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On Self-commutator Approximants

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Abstract. Let $B(\mathcal{X})$ denote the algebra of operators on a complex Banach space $\mathcal{X}$, $H(\mathcal{X}) = \{h \in B(\mathcal{X}) : h \text{ is hermitian}\}$, and $J(\mathcal{X}) = \{x \in B(\mathcal{X}) : x = x_1 + ix_2, x_1 \text{ and } x_2 \in H(\mathcal{X})\}$. Let $\delta_a \in B(B(\mathcal{X}))$ denote the derivation $\delta_a(x) = ax - xa$. If $J(\mathcal{X})$ is an algebra and ${\delta_a}^{-1}(0) \subseteq \delta_{a^*}^{-1}(0)$ for some $a \in J(\mathcal{X})$, then $||a|| \leq ||a-(x^*x-xx^*)||$ for all $x \in J(\mathcal{X}) \cap {\delta_a}^{-1}(0)$. The cases $J(\mathcal{X}) = B(\mathcal{H})$, the algebra of operators on a complex Hilbert space, and $J(\mathcal{X}) = \mathcal{C}_p$, the von Neumann–Schatten p-class, are considered.

1. Introduction

An element $h \in B(\mathcal{X})$, $B(\mathcal{X})$ = the algebra of (bounded linear) operators on a complex Banach space $\mathcal{X}$, is *hermitian* if the the algebra numerical range $V(B(\mathcal{X}), h) = \{f(h) : f \in B(\mathcal{X})^*, f(I) = 1 = ||f||\}$ is a subset of the set of reals [3, Page 8]. Let

$$H(\mathcal{X}) = \{h \in B(\mathcal{X}) : h \ \text{ is hermitian}\},$$

and let

$$J(\mathcal{X}) = \{x \in B(\mathcal{X}) : x = x_1 + ix_2, x_1 \ \text{ and } \ x_2 \in H(\mathcal{X})\}.$$

Then each $x \in J(\mathcal{X})$ has a unique representation $x = x_1 + ix_2$, x_1 and $x_2 \in H(\mathcal{X})$, and we may define a mapping $x \longrightarrow x^*$ from $J(\mathcal{X})$ into itself by $x^* = x_1 - ix_2$ $(= (x_1 + ix_2)^*)$: $J(\mathcal{X})$ with the operator norm $||.||$ of $B(\mathcal{X})$ is a complex Banach space such that * is a continuous linear involution on $J(\mathcal{X})$ [3, Lemma 8, Page 50]. Recall that an operator $a \in B(\mathcal{X})$ is normal if $a = a_1 + ia_2 \in J(\mathcal{X})$ and $[a_1, a_2] = a_1a_2 - a_2a_1 = 0$. We say that an operator $a \in J(\mathcal{X})$ satisfies *the PF-property*, short for the Putnam–Fuglede property, if $a^{-1}(0) \subseteq {a^*}^{-1}(0)$. Normal operators satisfy the PF–property: if $a = a_1 + ia_2$ is normal, then $ax = 0$ implies $a_1x = a_2x = 0 \Longrightarrow a^*x = 0$ [4, Page 124].

Let $\delta_a \in B(B(\mathcal{X}))$ denote the *derivation* $\delta_a(x) = ax - xa = (L_a - R_a)x$, where L_a and R_a denote, respectively, the operators of left multiplication and right multiplication by a. If $a \in H(\mathcal{X})$, then L_a, R_a and $L_a - R_a \in H(\mathcal{X})$. Evidently, if $a = a_1 + ia_2$, then $\delta_a = \delta_{a_1} + i\delta_{a_2}$, where $[\delta_{a_1}, \delta_{a_2}] = 0$ whenever $[a_1, a_2] = 0$.

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Hence, if a is normal then δ_a is normal, and this by [12, Corollary 8] implies that

$$-2\sqrt{||\delta_a(x)||||y||} + ||x|| \leq ||x - \delta_a(y)||$$

for all $x, y \in B(\mathcal{X})$. In particular, if $x \in {\delta_a}^{-1}(0)$, then (for all $y \in B(\mathcal{X})$)

$$||x|| \leq ||x - \delta_a(y)||, \tag{1}$$

i.e., the kernel ${\delta_a}^{-1}(0)$ of δ_a is orthogonal to the range $\delta_a(B(\mathcal{X}))$ of δ_a in the sense of G. Birkhoff and R. C. James [9, page 93]. Kernel-range inequalities of type (1), especially in the setting of the algebra $B(\mathcal{H})$ (of operators on a complex Hilbert space $\mathcal{H}$) and the von Neumann–Schatten p-classes $\mathcal{C}_p = \mathcal{C}_p(\mathcal{H})$ ($\mathcal{H}$ separable, $1 \leq p < \infty$) have been considered by a number of authors (see [2], [6], [7], [11], [12], [14] for further references). In this paper we look at the equation $\delta_a(x) = 0$ from the view point that $x \in {\delta_a}^{-1}(0) \iff a \in \delta_x^{-1}(0)$, and prove some results on self-commutator approximants of the type recently proved by P. J. Maher [13] (for self-adjoint a and $x \in {\delta_a}^{-1}(0)$). Assuming that $J(\mathcal{X})$ is an algebra (in particular, $J(\mathcal{X}) = B(\mathcal{H})$ or $J(\mathcal{X}) = \mathcal{C}_p$ for some $1 \leq p < \infty$) and the PF–property that ${\delta_a}^{-1}(0) \subseteq \delta_{a^*}^{-1}(0)$ for $a \in J(\mathcal{X})$, we prove that

$$||a|| \leq ||a - [x^*, x]|| \tag{2}$$

for all $x \in J(\mathcal{X}) \cap {\delta_a}^{-1}(0)$. In the case in which $1 < p < \infty$, ${\delta_a}^{-1}(0) \subseteq \delta_{a^*}^{-1}(0)$ and $a \in \mathcal{C}_p$, it is proved that

$$||a||_p \leq \min\{||a - \delta_{x_1}(y)||_p, ||a - \delta_{x_2}(y)||_p\} \tag{3}$$

for all $x = x_1 + ix_2$ and $y \in B(\mathcal{H})$ such that $\delta_{x_j}(y) \in \mathcal{C}_p$ $(j = 1, 2)$ if and only if $x \in {\delta_a}^{-1}(0)$. We also prove that inequality (2) holds for essentially normal operators $x \in B(\mathcal{H}) \cap {\delta_a}^{-1}(0)$ such that $||a||$ equals the essential norm $||a||_e$ of a.

2. Results

Evidently, $x \in {\delta_a}^{-1}(0) \iff a \in \delta_x^{-1}(0)$ for all $a, x \in B(\mathcal{X})$. Since $h \in H(\mathcal{X})$ does not (in general) imply that $h^2 \in H(\mathcal{X})$ [3, Example 1, Page 58], $J(\mathcal{X})$ is not (in general) a subalgebra of $B(\mathcal{X})$. If however $J(\mathcal{X})$ is an algebra, then $h, k \in H(\mathcal{X})$ implies that h^2 and $hk + kh \in H(\mathcal{X})$ [3, Theorem 3, Page 59]. Recall that $i(a_1a_2 - a_2a_1) \in H(\mathcal{X})$ whenever $a_1, a_2 \in H(\mathcal{X})$ [3, Lemma 4, Page 47]. Let $a = a_1 + ia_2$ and $b = b_1 + ib_2 \in J(\mathcal{X})$, and assume that $J(\mathcal{X})$ is an algebra. Then both

$$ab + b^*a^* = \{(a_1b_1 + b_1a_1) - (a_2b_2 + b_2a_2)\} + i\{(a_2b_1 - b_1a_2) + (a_1b_2 - b_2a_1)\}$$

and

$$i(ab - b^*a^*) = i\{(a_1b_1 - b_1a_1) - (a_2b_2 - b_2a_2)\} - \{((a_2b_1 + b_1a_2) + (a_1b_2 + b_2a_1)\}$$

are in $H(\mathcal{X})$. Hence

$$(ab)^* = \frac{1}{2}(ab + b^*a^*) + \frac{i}{2}(ab - b^*a^*) = b^*a^*.$$

Theorem 2.1. *If $J(\mathcal{X})$ is an algebra and ${\delta_a}^{-1}(0) \subseteq \delta_{a^*}^{-1}(0)$ for some $a \in J(\mathcal{X})$, then $||a|| \le ||a - [x^*, x]||$ for all $x \in J(\mathcal{X}) \cap {\delta_a}^{-1}(0)$.*

Proof. The hypotheses $J(\mathcal{X})$ is an algebra and ${\delta_a}^{-1}(0) \subseteq \delta_{a^*}^{-1}(0)$ imply that $\delta_x(a) = \delta_{x^*}(a) = 0$ for every $x \in J(\mathcal{X}) \cap {\delta_a}^{-1}(0)$. Hence, upon letting $x = x_1 + ix_2$, $\delta_{x_1}(a) = \delta_{x_2}(a) = 0$. Since $x_j \in H(\mathcal{X})$, $j = 1, 2$, it follows that

$$||a|| \le \min\{||a - \delta_{x_1}(y)||, ||a - \delta_{x_2}(y)||\}$$

for all $y \in J(\mathcal{X})$ [12, Corollary 8]. Choose $y = 2ix_2$ (in $\delta_{x_1}(y)$); then $\delta_{x_1}(y) = [x^*, x]$ and $||a|| \le ||a - [x^*, x]||$ for all $x \in J(\mathcal{X}) \cap {\delta_a}^{-1}(0)$. $\square$

The following corollary is immediate from Theorem 2.1.

Corollary 2.2. *If $a \in B(\mathcal{H})$ is such that ${\delta_a}^{-1}(0) \subseteq \delta_{a^*}^{-1}(0)$, then $||a|| \le ||a - [x^*, x]||$ for all $x \in B(\mathcal{H}) \cap {\delta_a}^{-1}(0)$.*

An operator $a \in B(\mathcal{H})$ is *essentially normal* if $\pi(a)$ is normal, where $\pi : B(\mathcal{H}) \longrightarrow B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the Calkin map. (Equivalently, a is essentially normal if $\pi([a^*, a]) = 0$.) For essentially normal $x \in {\delta_a}^{-1}(0)$, we have the following.

Theorem 2.3. *If $x \in {\delta_a}^{-1}(0) \cap B(\mathcal{H})$ is essentially normal, then $||\pi(a)|| \le ||a - [x^*, x]||$.*

Proof. If $x \in {\delta_a}^{-1}(0)$, then $\pi(a) \in \delta_{\pi(x)}^{-1}(0)$. Since $B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is a C^*-algebra, there exists a Hilbert space $\mathcal{H}_0$ and a $*$-isometric isomorphism $\psi : B(\mathcal{H})/\mathcal{K}(\mathcal{H}) \longrightarrow B(\mathcal{H}_0)$ such that $x_0 = \psi(\pi(x))$ is a normal element of $B(\mathcal{H}_0)$. Letting $a_0 = \psi(\pi(a))$, it follows that

$$||\pi(a)|| = ||a_0|| \le ||a_0 - \delta_{x_0}(\psi(\pi(y))|| = ||\pi(a - \delta_x(y))|| \le ||a - \delta_x(y)||$$

for all $y \in B(\mathcal{H})$. Choose $y = -x^*$. $\square$

In general, $||\pi(a)|| \neq ||a||$. However, if $a \in B(\mathcal{H})$ is hyponormal (i.e., $|a^*|^2 \le |a|^2$), or normaloid ($||a||$ equals the spectral radius of a) and without eigen-values of finite multiplicity, then $||\pi(a)|| = ||a||$ (see [8, Page 1730]): for such $a \in B(\mathcal{H})$, $||a|| \le ||a - [x^*, x]||$.

A version of Theorems 2.1 has been proved by Maher [13, Theorems 4.1(a) and 4.2] for the von Neumann-Schatten p-classes $(\mathcal{C}_p, ||.||_p)$; $1 \le p < \infty$. Observe from the proof of Theorem 2.1 that if ${\delta_a}^{-1}(0) \subseteq \delta_{a^*}^{-1}(0)$, then $||a||_p \le ||a - [x^*, x]||_p$ for all $a \in \mathcal{C}_p$ and $x \in {\delta_a}^{-1}(0)$ such that $[x^*, x] \in \mathcal{C}_p$. The following theorem proves that the condition $x = x_1 + ix_2 \in {\delta_a}^{-1}(0)$ is necessary for $||a||_p \le \min\{||a -$

$\delta_{x_1}(y)||_p, ||a-\delta_{x_2}(y)||_p\}$ in the case in which $1<p<\infty$. But before that we introduce some terminology. If $(\mathcal{V}, ||.||)$ is a Banach space, then $||.||$ is said to be Gateaux–differentiable at a non-zero $x \in \mathcal{V}$ if

$$\lim_{t\longrightarrow 0} \frac{||x+ty||-||x||}{t} = \mathrm{Re}D_x(y)$$

exists for all $y \in \mathcal{V}$. Here $t \in \mathcal{R}$ (= the set of reals), Re denotes the *real part* and D_x is the unique support functional in the dual space $\mathcal{V}^*$ such that $||D_x|| = 1$ and $||D_x(x)|| = ||x||$. The Gateaux–differentiability of $||.||$ at x implies that x is a smooth point of the sphere with radius $||x||$. If an $a \in \mathcal{C}_p$, $1<p<\infty$, has the polar decomposition $a = u|a|$, then $|a|^{p-1}u^* \in \mathcal{C}_{p'}$, $\frac{1}{p}+\frac{1}{p'} = 1$, and $D_a(y) = \mathrm{tr}\{|a|^{p-1}u^*y/||a||_p^{p-1})$ for every $y \in \mathcal{C}_p$ [1, Theorem 2.3]. (As usual, tr denotes the trace functional.) Recall from [10] that if $a, b \in \mathcal{V}$ and a is a smooth point of $\mathcal{V}$, then $||a|| \le ||a+tb||$ for all complex t if and only if $D_a(b) = 0$.

Theorem 2.4. *If* ${\delta_a}^{-1}(0) \subseteq \delta_{a^*}^{-1}(0)$ *then*

$$||a||_p \le \min\{||a-\delta_{x_1}(y)||_p, ||a-\delta_{x_2}(y)||_p\} \tag{4}$$

for all $a \in \mathcal{C}_p$ *and* $y \in B(\mathcal{H})$ *such that* $\delta_{x_j}(y) \in \mathcal{C}_p$, $j = 1,2$ *and* $1<p<\infty$, *if and only if* $x = x_1 + ix_2 \in {\delta_a}^{-1}(0)$.

Proof. The 'if part' being evident (from $\delta_a(x) = 0 \Longrightarrow \delta_{x_1}(a) = \delta_{x_2}(a) = 0$), we prove the 'only if' part. Recall that $\mathcal{C}_p$, $1<p<\infty$, is uniformly convex; hence operators $a \in \mathcal{C}_p$ are smooth points of $\mathcal{C}_p$. Let a have the polar decomposition $a = u|a|$. Then, [10], the inequality of the statement of the theorem holds for all $y \in B(\mathcal{H})$ such that $\delta_{x_j}(y) \in \mathcal{C}_p$, $j = 1,2$, if and only if the support functional $D_a(\delta_{x_j}(y)) = \mathrm{tr}(|a|^{p-1}u^*\delta_{x_j}(y)/||a||_p^{p-1}) = 0$. Set $|a|^{p-1}u^* = \tilde{a}$; then $\tilde{a} \in \mathcal{C}_{p'}$, $\frac{1}{p}+\frac{1}{p'} = 1$. Choose y to be the rank one operator $y = e \otimes f$ for some $e, f \in \mathcal{H}$. Then $\delta_{x_j}(y) \in \mathcal{C}_p$ and

$$\begin{aligned} \mathrm{tr}(\tilde{a}\delta_{x_j}(y)) &= \mathrm{tr}(\tilde{a}(x_jy - yx_j)) = \mathrm{tr}((\tilde{a}x_j - x_j\tilde{a})y) \\ &= \mathrm{tr}(\delta_{\tilde{a}}(x_j)e \otimes f) = (\delta_{\tilde{a}}(x_j)e, f) = 0 \end{aligned}$$

for all $e, f \in \mathcal{H}$. Hence $\delta_{\tilde{a}}(x_j) = 0$; $j = 1,2$. The operator x_j being self–adjoint

$$u|a|^{p-1}x_j = x_ju|a|^{p-1} \Longrightarrow |a|^{2(p-1)}x_j = |a|^{p-1}u^*x_ju|a|^{p-1} = x_j|a|^{2(p-1)}.$$

Hence $[x_j, |a|] = 0$. Since $\tilde{a}x_j = x_j\tilde{a}$ implies $[x_j, u]|_{\overline{\mathrm{ran}|a|^{p-1}}} = 0$, it follows that

$$ax_j = u|a|x_j = ux_j|a| = x_ju|a| = x_ja.$$

Hence $\delta_a(x_1) + i\delta_a(x_2) = \delta_a(x) = 0$. □

A stronger result is possible in the case in which $p = 2$.

Corollary 2.5. *If $a \in \mathcal{C}_2$, then*

$$||a + \delta_{x_j}(y)||_2^2 = ||a||_2^2 + ||\delta_{x_j}(y)||_2^2 = ||a^* + \delta_{x_j}(y)||_2^2, \quad j = 1, 2,$$

for all $y \in \mathcal{C}_2$ if and only if $x = x_1 + ix_2 \in {\delta_a}^{-1}(0) \cap \delta_{a^*}^{-1}(0)$.

Proof. $\mathcal{C}_2$ has a Hilbert space structure with inner product $(s, t) = \mathrm{tr}(t^* s)$. Since

$$\begin{aligned}
&||a + \delta_{x_j}(y)||_2^2 = ||a||_2^2 + ||\delta_{x_j}(y)||_2^2 + 2\mathrm{Re}(\delta_{x_j}(y), a),\\
&||a^* + \delta_{x_j}(y)||_2^2 = ||a||_2^2 + ||\delta_{x_j}(y)||_2^2 + 2\mathrm{Re}(a, \delta_{x_j}^*(y)) = ||a||_2^2 + ||\delta_{x_j}(y)||_2^2 + 2\mathrm{Re}(a, \delta_{x_j}(y)),\\
&(\delta_{x_j}(y), a) = \mathrm{tr}(a^* \delta_{x_j}(y)) = \mathrm{tr}(\delta_{a^*}(x_j)y) = \mathrm{tr}(\delta_a^*(x_j)y),
\end{aligned}$$

and

$$(a, \delta_{x_j}(y)) = \mathrm{tr}(\delta_{x_j}(y)a) = \mathrm{tr}(\delta_a(x_j)y),$$

it follows that if $x \in {\delta_a}^{-1}(0) \cap \delta_{a^*}^{-1}(0)$, then $||a + \delta_{x_j}(y)||_2^2 = ||a||_2^2 + ||\delta_{x_j}(y)||_2^2 = ||a^* + \delta_{x_j}(y)||_2^2$ for all $y \in \mathcal{C}_2$. Conversely, if this equality is satisfied, then the argument of the proof Theorem 2.4 (with $p = 2$) applied to the inequalities $||a||_2 \leq ||a - \delta_{x_j}(y)||_2$ and $||a^*||_2 \leq ||a^* - \delta_{x_j}(y)||_2$ implies that x_j, and so also x, $\in {\delta_a}^{-1}(0) \cap \delta_{a^*}^{-1}(0)$. $\square$

The elementary operator $\triangle_a(x) = axa - x$. We close this note with a remark on the elementary operator $\triangle_a$. If $a \in B(\mathcal{X})$ is a contraction, then $L_a R_a$ is a contraction. Hence

$$V(B(B(\mathcal{X}))), L_a R_a) \subseteq \{\lambda \in C : |\lambda| \leq 1\}$$

and

$$V(B(B(\mathcal{X})), \triangle_a) = V(B(B(\mathcal{X})), L_a R_a - I) \subseteq \{\lambda \in C : |\lambda + 1| \leq 1\}$$

[5, Proposition 4, Page 52]. (Here C denotes the complex plane.) This implies that the operator $\triangle_a$ is *dissipative* [3, Page 30], and hence

$$||x|| \leq ||x - \triangle_a(y)||$$

for all $x \in \triangle_a^{-1}(0)$ and $y \in B(\mathcal{X})$ [12, Theorem 7]. Although $\triangle_a$ may not be normal even for normal $a \in B(\mathcal{X})$, see [7, Example 2.1], a number of kernel–range orthogonality results for the elementary operator $\triangle_a \in B(B(\mathcal{H}))$ and $\triangle_a \in B(\mathcal{C}_p)$ are to be found in the extant literature; see for example [6], [7], [11], [14]. Seemingly, self–commutator approximant inequalities of the type (2) are not possible for $\triangle_a$. However, one does have the following interesting result.

Theorem 2.6. *Assume that $\triangle_a^{-1}(0) \subseteq \triangle_{a^*}^{-1}(0)$. If $a \in B(\mathcal{H})$ (resp., $a \in \mathcal{C}_p$), then $||a|| \leq ||a - [|x|, |x^*|]||$ for all $x \in B(\mathcal{H}) \cap \triangle_a^{-1}(0)$ (resp., $||a||_p \leq ||a - [|x|, |x^*|]||_p$ for all $x \in \mathcal{C}_p \cap \triangle_a^{-1}(0)$).*

Proof. If $x \in \triangle_a^{-1}(0)$, then $axa = x$ and $a^*xa^* = x$ ($\Longleftrightarrow ax^*a = x^*$). Since

$$ax^*x = (ax^*)axa = (ax^*a)xa = x^*xa,$$

$[a, |x|] = 0$. Hence $\delta_{|x|}(a) = 0$, which, since $|x| \geq 0$, implies that $||a|| \leq ||a - \delta_{|x|}(y)||$ for all $y \in B(\mathcal{H})$ (resp., $||a||_p \leq ||a - \delta_{|x|}(y)||_p$ for all $y \in B(\mathcal{H})$ such that $\delta_{|x|}(y) \in \mathcal{C}_p$). Choose $y = |x^*|$. Since $x \in \mathcal{C}_p$ implies $|x| \in \mathcal{C}_p$, the proof is complete. $\square$

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