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# On Sufficient Conditions for Certain Subclass of Analytic Functions Defined by Convolution

Paramasivam Sooriyakala*
*Department of Mathematics, Jaya College of Arts and Science, Thiruninravur, 602 204, India*
*e-mail* : marikkannan_1947@yahoo.co.in

Natarajan Marikkannan
*Department of Mathematics, Government Arts College, Paramakudi 623 707, India*
*e-mail* : natarajan.marikkannan@gmail.com

Abstract. In the present investigation sufficient conditions are found for certain subclass of normalized analytic functions defined by Hadamard product. Differential sandwich theorems are also obtained. As a special case of this we obtain results involving Ruscheweyh derivative, Sălăgean derivative, Carlson-shaffer operator, Dziok-Srivatsava linear operator, Multiplier transformation.

## 1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

For two functions $f(z)$ defined as in (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ the Hadamard product or convolution of $f(z)$ and $g(z)$, denoted by $(f * g)(z)$, is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

For $\alpha_j \in \mathbb{C}, (j = 1, 2, \cdots, l)$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, -3, \cdots\}, (j = 1, 2, \cdots, m)$, the Dziok-Srivatsava linear operator [7] for functions in $\mathcal{A}$ is defined as follows:

$$H^{l,m}(\alpha_1)f(z) := z + \sum_{n=2}^{\infty} \Gamma_n a_n z^n,$$

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where

$$\Gamma_n := \frac{(\alpha_1)_{n-1}\cdots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}\cdots(\beta_m)_{n-1}(1)_{n-1}}, \tag{1.2}$$

where $(\lambda)_n$ is the Pocchhammer symbol defined by

$$(\lambda)_n := \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1) & (n=1,2,3,\cdots). \end{cases}$$

On defining $g(z) = z + \sum_{n=2}^{\infty} \Gamma_n z^n$, we see that $(f * g)(z) = H^{l,m}(\alpha_1)f(z)$.

By taking $l = 2, m = 1, \alpha_1 = a, \alpha_2 = 1$ and $\beta_1 = c$ we see that

$$(f * g)(z) = H^{2,1}(a)f(z) = L(a,c)f(z),$$

where $L(a,c)f(z)$ denotes the Carlson-Shaffer linear operator [5].

On choosing $\frac{z}{(1-z)^{\lambda+1}}(\lambda > -1)$, $z + \sum_{n=2}^{\infty} n^m a_n z^n$ and $z + \sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^m z^n$ as $g(z)$, we find $(f * g)(z)$ as $D^{\lambda}f(z), \mathcal{D}^m f(z)$ and $I(r,\lambda)f(z)$ respectively, where $D^{\lambda}, \mathcal{D}^m$, and $I(m,\lambda)$ denotes Ruscheweyh derivative of order $\lambda$ , Sălăgean derivative of order $m$ and Multiplier transformation.

Let $\mathcal{H}$ denotes the class of all analytic functions defined on the open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisiting of functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots$. For two analytic functions $f$ and $F$, we say $F$ is superordinate to $f$, if $f$ is subordinate to $F$. Let $p, h \in \mathcal{H}$ and let $\phi(r,s,t;z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}$. If $p$ and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if $p$ satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z), \tag{1.3}$$

then $p$ is the solution of the differential superordination (1.3). An analytic function $q(z)$ is called *subordinant*, if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.3). A univalent subordinant $\tilde{q(z)}$ that satisfies $q(z) \prec \tilde{q(z)}$ for all subordinants $q(z)$ of (1.3), is said to be *best subordinant*. Recently Miller and Mocanu [3] considered certain first and second order differential superordinations. Using the results of Miller and Mocanu [3], Bulboacă have considered certain classes of first order differential superordinations [2] as well as superordination preserving integral operators [1].

In the present investigation we obtain the sufficient conditions for normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec \frac{z^2(f*g)'(z)}{[(f*g)(z)]^2} \prec q_2(z),$$

where $g(z)$ is the fixed analytic function in $\mathcal{A}$.

## 2. Preliminaries

For the present study we may need the following definitions and results.

**Definition 2.1** ([3, Definition 2, p.817]) **.** Denote by $\mathcal{Q}$, the set of all functions $f(z)$ that are analytic and univalent in $\overline{\Delta} \setminus E(f)$, where

$$E(f) := \{\zeta \in \partial\Delta : \lim_{z\to\zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta \setminus E(f)$.

**Theorem 2.1** (cf. Miller and Mocanu [4, Theorem 3.4h, p.132]) **.** *Let $q(z)$ be univalent in $\Delta$ and $\theta$ and $\phi$ be analytic in a domain $D$ containing $q(\Delta)$ with $\phi(w) \neq 0$, when $w \in q(\Delta)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that*
(i) *$Q(z)$ is starlike univalent in $\Delta$ and*
(ii) $\Re\left\{\dfrac{zh'(z)}{Q(z)}\right\} > 0$ *for $z \in \Delta$.*
*If $p$ is analytic in $\Delta$ with $p(\Delta) \subseteq D$ and*

$$\theta(p(z)) + zp'(z)\phi((p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) \tag{2.1}$$

*then*

$$p(z) \prec q(z)$$

*and $q(z)$ is the best dominant.*

**Theorem 2.2 ([2]).** *Let $q(z)$ be univalent in $\Delta$ and $\theta$ and $\phi$ be analytic in domain $D$ containing $q(\Delta)$. Suppose that*
(i) $\Re\left(\dfrac{\theta'(q(z))}{\phi(q(z))}\right) \geq 0$ *for $z \in \Delta$ and*
(ii) *$Q(z) = zq'(z)\phi(q(z))$ is starlike univalent in $\Delta$.*
*If $p \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$ with $p(\Delta) \subseteq D$ and $\theta(p(z)) + zp'(z)\phi(p(z))$ is univalent in $\Delta$, and*

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)), \tag{2.2}$$

*then*

$$q(z) \prec p(z)$$

*and $q(z)$ is the best subordinant.*

## 3. Main results

Throughout this paper we assume that $\alpha, \beta, \gamma$ and $\delta$ are complex numbers and $\delta \neq 0$.

**Theorem 3.1.** *Let $q(z)$ be a convex univalent in $\Delta$ with $q(0) = 1$. Assume that*

$$\Re\left\{\frac{\beta q(z) + 2\gamma q^2(z)}{\delta} - \frac{zq'(z)}{q(z)}\right\} > 0. \tag{3.1}$$

*Let*

$$\psi(z) := \alpha+\beta\frac{z^2(f*g)'(z)}{[(f*g)(z)]^2}+\gamma\left(\frac{z^2(f*g)'(z)}{[(f*g)(z)]^2}\right)^2 +\delta\left[\frac{(z(f*g)(z))''}{(f*g)'(z)}-\frac{2z(f*g)'(z)}{(f*g)(z)}\right]. \tag{3.2}$$

*If* $f\in\mathcal{A}$ *and*

$$\psi(z)\prec\alpha+\beta q(z)+\gamma q^2(z)+\delta\frac{zq'(z)}{q(z)}, \tag{3.3}$$

*then*

$$\frac{z^2(f*g)'(z)}{[(f*g)(z)]^2}\prec q(z)$$

*and* $q(z)$ *is the best dominant.*

*Proof.* Define the functions $p(z)$ by

$$p(z) := \frac{z^2(f*g)'(z)}{[(f*g)(z)]^2}. \tag{3.4}$$

Then clearly $p(z)$ is analytic in $\Delta$. Also by a simple computation, we find from (3.4) that

$$\frac{zp'(z)}{p(z)}=\frac{(z(f*g)(z))''}{(f*g)'(z)}-\frac{2z(f*g)'(z)}{(f*g)(z)}.$$

Also we find that

$$\psi(z) := \alpha+\beta\frac{z^2(f*g)'(z)}{[(f*g)(z)]^2}+\gamma\left(\frac{z^2(f*g)'(z)}{[(f*g)(z)]^2}\right)^2 +\delta\left[\frac{(z(f*g)(z))''}{(f*g)'(z)}-\frac{2z(f*g)'(z)}{(f*g)(z)}\right]=\alpha+\beta p(z)+\gamma p^2(z)+\delta\frac{zp'(z)}{p(z)}. \tag{3.5}$$

In view of (3.5) the subordination (3.3) becomes

$$\alpha+\beta p(z)+\gamma p^2(z)+\delta\frac{zp'(z)}{p(z)}\prec\alpha+\beta q(z)+\gamma q^2(z)+\delta\frac{zq'(z)}{q(z)}$$

and this can be rewritten as (2.1), where $\theta(w) := \alpha+\beta w+\gamma w^2$ and $\phi(w)=\dfrac{\delta}{w}$. Note that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}\setminus\{0\}$. Since $\delta\neq 0$, we have $\phi(w)\neq 0$. Let the functions $Q(z)$ and $h(z)$ defined as

$$Q(z) := zq'(z)\phi(q(z))=\delta\frac{zq'(z)}{q(z)},$$

$$h(z) := \alpha + \beta q(z) + \gamma q^2(z) + \delta \frac{zq'(z)}{q(z)}.$$

In light of hypothesis of Theorem 2.1, we see that $Q(z)$ is starlike and

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{\frac{\beta q(z) + 2\gamma q^2(z)}{\delta} - \frac{zq'(z)}{q(z)} + (1 + \frac{zq''(z)}{q'(z)})\right\}.$$

Hence the result follows as an application of Theorem 2.1. □

By taking $\alpha = \beta = \gamma = 0$ and $\delta = 1$ in Theorem  we get the following result of Ravichandran *et.al.*[10].

**Corollary 3.2.** *If* $f(z) \in \mathcal{A}$ *and*

$$\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \prec \frac{zq'(z)}{q(z)},$$

*then*

$$\frac{z^2 f'(z)}{f^2(z)} \prec q(z).$$

**Theorem 3.3.** *Let* $q(z)$ *be convex univalent in* $\Delta$ *with* $q(0) = 1$ *and satisfies*

$$\Re\left\{\frac{\beta q(z) + 2\gamma(q(z))^2}{\delta}\right\} > 0. \tag{3.6}$$

*If* $f \in \mathcal{A}, 0 \neq \frac{z^2(f*g)'(z)}{[(f*g)(z)]^2} \in \mathcal{H}[1,1] \cap \mathcal{Q}$ *and* $\psi(z)$ *as defined by* (3.2) *is univalent in* $\Delta$*, then*

$$\alpha + \beta q(z) + \gamma q^2(z) + \delta \frac{zq'(z)}{q(z)} \prec \psi(z) \tag{3.7}$$

*implies*

$$q(z) \prec \frac{z^2(f*g)'(z)}{[(f*g)(z)]^2}$$

*and* $q(z)$ *is best subordinant.*

*Proof.* In view of (3.5) the superordination (3.7) becomes

$$\alpha + \beta q(z) + \gamma q^2(z) + \delta \frac{zq'(z)}{q(z)} \prec \alpha + \beta p(z) + \gamma p^2(z) + \delta \frac{zp'(z)}{p(z)}$$

and this can be written as (2.2), where $\theta(w) = \alpha + \beta w + \gamma w^2$ and $\phi(w) = \frac{\delta}{w}$. Note that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C} \setminus \{0\}$. In light of hypothesis of Theorem 2.2, we see that

$$\Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} = \Re\left\{\frac{\beta q(z) + 2\gamma(q(z))^2}{\delta}\right\}.$$

Hence the result follows as an application of Theorem 2.2. □

By combining Theorem 3.1 and Theorem 3.3 we get the following sandwich result.

**Theorem 3.4.** *Let $q_1(z)$ and $q_2(z)$ be convex univalent functions defined on $\Delta$ with $q_1(0) = q_2(0) = 1$ where $q_1(z)$ satisfies* (3.6) *and $q_2(z)$ satisfies* (3.1). *Let $f \in \mathcal{A}$ and $0 \neq \frac{z^2(f*g)'(z)}{[(f*g)(z)]^2} \in \mathcal{H}[1,1] \cap \mathcal{Q}$ and $\psi(z)$ as defined by* (3.2) *is univalent in $\Delta$, then*

$$\alpha + \beta q_1(z) + \gamma q_1^2(z) + \delta \frac{zq_1'(z)}{q_1(z)} \prec \psi(z) \prec \alpha + \beta q_2(z) + \gamma q_2^2(z) + \delta \frac{zq_2'(z)}{q_2(z)}$$

*implies*

$$q_1(z) \prec \frac{z^2(f*g)'(z)}{[(f*g)(z)]^2} \prec q_2(z),$$

*where $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and best dominant.*

By taking $g(z) = z + \sum_{n=2}^{\infty} \Gamma_n z^n$ in Theorem 3.4, where $\Gamma_n$ is as defined in (1.2), we get the following result involving Dziok-Srivatsava operator.

**Corollary 3.5.** *Let $q_1(z)$ and $q_2(z)$ be convex univalent functions defined on $\Delta$ with $q_1(0) = q_2(0) = 1$ where $q_1(z)$ satisfies (3.6) and $q_2(z)$ satisfies (3.1). Let $f \in \mathcal{A}$ and $0 \neq \frac{z^2[H^{l,m}(\alpha_1)f(z)]'}{[H^{l,m}(\alpha_1)f(z)]^2} \in \mathcal{H}[1,1] \cap \mathcal{Q}$ and*

$$\begin{aligned}\psi(z) \quad = \quad & \alpha + \beta \frac{z^2[H^{l,m}(\alpha_1)f(z)]'}{[H^{l,m}(\alpha_1)f(z)]^2} + \gamma \left[\frac{z^2[H^{l,m}(\alpha_1)f(z)]'}{[H^{l,m}(\alpha_1)f(z)]^2}\right]^2 \\ & + \delta \left[\frac{(zH^{l,m}(\alpha_1)f(z))''}{(H^{l,m}(\alpha_1)f(z))'} - \frac{2z(H^{l,m}(\alpha_1)f(z))'}{H^{l,m}(\alpha_1)f(z)}\right]\end{aligned}$$

*is univalent in $\Delta$ then*

$$\alpha + \beta q_1(z) + \gamma q_1^2(z) + \delta \frac{zq_1'(z)}{q_1(z)} \prec \psi(z) \prec \alpha + \beta q_2(z) + \gamma q_2^2(z) + \delta \frac{zq_2'(z)}{q_2(z)}$$

*implies*

$$q_1(z) \prec \frac{z^2[H^{l,m}(\alpha_1)f(z)]'}{[H^{l,m}(\alpha_1)f(z)]^2} \prec q_2(z),$$

*where $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and best dominant.*

By taking $l = 2, m = 1, \alpha_1 = a, \alpha_2 = 1$ and $\beta_1 = c$ in Corollary 3.5 we get the following result involving Carlson-Shaffer linear operator.

**Corollary 3.6.** *Let $q_1(z)$ and $q_2(z)$ be convex univalent functions defined on $\Delta$*

*with* $q_1(0) = q_2(0) = 1$, *where* $q_1(z)$ *satisfies* (3.6) *and* $q_2(z)$ *satisfies* (3.1). *Let* $f \in \mathcal{A}$ *and* $0 \neq \frac{z^2(L(a,c)f(z))'}{(L(a,c)f(z))^2} \in \mathcal{H}[1,1] \cap \mathcal{Q}$ *and*

$$\begin{aligned}\psi(z) \quad := \quad & \alpha + \beta\frac{z^2(L(a,c)f(z))'}{(L(a,c)f(z))^2} + \gamma\left[\frac{z^2(L(a,c)f(z))'}{(L(a,c)f(z))^2}\right]^2 \\ & +\delta\left[\frac{(zL(a,c)f(z))''}{(L(a,c)f(z))'} - \frac{2z(L(a,c)f(z))'}{L(a,c)f(z)}\right]\end{aligned}$$

*is univalent in* $\Delta$ *then*

$$\alpha + \beta q_1(z) + \gamma q_1^2(z) + \delta\frac{zq_1'(z)}{q_1(z)} \prec \psi(z) \prec \alpha + \beta q_2(z) + \gamma q_2^2(z) + \delta\frac{zq_2'(z)}{q_2(z)}$$

*implies*

$$q_1(z) \prec \frac{z^2(L(a,c)f(z))'}{(L(a,c)f(z))^2} \prec q_2(z),$$

*where* $q_1(z)$ *and* $q_2(z)$ *are respectively the best subordinant and best dominant.*

By fixing $g(z) = \frac{z}{(1-z)^{\lambda+1}}$ in Theorem 3.4 we get the following result involving Ruscheweyh derivative.

**Corollary 3.7** *Let* $q_1(z)$ *and* $q_2(z)$ *be convex univalent functions defined on* $\Delta$ *with* $q_1(0) = q_2(0) = 1$ *where* $q_1(z)$ *satisfies* (3.6) *and* $q_2(z)$ *satisfies* (3.1). *Let* $f \in \mathcal{A}$ *and* $0 \neq \frac{z^2(D^\lambda f(z))'}{(D^\lambda f(z))^2} \in \mathcal{H}[1,1] \cap \mathcal{Q}$ *and*

$$\psi(z) := \alpha + \beta\frac{z^2(D^\lambda f(z))'}{(D^\lambda f(z))^2} + \gamma\left[\frac{z^2(D^\lambda f(z))'}{(D^\lambda f(z))^2}\right]^2 + \delta\left[\frac{(zD^\lambda f(z))''}{(D^\lambda f(z))'} - \frac{2z(D^\lambda f(z))'}{D^\lambda f(z)}\right]$$

*is univalent in* $\Delta$, *then*

$$\alpha + \beta q_1(z) + \gamma q_1^2(z) + \delta\frac{zq_1'(z)}{q_1(z)} \prec \psi(z) \prec \alpha + \beta q_2(z) + \gamma q_2^2(z) + \delta\frac{zq_2'(z)}{q_2(z)}$$

*implies*

$$q_1(z) \prec \frac{z^2(D^\lambda f(z))'}{(D^\lambda f(z))^2} \prec q_2(z),$$

*where* $q_1(z)$ *and* $q_2(z)$ *are respectively the best subordinant and best dominant.*

By fixing $g(z) = z + \sum_{n=2}^{\infty}\left(\frac{\lambda+n}{1+\lambda}\right)^m z^n$ we get the following result involving Multiplier transformation.

**Corollary 3.8.** *Let* $q_1(z)$ *and* $q_2(z)$ *be convex univalent functions defined on* $\Delta$

*with* $q_1(0) = q_2(0) = 1$, *where* $q_1(z)$ *satisfies* (3.6) *and* $q_2(z)$ *satisfies* (3.1). *Let* $f \in \mathcal{A}$ *and* $0 \neq \frac{z^2(I(m,\lambda)f(z))'}{[I(m,\lambda)f(z)]^2} \in \mathcal{H}[1,1] \cap \mathcal{Q}$ *and*

$$\begin{aligned}\psi(z) \quad := \quad & \alpha + \beta \frac{z^2(I(m,\lambda)f(z))'}{[I(m,\lambda)f(z)]^2} + \gamma \left[\frac{z^2(I(m,\lambda)f(z))'}{[I(m,\lambda)f(z)]^2}\right]^2 \\ & + \delta \left[\frac{(zI(m,\lambda)f(z))''}{(I(m,\lambda)f(z))'} - \frac{2z(I(m,\lambda)f(z))'}{I(m,\lambda)f(z)}\right]\end{aligned}$$

*is univalent in* $\Delta$, *then*

$$\alpha + \beta q_1(z) + \gamma q_1^2(z) + \delta \frac{zq_1'(z)}{q_1(z)} \prec \psi(z) \prec \alpha + \beta q_2(z) + \gamma q_2^2(z) + \delta \frac{zq_2'(z)}{q_2(z)}$$

*implies*

$$q_1(z) \prec \frac{z^2(I(m,\lambda)f(z))'}{[I(m,\lambda)f(z)]^2} \prec q_2(z),$$

*where* $q_1(z)$ *and* $q_2(z)$ *are respectively the best subordinant and best dominant.*

By taking $\lambda = 0$ in the Corollory 3.8 we get the following result involving Sălăgean derivative.

**Corollary 3.9.** *Let* $q_1(z)$ *and* $q_2(z)$ *be convex univalent functions defined on* $\Delta$ *with* $q_1(0) = q_2(0) = 1$, *where* $q_1(z)$ *satisfies* (3.6) *and* $q_2(z)$ *satisfies* (3.1). *Let* $f \in \mathcal{A}$ *and* $0 \neq \frac{z\mathcal{D}^{m+1}f(z)}{[\mathcal{D}^m f(z)]^2} \in \mathcal{H}[1,1] \cap \mathcal{Q}$ *and*

$$\psi(z) := \alpha + \beta \frac{z\mathcal{D}^{m+1}f(z)}{[\mathcal{D}^m f(z)]^2} + \gamma \left[\frac{z\mathcal{D}^{m+1}f(z)}{[\mathcal{D}^m f(z)]^2}\right]^2 + \delta \left[\frac{z(z\mathcal{D}^m f(z))''}{\mathcal{D}^{m+1}f(z)} - \frac{2\mathcal{D}^{m+1}f(z)}{\mathcal{D}^m f(z)}\right]$$

*is univalent in* $\Delta$, *then*

$$\alpha + \beta q_1(z) + \gamma q_1^2(z) + \delta \frac{zq_1'(z)}{q_1(z)} \prec \psi(z) \prec \alpha + \beta q_2(z) + \gamma q_2^2(z) + \delta \frac{zq_2'(z)}{q_2(z)}$$

*implies*

$$q_1(z) \prec \frac{z\mathcal{D}^{m+1}f(z)}{[\mathcal{D}^m f(z)]^2} \prec q_2(z),$$

*where* $q_1(z)$ *and* $q_2(z)$ *are respectively the best subordinant and best dominant.*

By taking $g(z) = \frac{z}{1-z}, \alpha = 0, \beta = 1$ and $\gamma = 0$ we get the following result.

**Corollary 3.10.** *Let* $q_1(z)$ *and* $q_2(z)$ *be convex univalent functions defined on* $\Delta$ *with* $q_1(0) = q_2(0) = 1$ *where* $q_1(z)$ *satisfies*

$$\Re\left\{\frac{q_1(z)}{\delta}\right\} > 0$$

*and* $q_2(z)$ *satisfies*

$$\Re\left\{\frac{q_2(z)}{\delta}-\frac{zq_2'(z)}{q_2(z)}\right\}>0.$$

*Let* $f\in\mathcal{A}$ *and* $0\neq\frac{z^2f'(z)}{(f(z))^2}\in\mathcal{H}[1,1]\cap\mathcal{Q}$ *and*

$$\psi(z):=\frac{z^2f'(z)}{(f(z))^2}+\delta\left[\frac{(zf(z))''}{f'(z)}-\frac{2zf'(z)}{f(z)}\right]$$

*is univalent in* $\Delta$ *then*

$$q_1(z)+\delta\frac{zq_1'(z)}{q_1(z)}\prec\psi(z)\prec q_2(z)+\delta\frac{zq_2'(z)}{q_2(z)}$$

*implies*

$$q_1(z)\prec\frac{z^2f'(z)}{(f(z))^2}\prec q_2(z),$$

*where* $q_1(z)$ *and* $q_2(z)$ *are respectively the best subordinant and best dominant.*

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