KYUNGPOOK Math. J. 49(2009), 31-39

Fractional Integrals and Generalized Olsen Inequalities

Hendra Gunawan*
Department of Mathematics, Bandung Institute of Technology, Bandung 40132, Indonesia
e-mail : hgunawan@math.itb.ac.id

Eridani
Department of Mathematics, Airlangga University Surabaya 60115, Indonesia
e-mail : eridani@unair.ac.id

Abstract. Let T_ρ be the generalized fractional integral operator associated to a function $\rho : (0,\infty) \to (0,\infty)$, as defined in [16]. For a function W on $\mathbb{R}^n$, we shall be interested in the boundedness of the multiplication operator $f \mapsto W \cdot T_\rho f$ on generalized Morrey spaces. Under some assumptions on ρ, we obtain an inequality for $W \cdot T_\rho$, which can be viewed as an extension of Olsen's and Kurata-Nishigaki-Sugano's results.

1. Introduction

For $0 < \alpha < n$, let I_α denote the Riesz potential or the (classical) fractional integral operator, which is given by the formula

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

Formally, through its Fourier transform, the operator I_α can be recognized as a multiple of the Laplacian to the power of $-\frac{\alpha}{2}$, that is,

$$I_\alpha f = \kappa(-\Delta)^{-\alpha/2} f,$$

where $\kappa = \kappa(n,\alpha)$ (see, for instance, [2], [13], [22], [24]). A well-known result for I_α is the Hardy-Littlewood-Sobolev inequality, which was proved by Hardy and Littlewood [8], [10] and Sobolev [23] around the 1930's.

Theorem 1.1 (Hardy-Littlewood; Sobolev). *For $1 < p < \frac{n}{\alpha}$, we have the inequality*

$$\|I_\alpha f\|_q \le C_p \|f\|_p, \tag{1.1}$$

* Corresponding author.

Received 23 April 2007; received in revised form 24 April 2007; accepted 1 May 2007.
2000 Mathematics Subject Classification: 26A33, 47B38, 42B20, 42B25, 26D10.
Key words and phrases: Fractional integral operators, Hardy-Littlewood maximal operators, multiplication operators, Olsen inequality, Morrey spaces.

that is, I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, provided that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

As an immediate consequence of this inequality, one has the following estimate for $(-\Delta)^{-1}$:

$$\|(-\Delta)^{-1} f\|_{np/(n-2)} \le C_p \|f\|_p,$$

for $1 < p < \frac{n}{2}$, $n \ge 3$. Here $u := (-\Delta)^{-1} f$ is a solution of the Poisson equation $-\Delta u = f$. From (1.1) one can also prove Sobolev's embedding theorems (see [24]).

Decades later, the inequality has been extended from Lebegues spaces to Morrey spaces. For $1 \le p < \infty$ and $0 \le \lambda \le n$, the (classical) Morrey space $L^{p,\lambda} = L^{p,\lambda}(\mathbb{R}^n)$ is defined to be the space of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{p,\lambda} := \sup_{B=B(a,r)} \Big(\frac{1}{r^\lambda} \int_B |f(y)|^p \, dy\Big)^{1/p} < \infty,$$

where $B(a,r)$ denotes the (open) ball centered at $a \in \mathbb{R}^n$ with radius $r > 0$ [14]. Here $\|\cdot\|_{p,\lambda}$ defines a semi-norm on $L^{p,\lambda}$. Note particularly that $L^{p,0} = L^p$ and $L^{p,n} = L^\infty$. For the structure of Morrey spaces and their generalizations, see the works of S. Campanato [3], J. Peetre [21], C. T. Zorko [26], and the references therein.

In the 1960's, S. Spanne proved that I_α is bounded from $L^{p,\lambda}$ to $L^{q,\lambda q/p}$ for $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $0 \le \lambda < n$, as stated in [21]. A stronger result was obtained by D. R. Adams [1] and reproved by F. Chiarenza and M. Frasca [4].

Theorem 1.2 (Adams; Chiarenza-Frasca)**.** *For $1 < p < \frac{n}{\alpha}$ and $0 \le \lambda < n - \alpha p$, we have the inequality*

$$\|I_\alpha f\|_{q,\lambda} \le C_{p,\lambda} \|f\|_{p,\lambda}$$

provided that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$.

The proof usually involves the properties of the Hardy-Littlewood maximal operator M, defined by the formula

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy,$$

where $|B(x,r)| = c\, r^n$ is the Lebesgue measure of $B(x,r)$. The operator M is known to be bounded on L^p for $1 < p \le \infty$ [9]. Chiarenza and Frasca [4] proved that M is also bounded on Morrey spaces.

Theorem 1.3 (Chiarenza-Frasca)**.** *The inequality*

$$\|Mf\|_{p,\lambda} \le C_{p,\lambda} \|f\|_{p,\lambda}$$

holds for $p > 1$ and $0 \le \lambda < n$.

For $1 \le p < \infty$ and a suitable function $\phi : (0,\infty) \to (0,\infty)$, we define the (generalized) Morrey space $\mathcal{M}_{p,\phi} = \mathcal{M}_{p,\phi}(\mathbb{R}^n)$ to be the space of all functions $f \in L^p_{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{p,\phi} := \sup_{B=B(a,r)} \frac{1}{\phi(r)} \left(\frac{1}{|B|} \int_B |f(y)|^p dy \right)^{1/p} < \infty.$$

Note that for $\phi(t) = t^{(\lambda-n)/p}$, $0 \le \lambda \le n$, we have $\mathcal{M}_{p,\phi} = L^{p,\lambda}$ — the classical Morrey space. Unless stated otherwise, we assume hereafter that the function ϕ satisfies the following two conditions:

$$\frac{1}{2} \le \frac{r}{s} \le 2 \Rightarrow \frac{1}{C_1} \le \frac{\phi(r)}{\phi(s)} \le C_1. \tag{1.1}$$

$$\int_r^\infty \frac{\phi^p(t)}{t}\, dt \le C_2\, \phi^p(r) \text{ for } 1 < p < \infty. \tag{1.2}$$

The condition (1.1) is known as *the doubling condition* (with *a doubling constant* C_1). Note that for any function ψ that satisfies the doubling condition, we have

$$\int_{2^k r}^{2^{k+1} r} \frac{\psi(t)}{t}\, dt \sim \psi(2^k r),$$

for every integer k and $r > 0$.

Now, for a given function $\rho : (0,\infty) \to (0,\infty)$, we define the (generalized) fractional integral operator T_ρ by

$$T_\rho f(x) := \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy.$$

For $\rho(t) = t^\alpha$, $0 < \alpha < n$, we have $T_\rho = I_\alpha$ — the classical fractional integral operator. The boundedness of the operator T_ρ on the generalized Morrey space $\mathcal{M}_{p,\phi}$ was first studied by Nakai [16]. Recent results on T_ρ can be found in [5], [6], [7], [17], [18], [19].

In this paper, we shall be interested in the boundedness of the multiplication operators $f \mapsto W \cdot I_\alpha f$ and $f \mapsto W \cdot T_\rho f$ on generalized Morrey spaces. In both cases, W is just a function on $\mathbb{R}^n$. We prove an inequality for $W \cdot I_\alpha$ [Theorem 3.3] and, under some assumptions on ρ, we also obtain an inequality for $W \cdot T_\rho$ [Theorem 3.5]. Our results can be viewed as an extension of Olsen's and Kurata-Nishigaki-Sugano's results. Indeed, for $\rho(t) = t^\alpha$, $0 < \alpha < n$, the inequalities for $W \cdot T_\rho$ reduce to those for the classical fractional integral operator $W \cdot I_\alpha$.

2. Inequalities for I_α and T_ρ

In [15], E. Nakai proved the boundedness of the Hardy-Littlewood maximal operator on generalized Morrey spaces.

Theorem 2.1 (Nakai)**.** *The inequality*

$$\|Mf\|_{p,\phi} \le C_{p,\phi}\|f\|_{p,\phi}$$

holds for $1 < p < \infty$.

Nakai also obtained the boundedness of I_α on generalized Morrey spaces, which can be viewed as an extension of Spanne's result. A similar result was also obtained by Sugano-Tanaka [25]. The following theorem can be considered as an extension of Adams-Chiarenza-Frasca's result.

Theorem 2.2. *Suppose that, in addition to the condition* (1.1) *and* (1.2)*,* ϕ *satisfies the inequality* $\phi(t) \le Ct^\beta$ *for* $-\frac{n}{p} \le \beta < -\alpha,\ 1 < p < \frac{n}{\alpha}$*. Then, for* $q = \frac{\beta p}{\alpha+\beta}$*, we have*

$$\|I_\alpha f\|_{q,\phi^{p/q}} \le C_{p,\beta}\|f\|_{p,\phi}.$$

Proof. As before, we assume that $f \ne 0$ and Mf is finite everywhere. For each $x \in \mathbb{R}^n$, write $I_\alpha f(x) = I_1(x) + I_2(x)$ where

$$I_1(x) := \int_{|x-y|<R} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy \quad \text{and} \quad I_2(x) := \int_{|x-y|\ge R} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy,$$

with R being an arbitrary positive number. Then, $|I_1(x)| \le C\,R^\alpha Mf(x)$, while for I_2 we have

$$\begin{aligned}
|I_2(x)| &\le \sum_{k=0}^{\infty} \int_{2^kR\le|x-y|<2^{k+1}R} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy \\
&\le \sum_{k=0}^{\infty} (2^kR)^{\alpha-n} \int_{B(x,2^{k+1}R)} |f(y)|\,dy \\
&\le C\sum_{k=0}^{\infty} (2^kR)^{\alpha-\frac{n}{p}} \Big(\int_{B(x,2^{k+1}R)} |f(y)|^p dy\Big)^{1/p} \\
&\le C\sum_{k=0}^{\infty} (2^kR)^{\alpha}\phi(2^kR)\,\|f\|_{p,\phi} \\
&\le C\,\|f\|_{p,\phi}\sum_{k=0}^{\infty} (2^kR)^{\alpha+\beta} \\
&\le C\,R^{\alpha+\beta}\|f\|_{p,\phi}.
\end{aligned}$$

Now choose $R = \Big(\frac{Mf(x)}{\|f\|_{p,\phi}}\Big)^{1/\beta}$ to get

$$|I_\alpha f(x)| \le |I_1(x)| + |I_2(x)| \le C\,[Mf(x)]^{(\alpha+\beta)/\beta}\|f\|_{p,\phi}^{-\alpha/\beta} = C\,[Mf(x)]^{p/q}\|f\|_{p,\phi}^{1-p/q}.$$

The inequality then follows from this and Theorem 2.1. □

Remark. Observe that when $\phi(t) = t^{(\lambda-n)/p}$, $0 \leq \lambda < n - \alpha p$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$, Theorem 2.2 reduces to Theorem 1.2.

A slight modification of Theorem 2.2 may be formulated for T_ρ as follows. We leave its proof to the reader.

Theorem 2.3. *Suppose that* $\rho(t) \leq C_1 t^\alpha$ *for some* $0 < \alpha < n$, *and, in addition to the condition* (1.1) *and* (1.2), $\phi(t) \leq C_2 t^\beta$ *for* $-\frac{n}{p} \leq \beta < -\alpha$, $1 < p < \frac{n}{\alpha}$. *Then, for* $q = \frac{\beta p}{\alpha + \beta}$, *we have*

$$\|T_\rho f\|_{q,\phi^{p/q}} \leq C_{p,\beta} \|f\|_{p,\phi}.$$

The following result of H. Gunawan [7] gives a further generalization of Theorem 1.2.

Theorem 2.4 (Gunawan)**.** *Suppose that, in addition to the condition* (1.1) *and* (1.2), ϕ *is surjective. If* ρ *satisfies the doubling condition and*

$$\int_0^r \frac{\rho(t)}{t}\, dt \leq C\phi(r)^{(p-q)/q} \quad \text{and} \quad \int_r^\infty \frac{\rho(t)\phi(t)}{t} dt \leq C\phi(r)^{p/q},$$

for $1 < p < q < \infty$, *then we have*

$$\|T_\rho f\|_{q,\phi^{p/q}} \leq C_{p,\phi} \|f\|_{p,\phi},$$

that is, T_ρ *is bounded from* $\mathcal{M}_{p,\phi}$ *to* $\mathcal{M}_{q,\phi^{p/q}}$.

3. Inequalities for $W \cdot I_\alpha$ and $W \cdot T_\rho$

In studying a Schrödinger equation with perturbed potentials W on $\mathbb{R}^n$ (particularly for $n = 3$), P. A. Olsen [20] proved the following result.

Theorem 3.1 (Olsen)**.** *For* $1 < p < \frac{n}{\alpha}$ *and* $0 \leq \lambda < n - \alpha p$, *we have*

$$\|W \cdot I_\alpha f\|_{p,\lambda} \leq C_{p,\lambda} \|W\|_{(n-\lambda)/\alpha,\lambda} \|f\|_{p,\lambda},$$

that is, $W \cdot I_\alpha$ *is bounded on* $L^{p,\lambda}$, *provided that* $W \in L^{(n-\lambda)/\alpha,\lambda}$.

As a consequence of Theorem 3.1, we see that for $1 < p < \frac{n}{2}$, $n \geq 3$, the estimate

$$\|W \cdot (-\Delta)^{-1} f\|_{p,\lambda} \leq C_{p,\lambda}\, \|W\|_{(n-\lambda)/2,\lambda} \|f\|_{p,\lambda},$$

holds provided that $W \in L^{(n-\lambda)/2,\lambda}$, $0 \le \lambda < n-2p$. In particular, when $\lambda = 0$, one has

$$\|W \cdot (-\Delta)^{-1} f\|_p \le C_p \, \|W\|_{n/2} \|f\|_p$$

provided that $W \in L^{n/2}$.

K. Kurata *et al.* [12] extended Olsen's result by proving that, for some $p > 1$ and a function ϕ satisfying several conditions (including the doubling condition), the operator $W \cdot I_\alpha$ is bounded on generalized Morrey spaces $\mathcal{M}_{p,\phi}$, provided that $W \in \mathcal{M}_{s_1,\phi} \cap \mathcal{M}_{s_2,\phi}$ for some indices s_1 and s_2. Their estimate, however, is rather complicated. We shall here present simpler estimates for $W \cdot I_\alpha$ on generalized Morrey spaces.

The first estimate below is a consequence of Theorem 2.2, while the second one is obtained directly without using Theorem 2.2.

Theorem 3.2. *Suppose that, in addition to the condition* (1.1) *and* (1.2)*,* ϕ *satisfies the inequality* $\phi(t) \le Ct^\beta$ *for* $-\frac{n}{p} \le \beta < -\alpha$, $1 < p < \frac{n}{\alpha}$*. Then, we have*

$$\|W \cdot I_\alpha f\|_{p,\phi} \le C_{p,\beta} \|W\|_{s,\phi^{p/s}} \|f\|_{p,\phi}$$

provided that $W \in \mathcal{M}_{s,\phi^{p/s}}$ *where* $s = -\frac{\beta p}{\alpha}$.

Proof. Use Hölder's inequality and Theorem 2.2. □

Theorem 3.3. *Suppose that* ϕ *satisfies the doubling condition and the inequality*

$$\int_r^\infty t^{\alpha-1} \phi(t) \, dt \le C \, r^\alpha \phi(r).$$

Then, for $1 < p < \frac{n}{\alpha}$*, we have*

$$\|W \cdot I_\alpha f\|_{p,\phi} \le C_{p,\phi} \|W\|_{n/\alpha} \|f\|_{p,\phi},$$

provided that $W \in L^{n/\alpha}$.

Proof. For $a \in \mathbb{R}^n$ and $r > 0$, let $B = B(a,r)$, $\tilde{B} = B(a,2r)$, and write $f = f_1 + f_2 := f\chi_{\tilde{B}} + f\chi_{\tilde{B}^c}$. We observe that $f_1 \in L^p$ with

$$\|f_1\|_p = \Big(\int_{\mathbb{R}^n} |f_1(y)|^p dy\Big)^{1/p} = \Big(\int_{\tilde{B}} |f(y)|^p dy\Big)^{1/p} \le C \, r^{n/p} \phi(r) \|f\|_{p,\phi}.$$

Hence, by applying Theorem 3.1 for $\lambda = 0$, we get

$$\Big(\int_B |W \cdot I_\alpha f_1(x)|^p dx\Big)^{1/p} \le \|W \cdot I_\alpha f_1\|_p \le C \, \|W\|_{n/\alpha} \|f_1\|_p \le C \, r^{n/p} \phi(r) \|W\|_{n/\alpha} \|f\|_{p,\phi},$$

whence

$$\frac{1}{\phi(r)} \Big(\frac{1}{|B|} \int_B |W \cdot I_\alpha f_1(x)|^p dx\Big)^{1/p} \le C \, \|W\|_{n/\alpha} \|f\|_{p,\phi}.$$

Next, for $x \in B$, we have

$$|I_\alpha f_2(x)| \le \int_{\tilde{B}^c} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \le \int_{|x-y|\ge r} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy.$$

Then, as in the proof of Theorem 2.4, we shall obtain

$$|I_\alpha f_2(x)| \le C\, \|f\|_{p,\phi} \int_r^\infty t^{\alpha-1}\phi(t)\, dt \le C\, r^\alpha \phi(r)\|f\|_{p,\phi}.$$

Hence

$$\begin{aligned}\Big(\frac{1}{|B|}\int_B |W\cdot I_\alpha f_2(x)|^p dx\Big)^{1/p} &\le C\, r^\alpha\phi(r)\|f\|_{p,\phi}\Big(\frac{1}{|B|}\int_B |W(x)|^p\, dx\Big)^{1/p}\\ &\le C\, r^\alpha\phi(r)\|f\|_{p,\phi}\Big(\frac{1}{|B|}\int_B |W(x)|^{n/\alpha} dx\Big)^{\alpha/n}\\ &\le C\,\phi(r)\|W\|_{n/\alpha}\|f\|_{p,\phi},\end{aligned}$$

and so

$$\frac{1}{\phi(r)}\Big(\frac{1}{|B|}\int_B |W\cdot I_\alpha f_2(x)|^p dx\Big)^{1/p} \le C\,\|W\|_{n/\alpha}\|f\|_{p,\phi}.$$

The desired estimate follows from the two estimates via Minkowski inequality. □

The following two theorems provide estimates for $W\cdot T_\rho$ on generalized Morrey spaces. The first is a consequence of Theorem 2.3, while the second follows from Theorem 2.4. We leave the proof of the former to the reader.

Theorem 3.4. *Suppose that $\rho(t) \le C_1 t^\alpha$ for some $0 < \alpha < n$, and, in addition to the condition (1.1) and (1.2), $\phi(t) \le C_2 t^\beta$ for $-\frac{n}{p} \le \beta < -\alpha$, $1 < p < \frac{n}{\alpha}$. Then, we have*

$$\|W\cdot T_\rho f\|_{p,\phi} \le C_{p,\beta}\|W\|_{s,\phi^{p/s}}\|f\|_{p,\phi}$$

provided that $W \in \mathcal{M}_{s,\phi^{p/s}}$ where $s = -\frac{\beta p}{\alpha}$.

Theorem 3.5. *Suppose that, in addition to the condition (1.1) and (1.2), ϕ is surjective. If ρ satisfies the doubling condition and*

$$\int_0^r \frac{\rho(t)}{t}\, dt \le C\phi(r)^{(p-q)/q} \quad \text{and} \quad \int_r^\infty \frac{\rho(t)\phi(t)}{t}\, dt \le C\phi(r)^{p/q},$$

for $1 < p < q < \infty$, then we have

$$\|W\cdot T_\rho f\|_{p,\phi} \le C_{p,\phi}\|W\|_{s,\phi^{p/s}}\|f\|_{p,\phi},$$

provided that $W \in \mathcal{M}_{s,\phi^{p/s}}$ where $\frac{1}{s} = \frac{1}{p} - \frac{1}{q}$.

Proof. Let $B = B(a,r)$ be an arbitrary ball in $\mathbb{R}^n$. By Hölder's inequality, we have

$$\frac{1}{|B|}\int_B |W\cdot T_\rho f(x)|^p dx \le \Big(\frac{1}{|B|}\int_B |W(x)|^s dx\Big)^{p/s}\Big(\frac{1}{|B|}\int_B |T_\rho f(x)|^q dx\Big)^{p/q},$$

with $\frac{p}{s}+\frac{p}{q}=1$. Now take the p-th roots and then divide both sides by $\phi(r)$ to get

$$\begin{aligned}\frac{1}{\phi(r)}\Big(\frac{1}{|B|}\int_B |W\cdot T_\rho f(x)|^p dx\Big)^{1/p} &\le \frac{1}{\phi(r)^{p/s}}\Big(\frac{1}{|B|}\int_B |W(x)|^s dx\Big)^{1/s}\\ &\quad\times\frac{1}{\phi(r)^{p/q}}\Big(\frac{1}{|B|}\int_B |T_\rho f(x)|^q dx\Big)^{1/q}\\ &\le C\,\|W\|_{s,\phi^{p/s}}\|T_\rho f\|_{q,\phi^{p/q}}.\end{aligned}$$

The desired inequality is obtained by taking the supremum over all balls B and using the fact that T_ρ is bounded from $\mathcal{M}_{p,\phi}$ to $\mathcal{M}_{q,\phi^{p/q}}$. □

Acknowledgment. The first author is supported by ITB Research Grant No. 0004/ K01.03.2/PL2.1.5/I/2006.

References

[1] D. R. Adams, *A note on Riesz potentials* , Duke Math. J., **42**(1975), 765–778.

[2] D. R. Adams and L. I. Hedberg, Functions Spaces and Potential Theory, Springer-Verlag, Berlin, 1996.

[3] S. Campanato, *Proprietà di una famiglia di spazi funzionali*, Ann. Scuola Norm. Sup. Pisa, **18**(1964), 137-160.

[4] F. Chiarenza and M. Frasca, *Morrey spaces and Hardy-Littlewood maximal function*, Rend. Mat., **7**(1987), 273-279.

[5] Eridani, *On the boundedness of a generalized fractional integral on generalized Morrey spaces*, Tamkang J. Math., **33**(2002), 335-340.

[6] Eridani, H. Gunawan, and E. Nakai, *On generalized fractional integral operators*, Sci. Math. Jpn., **60**(2004), 539-550.

[7] H. Gunawan, *A note on the generalized fractional integral operators*, J. Indones. Math. Soc. (MIHMI), **9**(2003), 39-43.

[8] G. H. Hardy and J. E. Littlewood, *Some properties of fractional integrals. I*, Math. Zeit., **27**(1927), 565-606.

[9] G. H. Hardy and J. E. Littlewood, *A maximal theorem with function-theoretic applications*, Acta Math., **54**(1930), 81-116.

[10] G. H. Hardy and J. E. Littlewood, *Some properties of fractional integrals. II*, Math. Zeit., **34**(1932), 403-439.

[11] L. I. Hedberg, *On certain convolution inequalities*, Proc. Amer. Math. Soc., **36**(1972), 505-510.

[12] K. Kurata, S. Nishigaki, and S. Sugano, *Boundedness of integral operators on generalized Morrey spaces and its application to Schrödinger operators*, Proc. Amer. Math. Soc., **128**(2002), 1125-1134.

[13] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, Vol. **204**, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.

[14] C. B. Morrey, *Functions of several variables and absolute continuity*, Duke Math. J., **6**(1940), 187-215.

[15] E. Nakai, *Hardy-Littlewood maximal operator, singular integral operators, and the Riesz potentials on generalized Morrey spaces*, Math. Nachr., **166**(1994), 95-103.

[16] E. Nakai, *On generalized fractional integrals*, Taiwanese J. Math., **5**(2001), 587-602.

[17] E. Nakai, On generalized fractional integrals on the weak Orlicz spaces, BMO_ϕ, the Morrey spaces and the Campanato spaces, in *Function Spaces, Interpolation Theory and Related Topics (Lund, 2000)*, 95-103, de Gruyter, Berlin, 2002.

[18] E. Nakai, *Recent topics on fractional integral operators* (Japanese), Sūgaku, **56**(2004), 260-280.

[19] E. Nakai, *Generalized fractional integrals on Orlicz-Morrey spaces*, in Banach and Function Spaces, 323-333, Yokohama Publ., Yokohama, 2004.

[20] P. A. Olsen, *Fractional integration, Morrey spaces and a Schrödinger equation*, Comm. Partial Differential Equations, **20**(1995), 2005-2055.

[21] J. Peetre, *On the theory of $\mathcal{L}_{p,\lambda}$ spaces*, J. Funct. Anal., **4**(1969), 71-87.

[22] B. Rubin, Fractional Integrals and Potentials, Addison-Wesley, Essex, 1996.

[23] S. L. Sobolev, *On a theorem in functional analysis* (Russian), Mat. Sob., **46**(1938), 471-497 [English translation in Amer. Math. Soc. Transl. ser. 2, **34**(1963), 39-68].

[24] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton, 1970.

[25] S. Sugano and H. Tanaka, *Boundedness of fractional integral operators on generalized Morrey spaces*, Sci. Math. Jpn., **58**(2003), 531-540.

[26] C. T. Zorko, *Morrey space*, Proc. Amer. Math. Soc., **98**(1986), 586-592.