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Representations of the Braid Group and Punctured Torus Bundles

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Abstract. In this short note, we consider a family of linear representations of the braid group and the fundamental group of a punctured torus bundle over the circle. We construct an irreducible (special) unitary representation of the fundamental group of a closed 3-manifold obtained by the Dehn filling.

1. Introduction

Representations of the fundamental group have played an important role in the study of 3-dimensional topology. For instance, studying the structure of the $SL(2, C)$-representation space gives us information about embedded surfaces in a 3-manifold. Representing a knot group into well-known groups, including $SU(2)$, to obtain geometric information of a knot has met with success in various context.

In [4], motivated by the volume conjecture due to Kashaev-Murakami-Murakami [7], the first author et al. introduced an infinite sequence of L^2-torsion invariants, which approximates the simplicial volume, for a surface bundle over the circle. It is defined by using the regular representations associated with the lower central series of the surface group. One of the results in [4] states that the geometric structure of a punctured torus bundle is detected by our first invariant corresponding to the homology representation. More precisely, the invariant is non-trivial if and only if a punctured torus bundle admits the hyperbolic structure.

In view of such a background, it seems natural to ask whether representations into other groups with the similar geometric information exist. In particular, we would like to construct an invariant of 3-manifolds derived from the representation

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in question. For the problem, we can find in this note a one-parameter family of irreducible $GL(4, C)$-representations of the fundamental group of a punctured torus bundle over the circle. We have not achieved our original attempt rigorously, but it will be useful to introduce an invariant of punctured torus bundles, which is minimized by the exterior of the figure eight knot in the 3-sphere.

The purpose of this short note is to discuss another application of this family. Namely, we give in Section 3 a sufficient condition that it induces a 4-dimensional irreducible (special) unitary representation of a closed 3-manifold obtained by the Dehn filling. To this end, we first consider a complex one-parameter family of n-dimensional linear representations for the braid group B_n (see the next section). The construction used here is based on Mangum-Shanahan's recipe for a family of $SL(3, C)$-representations [5]. See also [6] for a two-parameters family of $SL(6, C)$-representations.

2. A family of representations

The braid group B_n of n strings is generated by $n-1$ elements $\sigma_1, \cdots, \sigma_{n-1}$ which satisfy the two kinds of braid relations:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \ (|i-j| \geq 2),$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \ (1 \leq i \leq n-2).$$

Let $r : B_n \to GL(n, Z[s, s^{-1}])$ be a representation of B_n defined by the assignment

$$r(\sigma_i) = I_{i-1} \oplus \begin{pmatrix} 0 & -s \\ s & 0 \end{pmatrix} \oplus I_{n-i-1},$$

where I_n denotes the $n \times n$ identity matrix. We easily see that r satisfies all the relations of B_n. We can regard r as a one-parameter deformation of an $SO(n)$-representation ($s = 1$) or a $U(n)$-representation ($s = \sqrt{-1}$). In the following, we construct a family of 4-dimensional irreducible representations of a punctured torus bundle along the paper [5] (see also [6]).

Let W_f be an oriented punctured torus bundle over the circle with a monodromy $f \in SL(2, Z)$. Namely, it is the identification space $T \times R/(x, \tau) \sim (f(x), \tau + 1)$, where T is a once punctured torus. The fundamental group of W_f has a presentation of the form

$$\pi_1 W_f = \langle x, y, z \mid zxz^{-1} = f_*(x), \ zyz^{-1} = f_*(y)\rangle.$$

We first modify the representation r as $r'(\sigma_i) = (-1/s^2)^{1/4} r(\sigma_i)$. For a technical reason, we also put $t = s^{1/2}$ and then obtain a representation $\alpha : B_4 \to GL(4, C[t, t^{-1}])$ for $t \neq 0$. For each i, $\det \alpha(\sigma_i) = -1$ and further $\alpha\left((\sigma_1\sigma_2\sigma_3)^4\right) = I_4$ holds. Hence we have a representation

$$\bar{\alpha} : B_4/C_4 \to GL(4, C[t, t^{-1}]),$$

where C_4 is the center of B_4 generated by $(\sigma_1\sigma_2\sigma_3)^4$.

Next we review the construction of a homomorphism from $\pi_1 W_f$ to B_4/C_4. Since the free group $F_2 \cong \langle x, y\rangle$ injects into B_4 ($x \mapsto \sigma_1\sigma_3^{-1}$, $y \mapsto \sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}$) and the image is a normal subgroup of B_4 (see [2]), we have a homomorphism $h : B_4 \to \mathrm{Aut}(F_2)$. Moreover, the kernel of h is C_4 and the image of $\bar{h} : B_4/C_4 \to \mathrm{Aut}(F_2)$ is $\mathrm{Aut}^+(F_2)$ (see [3]), so that we have an isomorphism

$$\bar{h} : B_4/C_4 \to \mathrm{Aut}^+(F_2).$$

Here, $\mathrm{Aut}^+(F_2)$ denotes an index two subgroup of $\mathrm{Aut}(F_2)$, which is the preimage of $SL(2, Z)$ under the natural surjective homomorphism $\mathrm{Aut}(F_2) \to GL(2, Z)$. Using the isomorphism, we can define a homomorphism $\iota : \pi_1 W_f \to B_4/C_4$ by

$$\begin{aligned}
\iota(x) &= [\sigma_1\sigma_3^{-1}],\\
\iota(y) &= [\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}],\\
\iota(z) &= \bar{h}^{-1}(f_*).
\end{aligned}$$

The composite of ι and $\bar{\alpha}$ yields a family of representations

$$\rho_t : \pi_1 W_f \to GL(4, C).$$

A direct computation shows that

$$X = \rho_t(x) = \begin{pmatrix} 0 & -t^2 & 0 & 0 \\ t^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & t^{-2} \\ 0 & 0 & -t^{-2} & 0 \end{pmatrix}, \quad Y = \rho_t(y) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ t^4 & 0 & 0 & 0 \\ 0 & t^{-4} & 0 & 0 \end{pmatrix}$$

and $\det X = \det Y = 1$. Let P be a nonsingular matrix given by

$$P = \begin{pmatrix} 0 & 0 & -\sqrt{-1} & \sqrt{-1} \\ 0 & 0 & 1 & 1 \\ \sqrt{-1} & -\sqrt{-1} & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Taking conjugations of X and Y by P, we obtain

$$P^{-1}XP = \begin{pmatrix} -\sqrt{-1}t^{-2} & 0 & 0 & 0 \\ 0 & \sqrt{-1}t^{-2} & 0 & 0 \\ 0 & 0 & -\sqrt{-1}t^2 & 0 \\ 0 & 0 & 0 & \sqrt{-1}t^2 \end{pmatrix}, \tag{2.1}$$

$$P^{-1}YP = \begin{pmatrix} 0 & 0 & (t^{-4} - t^4)/2 & (t^{-4} + t^4)/2 \\ 0 & 0 & (t^{-4} + t^4)/2 & (t^{-4} - t^4)/2 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.2}$$

All the invariant subspaces for (2.1) are spanned by the coordinate axes. However, none of these subspaces is fixed by (2.2). Thus ρ_t is irreducible for all values of $t \in C$ except for $t = 0$. Summing up, we have the following.

Proposition 2.1. *There exists a complex one-parameter family ρ_t of irreducible representations of a punctured torus bundle over the circle into $GL(4, C)$.*

Remark 2.2. As we see in the next section, if we specialize the value of t (put $t = (-1)^{1/4}$ for example), then we can obtain an irreducible $SU(4)$-representation of $\pi_1 W_f$ under a certain condition.

3. Unitary representations for closed 3-manifolds

In this section, we construct representations of closed 3-manifolds obtained from W_f by Dehn fillings. To this end, we first explain how to compute $\bar{h}^{-1}(f_*)$ for any $f_* \in \mathrm{Aut}^+(F_2)$ (see also [5] Remark 4). Put $S = \bar{h}(\sigma_2\sigma_3\sigma_2^{-1})$ and $R = \bar{h}(\sigma_2\sigma_1\sigma_3)$. Checking the conjugate action of S and R on F_2, we see that

$$S(x) = xy, \quad S(y) = y, \quad R(x) = y, \quad R(y) = x^{-1}.$$

By suitable application of S, R and their inverses to $f_*(x)$ and $f_*(y)$, we obtain x and y as a result. Namely, if we denote the word in R and S by $\mathcal{F}$, then $\mathcal{F} \circ f_*$ is the identity in $\mathrm{Aut}^+(F_2)$.

Now let us consider a monodromy $f = \begin{pmatrix} 0 & -1 \\ 1 & m \end{pmatrix} \in SL(2, Z)$ as our object and fix a group presentation

$$\pi_1 W_f = \langle x, y, z \mid zxz^{-1} = y^{-1},\ zyz^{-1} = yxy^{m-1} \rangle,$$

where m is an integer.

Remark 3.1. The conjugacy classes in $SL(2, Z)$ are of three types: (i) elliptic, (ii) parabolic and (iii) hyperbolic. Our monodromy f contains all the types and is conjugate to $f' = \begin{pmatrix} a & bc \\ 1 & d \end{pmatrix} \in SL(2, Z)$, where $m = a + d$. Moreover $W_{f'}$ is a $|c|$-fold covering of W_g with the monodromy $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. As is known, the conjugacy classes in $SL(2, Z)$ are not determined only by the trace and there are finitely many conjugacy classes with the same trace (see [1] and its references).

Lemma 3.2. *We have*

$$\mathcal{F}_m(y^{-1}) = x \quad \textit{and} \quad \mathcal{F}_m(yxy^{m-1}) = y$$

for the word $\mathcal{F}_m = R^{-1}S^{-1}R^2S^{1-m}$.

Proof. Straightforward calculations show that the image of x and y via $\mathcal{F}_m$ are

$$\mathcal{F}_m(x) = xyx^{m-1}, \quad \mathcal{F}_m(y) = x^{-1}.$$

Hence $\mathcal{F}_m(y^{-1}) = x$ and $\mathcal{F}_m(yxy^{m-1}) = x^{-1} \cdot xyx^{m-1} \cdot x^{1-m} = y$. $\square$

Thus we obtain $\bar{h}^{-1}(f_*) = \bar{h}^{-1}(\mathcal{F}_m^{-1})$ and then

$$\bar{h}^{-1}(f_*) = [\sigma_2 {\sigma_3}^{m-1} \sigma_2^{-1} \sigma_3^{-1} \sigma_1^{-1} \sigma_2^{-1} \sigma_3].$$

Therefore we have the matrix below corresponding to the monodromy f,

$$Z_m = \rho_t(z) = \begin{cases} \begin{pmatrix} 0 & 0 & 0 & a_k \\ b_k & 0 & 0 & 0 \\ 0 & 0 & c_k & 0 \\ 0 & d_k & 0 & 0 \end{pmatrix} & m = 2k-1, \\ \begin{pmatrix} 0 & 0 & 0 & a'_k \\ 0 & b'_k & 0 & 0 \\ 0 & 0 & c'_k & 0 \\ d'_k & 0 & 0 & 0 \end{pmatrix} & m = 2k, \end{cases}$$

where

$$\begin{aligned} a_k &= (-1)^{\frac{k}{2}}\, t^{-2k+2}, & a'_k &= (-1)^{\frac{2k+1}{4}}\, t^{-2k+1}, \\ b_k &= -(-1)^{\frac{3k}{2}}\, t^{2k-2}, & b'_k &= (-1)^{\frac{6k+1}{4}}\, t^{2k-1}, \\ c_k &= -(-1)^{\frac{k}{2}}\, t^{-2k+4}, & c'_k &= -(-1)^{\frac{2k+1}{4}}\, t^{-2k+3}, \\ d_k &= (-1)^{\frac{3k}{2}}\, t^{2k-4}, & d'_k &= (-1)^{\frac{6k+1}{4}}\, t^{2k-3}. \end{aligned}$$

It is easy to see that $\det Z_m = 1$, if m is odd. Hence we get a representation $\rho_t : \pi_1 W_f \to SL(4, C)$ in this case (see Remark 2.2).

Here we recall the definition of Dehn filling on a compact 3-manifold. We first note that W_f is homeomorphic to the interior of a compact 3-manifold with a torus boundary E. Let $\nu \in H_1(E)$ be a primitive homology element (i.e., ν is not a multiple of another $\nu' \in H_1(E)$). Let Σ be a solid torus with boundary $\partial\Sigma$ and $\phi : E \to \partial\Sigma$ be a homeomorphism so that $\phi_*(\nu) = 0$ in $H_1(\Sigma)$. Then the identification space $W_f^\nu = (W_f \cup \Sigma)/\phi$ is called the ν-Dehn filling. The homeomorphism type of W_f^ν only depends on ν. If there is some implicit identification $H_1(E) \cong Z^2$ carrying ν to (p, q), we call W_f^ν the (p, q)-Dehn filling and denote it by $W_{(p,q)}$ for simplicity.

Now we are ready to study the possibility that a family of representations ρ_t induces a non-trivial representation of the closed 3-manifold obtained by the (p, q)-Dehn filling. As an answer, we have the following theorem.

Theorem 3.3. *For a monodromy* $f = \begin{pmatrix} 0 & -1 \\ 1 & m \end{pmatrix} \in SL(2, Z)$ *and a coprime pair of integers* (p, q)*, if*

$$p \equiv \begin{cases} 0 \mod 3 & (m = 2k-1) \\ 0 \mod 2 & (m = 2k) \end{cases},$$

there exists an irreducible $SU(4)$*-representation* $(m = 2k-1)$ *or* $U(4)$*-representation* $(m = 2k)$ *of the fundamental group of the closed 3-manifold* $W_{(p,q)}$ *obtained from*

W_f by the (p,q)-Dehn filling.

Proof. First, we investigate the case where m is an odd integer. Let $\{\mu,\lambda\}$ be a basis of the fundamental group of the boundary torus. Namely, the basis are represented by z and $xyx^{-1}y^{-1}$ respectively. Two matrices $\rho_t(\mu)$ and $\rho_t(\lambda)$ are commutative, so that after suitable change of the basis we get two diagonal matrices M and L, whose diagonal components are

$$\left\{(-1)^{\frac{k+2}{2}}\, t^{-2k+4}, (-1)^{\frac{7k+2}{6}}\, t^{\frac{2k-4}{3}}, (-1)^{\frac{7k+6}{6}}\, t^{\frac{2k-4}{3}}, (-1)^{\frac{7k+10}{6}}\, t^{\frac{2k-4}{3}}\right\}$$

and $\{-t^{-12}, -t^4, -t^4, -t^4\}$ respectively. A representation $\rho_t : \pi_1 W_f \to SL(4,C)$ induces a representation $\bar{\rho}_t : \pi_1 W_{(p,q)} \to SL(4,C)$ if $\rho_t(\mu^p\lambda^q) = I_4$ (i.e., $M^pL^q = I_4$) holds. This condition can be expressed as the following four equations:

$$(-1)^{\frac{k+2}{2}p+q}\, t^{-\{(2k-4)p+12q\}} = 1, \tag{3.1}$$

$$(-1)^{\frac{7k+2}{6}p+q}\, t^{\frac{1}{3}\{(2k-4)p+12q\}} = 1, \tag{3.2}$$

$$(-1)^{\frac{7k+6}{6}p+q}\, t^{\frac{1}{3}\{(2k-4)p+12q\}} = 1, \tag{3.3}$$

$$(-1)^{\frac{7k+10}{6}p+q}\, t^{\frac{1}{3}\{(2k-4)p+12q\}} = 1. \tag{3.4}$$

The equations (3.2), (3.3) and (3.4) imply $p \equiv 0 \mod 3$. Since

$$(-1)^{\frac{k+2}{2}} = \begin{cases} (-1)^{\frac{7k+2}{6}} & k = 3l+1 \\ (-1)^{\frac{7k+6}{6}} & k = 3l \\ (-1)^{\frac{7k+10}{6}} & k = 3l+2 \end{cases}$$

hold, one of the three equations (3.2), (3.3), (3.4) can be written to be

$$(-1)^{\frac{k+2}{2}p+q}\, t^{\frac{1}{3}\{(2k-4)p+12q\}} = 1.$$

Substituting it for the first equation (3.1), we obtain

$$(-1)^{\frac{4k+8}{2}p+4q} = 1.$$

This equation always holds. Hence the condition $\rho_t(\mu^p\lambda^q) = I_4$ is described as $p \equiv 0$ mod 3. Conversely, if we put $t = (-1)^{1/4}$ and $p = 3l$ ($l \in \mathbb{Z}$), then four equations (3.1), (3.2), (3.3) and (3.4) are satisfied. Moreover, we can check that the matrices X, Y and Z_m satisfy the unitarity condition when $t = (-1)^{1/4}$.

Next, we consider the case where m is even. In this case, $\det Z_m = -1$ holds. After changing the basis, the diagonal components of the diagonal matrices M and L are

$$\left\{(-1)^{\frac{2k+5}{4}}\, t^{-2k+3}, (-1)^{\frac{6k+1}{4}} t^{2k-1}, (-1)^{\frac{1}{4}}\, t^{-1}, (-1)^{\frac{5}{4}}\, t^{-1}\right\}$$

and $\{-t^{-12}, -t^4, -t^4, -t^4\}$ respectively. The condition $\rho_t(\mu^p\lambda^q) = I_4$ is equivalent to the equations

$$(-1)^{\frac{2k+5}{4}p+q}\, t^{(-2k+3)p-12q} = 1, \tag{3.5}$$

$$(-1)^{\frac{6k+1}{4}p+q}\, t^{(2k-1)p+4q} = 1, \tag{3.6}$$

$$(-1)^{\frac{1}{4}p+q}\, t^{-p+4q} = 1, \tag{3.7}$$

$$(-1)^{\frac{5}{4}p+q}\, t^{-p+4q} = 1. \tag{3.8}$$

The last two equations (3.7), (3.8) imply $p \equiv 0 \mod 2$. Substituting (3.7) for (3.5) and (3.6), they can be written to be

$$(-1)^{\frac{k}{2}p}\, t^{-2kp} = 1, \qquad (-1)^{\frac{3k}{2}p} t^{2kp} = 1.$$

These equations always hold for $t = (-1)^{1/4}$. Hence the condition $\rho_t(\mu^p\lambda^q) = I_4$ is described as $p \equiv 0 \mod 2$ in this case. We can easily check that four equations (3.5), (3.6), (3.7) and (3.8) are satisfied if we put $t = (-1)^{1/4}$ and $p = 2l$ ($l \in \mathbb{Z}$). Moreover the matrix Z_{2k} satisfies the unitarity condition when $t = (-1)^{1/4}$. This completes the proof. $\square$

Remark 3.4. For a monodromy $f = \begin{pmatrix} 0 & -1 \\ 1 & m \end{pmatrix} \in SL(2, Z)$ with an odd trace, we obtain an irreducible Spin(6)-representation of $\pi_1 W_{(p,q)}$ in terms of the isomorphism between classical Lie groups $SU(4) \cong \mathrm{Spin}(6)$.

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