-u_n(x))}) \tag{35}$$

and

$$w_r(x)|L_n^*(f;x) - f(x)| \le M_6(r)\,\omega(f;C_r;\sqrt{2x(x-v_n(x))}), \quad n \ge 2r, \tag{36}$$

where $\omega(f;C_r)$is the modulus of continuity of f defined by (15).

Proof. Because the proofs of (35) and (36) are analogous, we shall prove only (35). We shall use the Stieklov function f_h of $f \in C_r$, i.e.

$$f_h(x) := \frac{1}{h}\int_0^h f(x+t)dt, \quad x,h > 0. \tag{37}$$

From (37) and (15) it follows that

$$\|f_h - f\|_r \le \omega(f;C_r;h), \tag{38}$$

$$\|f_h'\|_r \le h^{-1}\omega(f;C_r;h), \tag{39}$$

for every $f \in C_r$ and $h > 0$. These inequalities show that if $f \in C_r$ with a fixed $r \in N_0$, then $f_h \in C_r^1$ for every $h > 0$. Hence for $f \in C_r$ and $h > 0$ we can write

$$\begin{aligned} P_n^*(f(t);x) - f(x) = P_n^*(f(t) - f_h(t);x) + P_n^*(f_h(t);x) - f_h(x) \\ + f_h(x) - f(x) \quad \text{for } x \in I, \ n \in N. \end{aligned} \tag{40}$$

By (29), (26), (27) and (38)we see that there exists $M_7(r) = constant > 0$ such that

$$\begin{aligned} w_r(x)|P_n^*(f(t) - f_h(t);x)| &\le M_7(r)\,\|f - f_h\|_r \\ &\le M_7(r)\,\omega(f;C_r;h). \end{aligned} \tag{41}$$

Applying Theorem 1 for f_h and (39), we get

$$
\begin{aligned}
(42)\quad w_r(x)|P_n^*(f_h(t);x)-f_h(x)| &\le M_3(r)\|f_h'\|_r\sqrt{2x(x-u_n(x))}\\
&\le M_3(r)\,h^{-1}\,\omega(f;C_r;h))\sqrt{2x(x-u_n(x)}.
\end{aligned}
$$

Using (41), (42) and (38), we deduce from (40)

$$
(43)\quad w_r(x)|P_n^*(f;x)-f(x)| \le M_8(r)\,\omega(f;C_r;h)\times\left\{1+h^{-1}\sqrt{2x(x-u_n(x))}\right\}
$$

for $x>0$, $h>0$ and $n\in N$. Now, for given x and n setting $h=\sqrt{2x(x-u_n(x))}$ to (43), we obtain desired inequality (35) and we complete the proof. □

From Theorem 2 and Lemma 2 results the following

Corollary. *For every $f\in C_r$, $r\in N_0$, we have $\lim_{n\to\infty} P_n^*(f;x)=f(x)$, $x\in I$, and this convergence is uniform on every interval $[a,b]$, $a>0$.*

The above statement is also true for Stancu operators L_n^*.

3.2. Considering the Stancu operators L_n in polynomial weighted spaces C_r and using methods of proofs of Theorem 1 and Theorem 2, we can obtain the following estimation

$$
(44)\qquad w_r(x)|L_n(f;x)-f(x)| \le M_9(r)\,\omega\left(f;C_r;\sqrt{\frac{x(x+1)}{n-1}}\right),
$$

for every $f\in C_r$, $r\in N_0$, $x>0$ and $n\ge 2r+2$.

The inequalities (44), (36) and (12) show that the Stancu operators L_n^* have better approximation properties than L_n for functions $f\in C_r$, $r\in N_0$, and $n\ge 2r+2$. Moreover, by (20) and Lemma 2 we get for arguments of moduli of continuity of f given in (36) and (44)

$$
\begin{aligned}
&\sqrt{\frac{x(x+1)}{n-1}}-\sqrt{2x(x-v_n(x))}\\
=&\sqrt{\frac{x(x+1)}{n-1}}-\frac{\sqrt{4x^2(x+1)}}{\sqrt{2nx+1+\sqrt{1+4n(n-1)x^2}}}\\
=&\sqrt{\frac{x(x+1)}{n-1}}\left(1-\frac{\sqrt{4(n-1)x}}{\sqrt{2nx+1+\sqrt{1+4n(n-1)x^2}}}\right)
\end{aligned}
$$

$$= \sqrt{\frac{x(x+1)}{\sqrt{n-1}}}\, \frac{\sqrt{1+4n(n-1)x^2} - 2(n-1)x + 2x + 1}{\sqrt{2nx+1+\sqrt{1+4n(n-1)x^2}}}$$

$$\times \frac{1}{\left[\sqrt{2nx+1+\sqrt{1+4n(n-1)x^2}} + \sqrt{4(n-1)x}\right]}$$

$$> \sqrt{\frac{x(x+1)}{n-1}}\, \frac{2x+1}{\sqrt{4nx+2}\left[\sqrt{4nx+2} + \sqrt{4(n-1)x}\right]}$$

$$> \sqrt{\frac{x(x+1)}{n-1}}\, \frac{2x+1}{2(4nx+2)} > \sqrt{\frac{x(x+1)}{n-1}}\, \frac{1}{4n},$$

for all $x > 0$ and $n \geq 2r+2$.

Analogously, estimations (7), (35) and (24) show that P_n^*, $n \in N$, have better approximation properties than P_n for functions $f \in C_r$ (see [8]). Moreover, by (7), (35) and (18) we can obtain

$$\frac{x}{\sqrt{n}} - \sqrt{2x(x-u_n(x))} \geq \frac{x}{4(n+1)\sqrt{n}} \quad \text{for } x > 0 \text{ and } n \in N.$$

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