

## Uniqueness of Meromorphic Functions That Share Three Sets

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ABSTRACT. Dealing with a question of gross, we prove some uniqueness theorems concerning meromorphic functions with the notion of weighted sharing of sets. Our results will not only improve and supplement respectively two results of Lahiri-Banerjee [9] and Qiu and Fang [13] but also improve a very recent result of the present author [1].

### 1. Introduction and preliminaries

Let  $f$  and  $g$  be two nonconstant meromorphic functions defined in the open complex plane  $\mathbb{C}$ . We denote by  $T(r)$  the maximum of  $T(r, f)$  and  $T(r, g)$ . The notation  $S(r)$  denotes any quantity satisfying  $S(r) = o(T(r))$  as  $r \rightarrow \infty$ , outside a possible exceptional set of finite linear measure. If for some  $a \in \mathbb{C} \cup \{\infty\}$ ,  $f$  and  $g$  have the same set of  $a$ -points with same multiplicities then we say that  $f$  and  $g$  share the value  $a$  CM (counting multiplicities). If we do not take the multiplicities into account,  $f$  and  $g$  are said to share the value  $a$  IM (ignoring multiplicities). Let  $S$  be a set of distinct elements of  $\mathbb{C} \cup \{\infty\}$  and  $E_f(S) = \bigcup_{a \in S} \{z : f(z) - a = 0\}$ , where each zero is counted according to its multiplicity. If we do not count the multiplicity the set  $E_f(S) = \bigcup_{a \in S} \{z : f(z) - a = 0\}$  is denoted by  $\overline{E}_f(S)$ . If  $E_f(S) = E_g(S)$  we say that  $f$  and  $g$  share the set  $S$  CM. On the other hand if  $\overline{E}_f(S) = \overline{E}_g(S)$ , we say that  $f$  and  $g$  share the set  $S$  IM.

In 1976 F. Gross [3] posed the following question:

**Question A.** Can one find two finite sets  $S_j$  ( $j = 1, 2$ ) such that any two non-constant entire functions  $f$  and  $g$  satisfying  $E_f(S_j) = E_g(S_j)$  for  $j = 1, 2$  must be identical?

For meromorphic function it is natural to ask the following question.

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**Question B([15]).** Can one find three finite sets  $S_j$  ( $j = 1, 2, 3$ ) such that any two nonconstant meromorphic functions  $f$  and  $g$  satisfying  $E_f(S_j) = E_g(S_j)$  for  $j = 1, 2, 3$  must be identical?

During the last few years several authors investigate the possible answer of Question B and continuous efforts is being put in to relax the hypothesis of the result. (cf. [1], [2], [9], [12], [13], [15], [18]).

In the direction of Question B Fang and Xu [2] proved the following result.

**Theorem A([2]).** Let  $S_1 = \{z : z^3 - z^2 - 1 = 0\}$ ,  $S_2 = \{0\}$  and  $S_3 = \{\infty\}$ . Suppose that  $f$  and  $g$  are two nonconstant meromorphic functions satisfying  $\Theta(\infty; f) > \frac{1}{2}$  and  $\Theta(\infty; g) > \frac{1}{2}$ . If  $E_f(S_j) = E_g(S_j)$  for  $j = 1, 2, 3$  then  $f \equiv g$ .

Dealing with the question of Gross, Qiu and Fang [13] proved the following theorem.

**Theorem B([13]).** Let  $n \geq 3$  be a positive integer  $S_1 = \{z : z^n - z^{n-1} - 1 = 0\}$ ,  $S_2 = \{0\}$ , and let  $f$  and  $g$  be two nonconstant meromorphic functions whose poles are of multiplicities at least 2. If  $E_f(\{\infty\}) = E_g(\{\infty\})$  and  $E_f(S_i) = E_g(S_i)$  for  $i = 1, 2$  then  $f \equiv g$ .

They also gave example to show that the condition that the poles of  $f(z)$  and  $g(z)$  are of multiplicities at least 2 can not be removed in Theorem B. It should be noted that if two meromorphic functions  $f$  and  $g$  have no simple pole then clearly  $\Theta(\infty, f) \geq \frac{1}{2}$  and  $\Theta(\infty, g) \geq \frac{1}{2}$ . Lahiri and Banerjee [9] investigated the situation for  $\Theta(\infty, f) \leq \frac{1}{2}$  and  $\Theta(\infty, g) \leq \frac{1}{2}$  in Theorem A and proved the following result.

**Theorem C([9]).** Let  $S_1 = \{z : z^n + az^{n-1} + b = 0\}$ ,  $S_2 = \{0\}$  and  $S_3 = \{\infty\}$ , where  $a, b$  are nonzero constants such that  $z^n + az^{n-1} + b = 0$  has no repeated root and  $n$  ( $\geq 4$ ) is an integer. If for two nonconstant meromorphic functions  $f$  and  $g$   $E_f(S_i) = E_g(S_i)$  for  $i = 1, 2, 3$  and  $\Theta(\infty; f) + \Theta(\infty; g) > 0$  then  $f \equiv g$ .

In 2004 Yi and Lin [18] proved the following theorem.

**Theorem D([18]).** Let  $S_1 = \{z : z^n + az^{n-1} + b = 0\}$ ,  $S_2 = \{0\}$  and  $S_3 = \{\infty\}$ , where  $a, b$  are nonzero constants such that  $z^n + az^{n-1} + b = 0$  has no repeated root and  $n$  ( $\geq 3$ ) is an integer. If for two nonconstant meromorphic functions  $f$  and  $g$ ,  $E_f(S_i) = E_g(S_i)$  for  $i = 1, 2, 3$  and  $\Theta(\infty; f) > \frac{1}{2}$  then  $f \equiv g$ .

Yi and Lin [18] remarked that the assumption  $E_f(S_2) = E_g(S_2)$  in the above result can be relaxed to  $\overline{E}_f(S_2) = \overline{E}_g(S_2)$ . Clearly Theorem D is an improvement of Theorem B. Recently the present author [1] has investigated the situation of further relaxation of the nature of sharing the set  $S_1$  in Theorem D with the idea of gradation of sharing of values and sets known as weighted sharing as introduced in

[6], [7] which measures how close a shared value is to being shared IM or to being shared CM. We now give the definition.

**Definition 1.1** ([6],[7]). Let  $k$  be a nonnegative integer or infinity. For  $a \in \mathbb{C} \cup \{\infty\}$  we denote by  $E_k(a; f)$  the set of all  $a$ -points of  $f$ , where an  $a$ -point of multiplicity  $m$  is counted  $m$  times if  $m \leq k$  and  $k + 1$  times if  $m > k$ . If  $E_k(a; f) = E_k(a; g)$ , we say that  $f, g$  share the value  $a$  with weight  $k$ .

We write  $f, g$  share  $(a, k)$  to mean that  $f, g$  share the value  $a$  with weight  $k$ . Clearly if  $f, g$  share  $(a, k)$  then  $f, g$  share  $(a, p)$  for any integer  $p, 0 \leq p < k$ . Also we note that  $f, g$  share a value  $a$  IM or CM if and only if  $f, g$  share  $(a, 0)$  or  $(a, \infty)$  respectively.

**Definition 1.2** ([6]). Let  $S$  be a set of distinct elements of  $\mathbb{C} \cup \{\infty\}$  and  $k$  be a nonnegative integer or  $\infty$ . We denote by  $E_f(S, k)$  the set  $E_f(S, k) = \bigcup_{a \in S} E_k(a; f)$ .

Clearly  $E_f(S) = E_f(S, \infty)$  and  $\overline{E}_f(S) = E_f(S, 0)$ . Improving the result of Yi-Lin [17] the present author have recently proved the following result.

**Theorem E** ([1]). *Let  $S_1, S_2$  and  $S_3$  be defined as in Theorem D. If for two nonconstant meromorphic functions  $f$  and  $g$   $E_f(S_1, 6) = E_g(S_1, 6)$ ,  $E_f(S_2, 0) = E_g(S_2, 0)$  and  $E_f(S_3, \infty) = E_g(S_3, \infty)$  and  $\Theta(\infty; f) + \Theta(\infty; g) > 1$  then  $f \equiv g$ .*

In this paper we will concentrate our attention of further relaxation of the nature of sharing the set  $S_1$  in Theorem B, Theorem C and Theorem E respectively. We now state the following five theorems which are the main results of the paper.

**Theorem 1.1.** *Let  $S_1, S_2$  and  $S_3$  be defined as in Theorem C. If for two nonconstant meromorphic functions  $f$  and  $g$   $E_f(S_1, 3) = E_g(S_1, 3)$ ,  $E_f(S_2, \infty) = E_g(S_2, \infty)$  and  $E_f(S_3, \infty) = E_g(S_3, \infty)$  and  $\Theta(\infty; f) + \Theta(\infty; g) > 0$  then  $f \equiv g$ .*

**Theorem 1.2.** *Let  $S_1, S_2$  and  $S_3$  be defined as in Theorem D. If for two nonconstant meromorphic functions  $f$  and  $g$   $E_f(S_1, 5) = E_g(S_1, 5)$ ,  $E_f(S_2, 0) = E_g(S_2, 0)$  and  $E_f(S_3, \infty) = E_g(S_3, \infty)$  and  $\Theta(\infty; f) + \Theta(\infty; g) > 1$  then  $f \equiv g$ .*

**Theorem 1.3.** *Let  $S_1, S_2$  and  $S_3$  be defined as in Theorem D. If for two nonconstant meromorphic functions  $f$  and  $g$   $E_f(S_1, 4) = E_g(S_1, 4)$ ,  $E_f(S_2, \infty) = E_g(S_2, \infty)$  and  $E_f(S_3, \infty) = E_g(S_3, \infty)$  and  $\Theta(\infty; f) + \Theta(\infty; g) > 1$  then  $f \equiv g$ .*

**Remark 1.1.** Clearly Theorem 1.1, Theorem 1.2 and Theorem 1.3 are improvements of Theorem C, Theorem E and Theorem B respectively.

**Theorem 1.4.** *Let  $S_1, S_2$  and  $S_3$  be defined as in Theorem C. If for two nonconstant meromorphic functions  $f$  and  $g$   $E_f(S_1, 2) = E_g(S_1, 2)$ ,  $E_f(S_2, \infty) = E_g(S_2, \infty)$  and  $E_f(S_3, \infty) = E_g(S_3, \infty)$  and  $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{3n-5}$  then  $f \equiv g$ .*

**Theorem 1.5.** *Let  $S_1, S_2$  and  $S_3$  be defined as in Theorem D. If for two non-*

constant meromorphic functions  $f$  and  $g$   $E_f(S_1, 3) = E_g(S_1, 3)$ ,  $E_f(S_2, \infty) = E_g(S_2, \infty)$  and  $E_f(S_3, \infty) = E_g(S_3, \infty)$  and  $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4n}{7n-12}$  then  $f \equiv g$ .

**Remark 1.2.** When  $n \geq 4$  Theorem 1.5 is true for  $\Theta(\infty; f) + \Theta(\infty; g) > 1$  and hence in that case Theorem 1.5 is an improvement of Theorem B.

The following example shows that the condition  $\Theta(\infty; f) + \Theta(\infty; g) > 0$  is sharp in Theorem 1.1.

**Example 1.1.** Let

$$g = -a \frac{e^{(n-1)z} - 1}{e^{nz} - 1}, \quad f(z) = e^z g(z)$$

and  $S_i$ 's be as in Theorem 1.1. Then  $E_f(S_i, \infty) = E_g(S_i, \infty)$  for  $i = 1, 2, 3$  because  $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$  and  $f \equiv e^z g$ . Also  $\Theta(\infty; f) + \Theta(\infty; g) = 0$  and  $f \not\equiv g$ .

Though for the standard definitions and notations of the value distribution theory we refer to [4], we now explain some notations which are used in the paper.

**Definition 1.3([5]).** For  $a \in \mathbb{C} \cup \{\infty\}$  we denote by  $N(r, a; f | = 1)$  the counting function of simple  $a$  points of  $f$ . For a positive integer  $m$  we denote by  $N(r, a; f | \leq m)$  ( $N(r, a; f | \geq m)$ ) the counting function of those  $a$  points of  $f$  whose multiplicities are not greater (less) than  $m$  where each  $a$  point is counted according to its multiplicity.

$\bar{N}(r, a; f | \leq m)$  ( $\bar{N}(r, a; f | \geq m)$ ) are defined similarly, where in counting the  $a$ -points of  $f$  we ignore the multiplicities. Also  $N(r, a; f | < m)$ ,  $N(r, a; f | > m)$ ,  $\bar{N}(r, a; f | < m)$  and  $\bar{N}(r, a; f | > m)$  are defined analogously.

**Definition 1.4.** We denote by  $\bar{N}(r, a; f | = k)$  the reduced counting function of those  $a$ -points of  $f$  whose multiplicities is exactly  $k$ , where  $k \geq 2$  is an integer.

**Definition 1.5.** Let  $f$  and  $g$  be two nonconstant meromorphic functions such that  $f$  and  $g$  share  $(a, k)$  where  $a \in \mathbb{C} \cup \{\infty\}$ . Let  $z_0$  be a  $a$ -point of  $f$  with multiplicity  $p$ , a  $a$ -point of  $g$  with multiplicity  $q$ . We denote by  $\bar{N}_L(r, a; f)$  the counting function of those  $a$ -points of  $f$  and  $g$  where  $p > q$ , by  $\bar{N}_E^{(k+1)}(r, a; f)$  the counting function of those  $a$ -points of  $f$  and  $g$  where  $p = q \geq k+1$ ; each point in these counting functions is counted only once. In the same way we can define  $\bar{N}_L(r, a; g)$  and  $\bar{N}_E^{(k+1)}(r, a; g)$ .

**Definition 1.6([7]).** We denote by  $N_2(r, a; f)$  the sum  $\bar{N}(r, a; f) + \bar{N}(r, a; f | \geq 2)$ .

**Definition 1.7([6],[7]).** Let  $f, g$  share a value  $a$  IM. We denote by  $\bar{N}_*(r, a; f, g)$  the reduced counting function of those  $a$ -points of  $f$  whose multiplicities differ from the multiplicities of the corresponding  $a$ -points of  $g$ . Clearly  $\bar{N}_*(r, a; f, g) \equiv$

$$\overline{N}_*(r, a; g, f) \text{ and } \overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g).$$

**Definition 1.8**([10]). Let  $a, b \in \mathbb{C} \cup \{\infty\}$ . We denote by  $N(r, a; f \mid g = b)$  the counting function of those  $a$ -points of  $f$ , counted according to multiplicity, which are  $b$ -points of  $g$ .

**Definition 1.9**([10]). Let  $a, b_1, b_2, \dots, b_q \in \mathbb{C} \cup \{\infty\}$ . We denote by  $N(r, a; f \mid g \neq b_1, b_2, \dots, b_q)$  the counting function of those  $a$ -points of  $f$ , counted according to multiplicity, which are not the  $b_i$ -points of  $g$  for  $i = 1, 2, \dots, q$ .

## 2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let  $F$  and  $G$  be two nonconstant meromorphic functions defined as follows.

$$(2.1) \quad F = \frac{f^{n-1}(f+a)}{-b}, \quad G = \frac{g^{n-1}(g+a)}{-b}.$$

Henceforth we shall denote by  $H$ ,  $\Phi$  and  $V$  the following three functions

$$H = \left( \frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G-1} \right),$$

$$\Phi = \frac{F'}{F-1} - \frac{G'}{G-1}$$

and

$$V = \left( \frac{F'}{F-1} - \frac{F'}{F} \right) - \left( \frac{G'}{G-1} - \frac{G'}{G} \right) = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}.$$

**Lemma 2.1**([7], Lemma 1). *Let  $F, G$  share  $(1, 1)$  and  $H \not\equiv 0$ . Then*

$$N(r, 1; F \mid = 1) = N(r, 1; G \mid = 1) \leq N(r, H) + S(r, F) + S(r, G).$$

**Lemma 2.2.** *Let  $S_1, S_2$  and  $S_3$  be defined as in Theorem 1.1 and  $F, G$  be given by (2.1). If for two nonconstant meromorphic functions  $f$  and  $g$   $E_f(S_1, 0) = E_g(S_1, 0)$ ,  $E_f(S_2, 0) = E_g(S_2, 0)$ ,  $E_f(S_3, 0) = E_g(S_3, 0)$  and  $H \not\equiv 0$  then*

$$N(r, H) \leq \overline{N}_*(r, 0, f, g) + \overline{N}(r, 0; f + a \mid \geq 2) + \overline{N}(r, 0; g + a \mid \geq 2) + \overline{N}_*(r, 1; F, G) \\ + \overline{N}_*(r, \infty; f, g) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G'),$$

where  $\overline{N}_0(r, 0; F')$  is the reduced counting function of those zeros of  $F'$  which are not the zeros of  $F(F-1)$  and  $\overline{N}_0(r, 0; G')$  is similarly defined.

*Proof.* Since  $E_f(S_1, 0) = E_g(S_1, 0)$  it follows that  $F$  and  $G$  share  $(1, 0)$ . We can easily verify that possible poles of  $H$  occur at (i) those zeros of  $f$  and  $g$  whose

multiplicities are distinct from the multiplicities of the corresponding zeros of  $g$  and  $f$  respectively, (ii) multiple zeros of  $f + a$  and  $g + a$ , (iii) those poles of  $f$  and  $g$  whose multiplicities are distinct from the multiplicities of the corresponding poles of  $g$  and  $f$  respectively, (iv) those 1-points of  $F$  and  $G$  with different multiplicities, (v) zeros of  $F'$  which are not the zeros of  $F(F - 1)$ , (vi) zeros of  $G'$  which are not zeros of  $G(G - 1)$ . Since  $H$  has only simple poles, the lemma follows from above. This proves the lemma.  $\square$

**Lemma 2.3**([14]). *Let  $f$  be a nonconstant meromorphic function and  $P(f) = a_0 + a_1f + a_2f^2 + \cdots + a_nf^n$ , where  $a_0, a_1, a_2, \dots, a_n$  are constants and  $a_n \neq 0$ . Then  $T(r, P(f)) = nT(r, f) + O(1)$ .*

**Lemma 2.4.** *Let  $F$  and  $G$  be given by (2.1). If  $f, g$  share  $(0, 0)$  and  $0$  is not an Picard exceptional value of  $f$  and  $g$ . Then  $\Phi \equiv 0$  implies  $F \equiv G$ .*

*Proof.* Suppose  $\Phi \equiv 0$ . Then by integration we obtain  $F - 1 \equiv C(G - 1)$ . It is clear that if  $z_0$  is a zero of  $f$  then it is a zero of  $g$ . So from (2.1) it follows that  $F(z_0) = 0$  and  $G(z_0) = 0$ . So  $C = 1$  and hence  $F \equiv G$ .  $\square$

**Lemma 2.5.** *Let  $F$  and  $G$  be given by (2.1),  $n \geq 3$  an integer and  $\Phi \not\equiv 0$ . If  $F, G$  share  $(1, m)$ ,  $f, g$  share  $(0, p)$ ,  $(\infty, k)$ , where  $0 \leq p < \infty$  then*

$$\begin{aligned} & [(n-1)p + n - 2] \overline{N}(r, 0; f | \geq p+1) \\ & \leq \overline{N}_*(r, 1; F, G) + \overline{N}_*(r, \infty; F, G) + S(r, f) + S(r, g). \end{aligned}$$

*Proof.* Suppose  $0$  is an e.v.P. (exceptional value Picard) of  $f$  and  $g$  then the lemma follows immediately. Next suppose  $0$  is not an e.v.P. of  $f$  and  $g$ . Let  $z_0$  is a zero of  $f$  with multiplicity  $q$  and a zero of  $g$  with multiplicity  $r$ . From (2.1) we know that  $z_0$  is a zero of  $F$  with multiplicity  $(n-1)q$  and a zero of  $G$  with multiplicity  $(n-1)r$ . We note that  $F$  and  $G$  have no zero of multiplicity  $t$  where  $(n-1)p < t < (n-1)(p+1)$ . So from the definition of  $\Phi$  it is clear that  $z_0$  is a zero of  $\Phi$  with multiplicity at least  $(n-1)(p+1) - 1$ . So we have

$$\begin{aligned} & [(n-1)p + n - 2] \overline{N}(r, 0; f | \geq p+1) \\ & = [(n-1)p + n - 2] \overline{N}(r, 0; g | \geq p+1) \\ & = [(n-1)p + n - 2] \overline{N}(r, 0; F | \geq (n-1)(p+1)) \\ & \leq N(r, 0; \Phi) \\ & \leq N(r, \infty; \Phi) + S(r, f) + S(r, g) \\ & \leq \overline{N}_*(r, \infty; F, G) + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g). \end{aligned}$$

The lemma follows from above.  $\square$

**Lemma 2.6.** *Let  $F$  and  $G$  be given by (2.1)  $f, g$  share  $(\infty, 0)$  and  $\infty$  is not an Picard exceptional value of  $f$  and  $g$ . Then  $V \equiv 0$  implies  $F \equiv G$ .*

*Proof.* Suppose  $V \equiv 0$ . Then by integration we obtain

$$1 - \frac{1}{F} \equiv A \left( 1 - \frac{1}{G} \right).$$

It is clear that if  $z_0$  is a pole of  $f$  then it is a pole of  $g$ . Hence from the definition of  $F$  and  $G$  we have  $\frac{1}{F(z_0)} = 0$  and  $\frac{1}{G(z_0)} = 0$ . So  $A = 1$  and hence  $F \equiv G$ .  $\square$

**Lemma 2.7.** *Let  $F, G$  be given by (2.1) and  $V \not\equiv 0$ . If  $f, g$  share  $(0, 0)$ ,  $(\infty, k)$ , where  $0 \leq k < \infty$ , and  $F, G$  share  $(1, m)$  then the poles of  $F$  and  $G$  are the zeros of  $V$  and*

$$\begin{aligned} & (nk + n - 1) \bar{N}(r, \infty; f | \geq k + 1) \\ &= (nk + n - 1) \bar{N}(r, \infty; g | \geq k + 1) \\ &\leq \bar{N}_*(r, 0; f, g) + \bar{N}(r, 0; f + a) + \bar{N}(r, 0; g + a) \\ &\quad + \bar{N}_L(r, 1; F) + \bar{N}_L(r, 1; G) + S(r, f) + S(r, g). \end{aligned}$$

*Proof.* Suppose  $\infty$  is an e.v.P. of  $f$  and  $g$  then the lemma follows immediately. Next suppose  $\infty$  is not an e.v.P. of  $f$  and  $g$ . Since  $f, g$  share  $(\infty; k)$ , it follows that  $F, G$  share  $(\infty; nk)$  and so a pole of  $F$  with multiplicity  $p(\geq nk + 1)$  is a pole of  $G$  with multiplicity  $r(\geq nk + 1)$  and vice versa. We note that  $F$  and  $G$  have no pole of multiplicity  $q$  where  $nk < q < nk + n$ . So using Lemma 2.3 and noting that  $f, g$  share  $(0, 0)$  and  $F, G$  share  $(1, m)$  we get from the definition of  $V$

$$\begin{aligned} & (nk + n - 1) \bar{N}(r, \infty; f | \geq k + 1) \\ &= (nk + n - 1) \bar{N}(r, \infty; g | \geq k + 1) \\ &= (nk + n - 1) \bar{N}(r, \infty; F | \geq nk + n) \\ &\leq N(r, 0; V) \\ &\leq N(r, \infty; V) + S(r, f) + S(r, g) \\ &\leq \bar{N}_*(r, 0; f, g) + \bar{N}(r, 0; f + a) + \bar{N}(r, 0; g + a) \\ &\quad + \bar{N}_*(r, 1; F, G) + S(r, f) + S(r, g). \end{aligned}$$

This proves the lemma.  $\square$

**Lemma 2.8**([1], Lemma 3). *Let  $f$  and  $g$  be two nonconstant meromorphic functions sharing  $(1, m)$ , where  $2 \leq m < \infty$ . Then*

$$\begin{aligned} & \bar{N}(r, 1; f | = 2) + 2\bar{N}(r, 1; f | = 3) + \cdots + (m - 1)\bar{N}(r, 1; f | = m) + m\bar{N}_L(r, 1; f) \\ &+ (m + 1)\bar{N}_L(r, 1; g) + m\bar{N}_E^{(m+1)}(r, 1; f) \leq N(r, 1; g) - \bar{N}(r, 1; g). \end{aligned}$$

**Lemma 2.9.** *Let  $F, G$  be given by (2.1) and they share  $(1, m)$ . If  $f, g$  share  $(0, p)$ ,  $(\infty, k)$ , where  $2 \leq m < \infty$  and  $H \not\equiv 0$ . Then*

$$\begin{aligned}
& T(r, F) \\
& \leq \bar{N}(r, 0; f) + \bar{N}(r, 0; g) + \bar{N}_*(r, 0; f, g) + N_2(r, 0; f + a) + N_2(r, 0; g + a) \\
& \quad + \bar{N}(r, \infty; f) + \bar{N}(r, \infty; g) + \bar{N}_*(r, \infty; f, g) - m(r, 1; G) - \bar{N}(r, 1; F | = 3) \\
& \quad - \cdots - (m-2)\bar{N}(r, 1; F | = m) - (m-2)\bar{N}_L(r, 1; F) - (m-1)\bar{N}_L(r, 1; G) \\
& \quad - (m-1)\bar{N}_E^{(m+1)}(r, 1; F) + S(r, F) + S(r, G).
\end{aligned}$$

*Proof.* By the second fundamental theorem we get

$$\begin{aligned}
(2.2) \quad & T(r, F) + T(r, G) \\
& \leq \bar{N}(r, 0; F) + \bar{N}(r, \infty; F) + \bar{N}(r, 0; G) + \bar{N}(r, \infty; G) \\
& \quad + \bar{N}(r, 1; F) + \bar{N}(r, 1; G) - N_0(r, 0; F') - N_0(r, 0; G') \\
& \quad + S(r, F) + S(r, G).
\end{aligned}$$

In view of Definition 1.7, using Lemmas 2.1, 2.2 and 2.8 we see that

$$\begin{aligned}
(2.3) \quad & \bar{N}(r, 1; F) + \bar{N}(r, 1; G) \\
& \leq N(r, 1; F | = 1) + \bar{N}(r, 1; F | = 2) + \bar{N}(r, 1; F | = 3) \\
& \quad + \cdots + \bar{N}(r, 1; F | = m) + \bar{N}_E^{(m+1)}(r, 1; F) \\
& \quad + \bar{N}_L(r, 1; F) + \bar{N}_L(r, 1; G) + \bar{N}(r, 1; G) \\
& \leq \bar{N}_*(r, 0; f, g) + \bar{N}(r, 0; f + a | \geq 2) + \bar{N}(r, 0; g + a | \geq 2) \\
& \quad + \bar{N}_*(r, \infty; f, g) + \bar{N}_L(r, 1; F) + \bar{N}_L(r, 1; G) \\
& \quad + \bar{N}(r, 1; F | = 2) + \cdots + \bar{N}(r, 1; F | = m) \\
& \quad + \bar{N}_E^{(m+1)}(r, 1; F) + \bar{N}_L(r, 1; F) + \bar{N}_L(r, 1; G) \\
& \quad + T(r, G) - m(r, 1; G) + O(1) - \bar{N}(r, 1; F | = 2) \\
& \quad - 2\bar{N}(r, 1; F | = 3) - (m-1)\bar{N}(r, 1; F | = m) - \cdots \\
& \quad - m\bar{N}_E^{(m+1)}(r, 1; F) - m\bar{N}_L(r, 1; F) - (m+1)\bar{N}_L(r, 1; G) \\
& \quad + \bar{N}_0(r, 0; F') + \bar{N}_0(r, 0; G') + S(r, F) + S(r, G) \\
& \leq \bar{N}_*(r, 0; f, g) + \bar{N}(r, 0; f + a | \geq 2) + \bar{N}(r, 0; g + a | \geq 2) \\
& \quad + \bar{N}_*(r, \infty; f, g) + T(r, G) - m(r, 1; G) - \bar{N}(r, 1; F | = 3) \\
& \quad - 2\bar{N}(r, 1; F | = 4) - \cdots - (m-2)\bar{N}(r, 1; F | = m) \\
& \quad - (m-2)\bar{N}_L(r, 1; F) - (m-1)\bar{N}_L(r, 1; G) \\
& \quad - (m-1)\bar{N}_E^{(m+1)}(r, 1; F) + \bar{N}_0(r, 0; F') + \bar{N}_0(r, 0; G') \\
& \quad + S(r, F) + S(r, G).
\end{aligned}$$

From (2.2) and (2.3) in view of Definition 1.6 the lemma follows.  $\square$

**Lemma 2.10**([9], Lemma 3). *Let  $f, g$  be two nonconstant meromorphic functions*



sharing  $(0, \infty)$ ,  $(\infty, \infty)$  and  $\Theta(\infty; f) + \Theta(\infty; g) > 0$ . Then  $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$  implies  $f \equiv g$ , where  $n (\geq 2)$  is an integer and  $a$  is a nonzero finite constant.

**Lemma 2.11**([8], Lemma 5). *If two nonconstant meromorphic functions  $f, g$  share  $(\infty, 0)$  then for  $n \geq 2$*

$$f^{n-1}(f+a)g^{n-1}(g+a) \not\equiv b^2,$$

where  $a, b$  are finite nonzero constants.

**Lemma 2.12**([17], Lemma 6). *If  $H \equiv 0$ , then  $F, G$  share  $(1, \infty)$ . If further  $F, G$  share  $(\infty, 0)$  then  $F, G$  share  $(\infty, \infty)$ .*

**Lemma 2.13**([11]). *If  $N(r, 0; f^{(k)} \mid f \neq 0)$  denotes the counting function of those zeros of  $f^{(k)}$  which are not the zeros of  $f$ , where a zero of  $f^{(k)}$  is counted according to its multiplicity then*

$$N(r, 0; f^{(k)} \mid f \neq 0) \leq k\bar{N}(r, \infty; f) + N(r, 0; f \mid < k) + k\bar{N}(r, 0; f \mid \geq k) + S(r, f).$$

**Lemma 2.14.** *Let  $F, G$  be given by (2.1),  $F, G$  share  $(1, m)$ ,  $0 \leq m < \infty$  and  $\omega_1, \omega_2 \cdots \omega_n$  are the distinct roots of the equation  $z^n + az^{n-1} + b = 0$  and  $n \geq 3$ . Then*

$$\bar{N}_L(r, 1; F) \leq \frac{1}{m+1} \left[ \bar{N}(r, 0; f) + \bar{N}(r, \infty; f) - N_{\otimes}(r, 0; f') \right] + S(r, f),$$

where  $N_{\otimes}(r, 0; f') = N(r, 0; f' \mid f \neq 0, \omega_1, \omega_2 \cdots \omega_n)$ .

*Proof.* Using Lemma 2.3 and Lemma 2.13 we see that

$$\begin{aligned} \bar{N}_L(r, 1; F) &\leq \bar{N}(r, 1; F \mid \geq m+2) \\ &\leq \frac{1}{m+1} (N(r, 1; F) - \bar{N}(r, 1; F)) \\ &\leq \frac{1}{m+1} \left[ \sum_{j=1}^n (N(r, \omega_j; f) - \bar{N}(r, \omega_j; f)) \right] \\ &\leq \frac{1}{m+1} \left( N(r, 0; f' \mid f \neq 0) - N_{\otimes}(r, 0; f') \right) \\ &\leq \frac{1}{m+1} \left[ \bar{N}(r, 0; f) + \bar{N}(r, \infty; f) - N_{\otimes}(r, 0; f') \right] + S(r, f). \end{aligned}$$

This proves the lemma.  $\square$

**Lemma 2.15**([16]). *Let  $F, G$  be two nonconstant meromorphic functions sharing  $(1, \infty)$  and  $(\infty, \infty)$ . If*

$$N_2(r, 0; F) + N_2(r, 0; G) + 2\bar{N}(r, \infty; F) < \lambda T_1(r) + S_1(r),$$

where  $\lambda < 1$  and  $T_1(r) = \max\{T(r, F), T(r, G)\}$  and  $S_1(r) = o(T_1(r))$ ,  $r \rightarrow \infty$ , outside a possible exceptional set of finite linear measure, then  $F \equiv G$  or  $FG \equiv 1$ .

**Lemma 2.16.** *Let  $F, G$  be given by (2.1) and  $n \geq 4$ . Also let  $F, G$  share  $(1, m)$ . If  $f, g$  share  $(0, 0), (\infty, k)$ , where  $2 \leq m < \infty, \Theta(\infty; f) + \Theta(\infty; g) > 0$  and  $H \equiv 0$ . Then  $f \equiv g$ .*

*Proof.* Since  $H \equiv 0$  we get from Lemma 2.12  $F$  and  $G$  share  $(1, \infty)$  and  $(\infty, \infty)$ . If possible let us suppose  $F \not\equiv G$ . Then from Lemma 2.4 and Lemma 2.5 we have

$$\overline{N}(r, 0; f) = \overline{N}(r, 0; g) = S(r).$$

Again from Lemma 2.6 and Lemma 2.7 we have

$$\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) \leq \frac{4}{n-1}T(r) + S(r).$$

Therefore we see that

$$\begin{aligned} (2.4) \quad & N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, \infty; F) \\ & \leq 2\overline{N}(r, 0; f) + 2\overline{N}(r, 0; g) + N_2(r, 0; f+a) + N_2(r, 0; g+a) + 2\overline{N}(r, \infty; f) \\ & \leq N_2(r, 0; f+a) + N_2(r, 0; g+a) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r). \end{aligned}$$

Using Lemma 2.3 we obtain

$$(2.5) \quad T_1(r) = n \max\{T(r, f), T(r, g)\} + O(1) = n T(r) + O(1).$$

So again using Lemma 2.3 we get from (2.4) and (2.5)

$$N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, \infty; F) \leq \frac{2 + \frac{4}{n-1}}{n} T_1(r) + S_1(r).$$

Since  $\frac{2 + \frac{4}{n-1}}{n} < 1$  for  $n \geq 4$  by Lemma 2.15 we have  $FG \equiv 1$ , which is impossible by Lemma 2.11. Hence  $F \equiv G$  i.e.  $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$ . This together with the assumption that  $f$  and  $g$  share  $(0, 0)$  implies that  $f$  and  $g$  share  $(0, \infty)$ . Now the lemma follows from Lemma 2.10.  $\square$

### 3. Proofs of the theorems

*Proof of Theorem 1.1.* Let  $F, G$  be given by (2.1). Then  $F$  and  $G$  share  $(1, 3), (\infty, \infty)$ . We consider the following cases.

**Case 1.** Let  $H \not\equiv 0$ . Then  $F \not\equiv G$ . Suppose 0 is not an e.v.P. of  $f$  and  $g$ . Then by Lemma 2.4 we get  $\Phi \not\equiv 0$ . Noting that  $f$  and  $g$  share  $(0, \infty)$  implies they share  $(0, 0)$ , from Lemmas 2.3, 2.5 and 2.9 we obtain for  $\varepsilon > 0$

$$\begin{aligned} (3.1) \quad & nT(r, f) \\ & \leq \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + N_2(r, 0; f+a) + N_2(r, 0; g+a) + \overline{N}(r, \infty; f) \\ & \quad + \overline{N}(r, \infty; g) - \overline{N}_L(r, 1; F) - 2\overline{N}_L(r, 1; G) + S(r, f) + S(r, g) \\ & \leq 2\overline{N}(r, 0; f) + T(r, f) + T(r, g) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) - \overline{N}_L(r, 1; F) \\ & \quad - 2\overline{N}_L(r, 1; G) + S(r, f) + S(r, g) \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{2}{n-2}[\overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G)] + 2T(r) + \{2 - \Theta(\infty; f) \\
 &\quad - \Theta(\infty; g) + 2\varepsilon\}T(r) - \overline{N}_L(r, 1; F) - 2\overline{N}_L(r, 1; G) \\
 &\quad + S(r, f) + S(r, g) \\
 &\leq [4 - \Theta(\infty; f) - \Theta(\infty; g) + 2\varepsilon]T(r) + S(r).
 \end{aligned}$$

If 0 is an e.v.P. of  $f$  and  $g$  then (3.1) automatically holds. In the same way we can obtain

$$(3.2) \quad nT(r, g) \leq [4 - \Theta(\infty; f) - \Theta(\infty; g) + 2\varepsilon] T(r) + S(r).$$

Combining (3.1) and (3.2) we see that

$$[n - 4 + \Theta(\infty; f) + \Theta(\infty; g) - 2\varepsilon] T(r) \leq S(r),$$

which leads to a contradiction for  $0 < \varepsilon < \frac{\Theta(\infty; f) + \Theta(\infty; g)}{2}$ .

**Case 2.** Let  $H \equiv 0$ . Then the theorem follows from Lemma 2.16.  $\square$

*Proof of Theorem 1.3.* Let  $F, G$  be given by (2.1). Then  $F$  and  $G$  share  $(1, 4), (\infty, \infty)$ . We consider the following cases.

**Case 1.** Let  $H \not\equiv 0$ . Then  $F \not\equiv G$ . Now using Lemmas 2.3, 2.4, 2.5 and 2.9 and proceeding in the same way as done in Case 1 of Theorem 1.1 we can deduce a contradiction.

**Case 2.** Let  $H \equiv 0$ . If  $n \geq 4$  the theorem follows from Lemma 2.16. So we will prove the theorem for  $n = 3$ . Clearly by Lemma 2.12  $F$  and  $G$  share  $(1, \infty)$  and  $(\infty, \infty)$ . Suppose  $F \not\equiv G$ . Proceeding as in the proof of Lemma 2.16 we can obtain (2.4) and (2.5). So using Lemma 2.3 we have

$$N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, \infty; F) \leq \frac{4 - \Theta(\infty; f) - \Theta(\infty; g)}{3} T_1(r) + S_1(r).$$

Noting that  $4 - \Theta(\infty; f) - \Theta(\infty; g) < 3$ , using Lemma 2.15 we  $FG \equiv 1$  which is impossible. Hence by Lemma 2.10 we get  $f \equiv g$ .  $\square$

*Proof of Theorem 1.2.* Let  $F, G$  be given by (2.1). Then  $F$  and  $G$  share  $(1, 5), (\infty, \infty)$ . We consider the following cases.

**Case 1.** Let  $H \not\equiv 0$ . Then  $F \not\equiv G$ . Suppose 0 is not an e.v.P. of  $f$  and  $g$ . Then by Lemma 2.4 we get  $\Phi \not\equiv 0$ . Noting that  $f$  and  $g$  share  $(0, 0)$  implies  $\overline{N}_*(r, 0; f, g) \leq \overline{N}(r, 0; f) = \overline{N}(r, 0; g)$  from Lemmas 2.3, 2.5 and 2.9 we get for  $\varepsilon > 0$

$$\begin{aligned}
 (3.3) \quad nT(r, f) &\leq 3\overline{N}(r, 0; f) + N_2(r, 0; f + a) + N_2(r, 0; g + a) + \overline{N}(r, \infty; f) \\
 &\quad + \overline{N}(r, \infty; g) - 3\overline{N}_L(r, 1; F) - 4\overline{N}_L(r, 1; G) + S(r, f) + S(r, g)
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{3}{n-2} [\overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G)] + 2T(r) + \{2 - \Theta(\infty; f) \\
&\quad - \Theta(\infty; g) + 2\varepsilon\} T(r) - 3\overline{N}_L(r, 1; F) - 4\overline{N}_L(r, 1; G) + S(r). \\
&\leq [4 - \Theta(\infty; f) - \Theta(\infty; g) + 2\varepsilon] T(r) + S(r).
\end{aligned}$$

If 0 is an e.v.P. of  $f$  and  $g$  then (3.3) automatically holds.

In the same way we can obtain

$$(3.4) \quad nT(r, g) \leq [4 - \Theta(\infty; f) - \Theta(\infty; g) + 2\varepsilon] T(r) + S(r).$$

From (3.3) and (3.4) we see that

$$[n - 4 + \Theta(\infty; f) + \Theta(\infty; g) - 2\varepsilon] T(r) \leq S(r),$$

which leads to a contradiction for  $0 < \varepsilon < \frac{n - 4 + \Theta(\infty; f) + \Theta(\infty; g)}{2}$ .

**Case 2.** Let  $H \equiv 0$ . Now proceeding in the same way as in the proof of Case 2 of Theorem 1.3 and noting that  $f, g$  share  $(0, 0)$  together with  $f^{n-1}(f+a) = g^{n-1}(g+a)$  implies  $f$  and  $g$  share  $(0, \infty)$  we can prove  $f \equiv g$ .  $\square$

*Proof of Theorem 1.4.* Let  $F, G$  be given by (2.1). Then  $F$  and  $G$  share  $(1, 2), (\infty; \infty)$ . We consider the following cases.

**Case 1.** Let  $H \not\equiv 0$ . Then  $F \not\equiv G$ . Suppose 0 is not an e.v.P. of  $f$  and  $g$ . Then by Lemma 2.4 and Lemma 2.5 we get  $\Phi \not\equiv 0$ . Noting that  $f$  and  $g$  share  $(\infty, \infty)$  implies  $\overline{N}(r, \infty; f) = \frac{1}{2}\overline{N}(r, \infty; f) + \frac{1}{2}\overline{N}(r, \infty; g)$  from Lemmas 2.3, 2.9 and 2.14 with  $m = 2$  we obtain for  $\varepsilon > 0$

$$\begin{aligned}
(3.5) \quad nT(r, f) &\leq \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + N_2(r, 0; f+a) + N_2(r, 0; g+a) + \overline{N}(r, \infty; f) \\
&\quad + \overline{N}(r, \infty; g) - \overline{N}_L(r, 1; G) + S(r, f) + S(r, g) \\
&\leq 2\overline{N}(r, 0; f) + T(r, f) + T(r, g) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) - \overline{N}_L(r, 1; G) \\
&\quad + S(r, f) + S(r, g) \\
&\leq \frac{2}{n-2} \overline{N}_L(r, 1; F) + 2T(r) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r, f) + S(r, g) \\
&\leq [2 + \frac{2}{3(n-2)}] T(r) + [1 + \frac{1}{3(n-2)}] \{\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g)\} \\
&\quad + S(r, f) + S(r, g) \\
&\leq [4 + \frac{4}{3(n-2)} - \frac{3n-5}{3(n-2)}] \{\Theta(\infty; f) + \Theta(\infty; g) - 2\varepsilon\} T(r) + S(r).
\end{aligned}$$

If 0 is an e.v.P. of  $f$  and  $g$  then (3.5) automatically holds.

In a similar manner we can obtain

$$(3.6) \quad nT(r, g) \leq [4 + \frac{4}{3(n-2)} - \frac{3n-5}{3(n-2)}] \{\Theta(\infty; f) + \Theta(\infty; g) - 2\varepsilon\} T(r) + S(r).$$

Combining (3.5) and (3.6) we see that

$$(3.7) \quad \left[ n - 4 - \frac{4}{3(n-2)} + \frac{3n-5}{3(n-2)} \{ \Theta(\infty; f) + \Theta(\infty; g) - 2\varepsilon \} \right] T(r) \leq S(r).$$

Since  $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{3n-5}$  there exists a  $\delta > 0$  such that

$$\Theta(\infty; f) + \Theta(\infty; g) = \frac{4}{3n-5} + \delta.$$

If we choose  $0 < \varepsilon < \frac{\delta}{2}$  then from (3.7) we can deduce a contradiction. Hence this subcase does not hold.

**Case 2.** Let  $H \equiv 0$ . Now using Lemma 2.16 we can prove  $f \equiv g$ .  $\square$

*Proof of Theorem 1.5.* Let  $F, G$  be given by (2.1). Then  $F$  and  $G$  share (1, 3),  $(\infty; \infty)$ . We consider the following cases.

**Case 1.** Let  $H \not\equiv 0$ . Then  $F \not\equiv G$ . Suppose 0 is not an e.v.P. of  $f$  and  $g$ . Then by Lemma 2.4 we get  $\Phi \not\equiv 0$ . Hence from Lemmas 2.3, 2.5, 2.9 and 2.14 with  $m = 3$  we obtain for  $\varepsilon > 0$

$$\begin{aligned} (3.8) \quad nT(r, f) &\leq \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + N_2(r, 0; f + a) + N_2(r, 0; g + a) + \overline{N}(r, \infty; f) \\ &\quad + \overline{N}(r, \infty; g) - \overline{N}_L(r, 1; F) - 2\overline{N}_L(r, 1; G) + S(r, f) + S(r, g) \\ &\leq \frac{2}{n-2} \{ \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) \} + T(r, f) + T(r, g) + \overline{N}(r, \infty; f) \\ &\quad + \overline{N}(r, \infty; g) - \overline{N}_L(r, 1; F) - 2\overline{N}_L(r, 1; G) + S(r, f) + S(r, g) \\ &\leq \frac{4-n}{n-2} \overline{N}_L(r, 1; F) + 2T(r) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) \\ &\quad + S(r, f) + S(r, g) \\ &\leq \left[ 2 + \frac{4-n}{4(n-2)} \right] T(r) + \left[ 1 + \frac{4-n}{8(n-2)} \right] \{ \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) \} \\ &\quad + S(r, f) + S(r, g) \\ &\leq \left[ 4 + \frac{4-n}{2(n-2)} - \frac{7n-12}{8(n-2)} \right] \{ \Theta(\infty; f) + \Theta(\infty; g) - 2\varepsilon \} T(r) + S(r) \\ &\leq \left[ 3 + \frac{n}{2(n-2)} - \frac{7n-12}{8(n-2)} \right] \{ \Theta(\infty; f) + \Theta(\infty; g) - 2\varepsilon \} T(r) + S(r). \end{aligned}$$

If 0 is an e.v.P. of  $f$  and  $g$  then (3.8) automatically holds.

In a similar manner we can obtain

$$(3.9) \quad nT(r, g) \leq \left[ 3 + \frac{n}{2(n-2)} - \frac{7n-12}{8(n-2)} \right] \{ \Theta(\infty; f) + \Theta(\infty; g) - 2\varepsilon \} T(r) + S(r).$$

Combining (3.8) and (3.9) we see that

$$(3.10) \quad \left[ n - 3 - \frac{n}{2(n-2)} + \frac{7n-12}{8(n-2)} \{ \Theta(\infty; f) + \Theta(\infty; g) - 2\varepsilon \} \right] T(r) \leq S(r).$$

Since  $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4n}{7n-12}$  there exists a  $\delta > 0$  such that

$$\Theta(\infty; f) + \Theta(\infty; g) = \frac{4n}{7n-12} + \delta.$$

If we choose  $0 < \varepsilon < \frac{\delta}{2}$  then from (3.10) we can obtain a contradiction. Hence this subcase does not hold.

**Case 2.** Let  $H \equiv 0$ . Now we can prove  $f \equiv g$  in the line of the proof of Case 2 of Theorem 1.2.  $\square$

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