복소다항식의 Schur 안정성

Schur Stability of Complex Polynomials

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Abstract: Determining the Schur stability of a polynomial is one of fundamental steps in many engineering problems including digital control system design or digital filter design. Due to its importance a variety of techniques have been reported in the literature for checking the Schur stability of a given polynomial. However most of them focus on real polynomials, and few results are available for complex polynomials. This paper concerns the Schur stability of complex polynomials. A simplified Jury's table for real polynomials is extended to complex polynomials.

Keywords: complex polynomial, schur stability, jury test

I. INTRODUCTION

Consider a linear time-invariant discrete system with the characteristic polynomial

$$D(z) = d_n z^n + d_{n-1} z^{n-1} + \cdots + d_1 z + d_0 \tag{1}$$

where $d_n \neq 0$. The stability of the system is decided by the location of zeros of $D(z)$ with respect to the unit circle in the complex plane. In order for the system to be stable it is required that its characteristic polynomial be a Schur stable polynomial, i.e., $D(z)$ has all its zeros inside the unit circle.

Determining the Schur stability of a polynomial is one of fundamental steps in many engineering problems including digital control system design or digital filter design. Due to its importance, many researchers have studied this topic and numerous techniques have been reported in the literature for checking the Schur stability of real (e.g., see [1] and references cited in [2]) or complex [3-5] polynomials. Among them the Jury test [1] provides a simple tabular form by which one can easily determine the Schur stability of real polynomials. As is well known the Jury test given in [1] has been considered as the most typical way of Schur stability test for real polynomials and it is introduced in most undergraduate textbooks. In [3] it was stated that a table similar to that of [1] can be used to test the Schur stability of complex polynomials. Then the table was utilized for the stability test of 2-D digital filters. A modified tabular form of [3] was given in [4] which can be used for the same purposes. In [5] a method based on a sequence of symmetric polynomials was presented for determining the root locations of complex polynomials. Later the test procedure of [5] was applied to the stability test of 2-D digital filters [6].

Although many techniques have been reported in the literature for checking the Schur stability of a given polynomial, most of them focus on real polynomials and few results (exceptions are [3-5]) are available for complex polynomials. It is because physical systems are usually represented by mathematical models with real coefficients. However testing methods for complex polynomials are important not only in academic point of view but also in engineering applications such as the stability test for 2-D digital filters [3-5].

This paper concerns the Schur stability of complex polynomials. The simplified Jury's table for real polynomials given in [7] is extended to complex polynomials. A division-free table is also given, which is appropriate for the stability test for 2-D digital filters as in [3,4]. The second stability table derived in this paper is very much similar to those of [3,4]. However a rigorous but simple proof is provided whereas it is not the case in [3,4]. This paper is organized as follows. In Section II a simplified Jury's table for real polynomials [7] is briefly reviewed. Main results of this paper are given in Section III, and Section IV contains some examples. Finally Section V concludes the paper.

II. SIMPLIFIED JURY'S TABLE FOR REAL POLYNOMIALS [4]

For an *n*th-order real polynomial $D(z)$ given in (1), form the table:

$$\begin{array}{ccccccc} d_n^{(n)} & d_{n-1}^{(n)} & d_{n-2}^{(n)} & \cdots & d_1^{(n)} & d_0^{(n)} & k_n \\ d_{n-1}^{(n-1)} & d_{n-2}^{(n-1)} & \cdots & d_1^{(n-1)} & d_0^{(n-1)} & & k_{n-1} \\ d_{n-2}^{(n-2)} & \cdots & d_1^{(n-2)} & d_0^{(n-2)} & & & k_{n-2} \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & & & & \vdots \\ d_1^{(1)} & d_0^{(1)} & & & & & k_1 \\ d_0^{(0)} & & & & & & \end{array} \tag{2}$$

where $d_i^{(n)} = d_i$, $k_i = d_0^{(i)} / d_i^{(i)}$ and

$$\begin{aligned} d_{n-i}^{(n-1)} &= d_{n-i+1}^{(n)} - k_n d_{i-1}^{(n)}, \qquad i = 1,2,\cdots,n \\ d_{n-i}^{(n-2)} &= d_{n-i+1}^{(n-1)} - k_{n-1} d_{i-2}^{(n-1)}, \qquad i = 2,3,\cdots,n \\ &\vdots \\ d_0^{(0)} &= d_1^{(1)} - k_1 d_0^{(1)} \end{aligned}$$

In [7] it has been shown that the table (2) is equivalent to Jury's tabular form given in [1]. Consequently we have that if $d_n > 0$, $D(z)$ is Schur stable if and only if $d_k^{(k)} > 0$ for $k = n-1$, $n-2, \cdots, 0$.

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As noted in [7], the table (2) is closely related to the recursive algorithm given in [8]. Consider a sequence of polynomials

$$D_k(z) = \hat{d}_k^{(k)} z^k + \hat{d}_{k-1}^{(k)} z^{k-1} + \cdots + \hat{d}_1^{(k)} z + \hat{d}_0^{(k)} \tag{3}$$

recursively obtained by

$$D_n(z) = D(z) \tag{4}$$

$$D_{k-1}(z) = \frac{1}{z}(D_k(z) + \gamma^{(k)} z^k D_k(z^{-1})) \tag{5}$$

where $\gamma^{(k)} = -\hat{d}_0^{(k)} / \hat{d}_k^{(k)}$. Since

$$z^k D_k(z^{-1}) = \hat{d}_0^{(k)} z^k + \hat{d}_1^{(k)} z^{k-1} + \cdots + \hat{d}_{k-1}^{(k)} z + \hat{d}_k^{(k)} \tag{6}$$

then

$$D_{k-1}(z) = (\hat{d}_k^{(k)} + \gamma^{(k)} \hat{d}_0^{(k)}) z^{k-1} + \cdots + (\hat{d}_1^{(k)} + \gamma^{(k)} \hat{d}_{k-1}^{(k)}) \tag{7}$$

and we have, for $i = 1, 2, \cdots, k$

$$\begin{aligned} \hat{d}_{k-i}^{(k-1)} &= \hat{d}_{k-i}^{(k)} + \gamma^{(k)} \hat{d}_i^{(k)} \\ &= \frac{1}{\hat{d}_k^{(k)}} (\hat{d}_k^{(k)} \hat{d}_{k-i+1}^{(k)} - \hat{d}_0^{(k)} \hat{d}_{i-1}^{(k)}) \end{aligned} \tag{8}$$

Now let $k = n$. Then

$$\begin{aligned} \hat{d}_{n-i}^{(n-1)} &= \frac{1}{d_n^{(n)}} (d_n^{(n)} d_{n-i+1}^{(n)} - d_0^{(n)} d_{i-1}^{(n)}) \\ &= d_{n-i}^{(n-1)}, \end{aligned} \qquad i = 1, 2, \cdots, n \tag{9}$$

Hence the second row in (2) constitutes the coefficients of $D_{n-1}(z)$. Similarly we can show that the third row of (2) constitutes the coefficients of $D_{n-2}(z)$, and the fourth row of (2) corresponds to the coefficients of $D_{n-3}(z)$, etc. In other words, the sequence of polynomials in (3) recursively obtained from (4) and (5) can be written in terns of table (2) as

$$D_k(z) = d_k^{(k)} z^k + d_{k-1}^{(k)} z^{k-1} + \cdots + d_1^{(k)} z + d_0^{(k)} \tag{10}$$

for $k = n, n-1, \cdots, 1, 0$.

On the other hand it has been shown in [8] that if $d_k^{(k)} > 0$, then the following two conditions are equivalent.

(i) The polynomial $D_k(z)$ is Schur stable.

(ii) The polynomial $D_{k-1}(z)$ is Schur stable and $d_{k-1}^{(k-1)} > 0$.

Repeatedly applying (i) and (ii), we have that if $d_n > 0$, $D(z)$ is Schur stable if and only if $d_k^{(k)} > 0$ for $k = n-1, n-2, \cdots, 0$ as in [7].

III. MAIN RESULTS

In this Section we extend the result of [7] to complex polynomials. To this end we need some preliminary results. Consider an nth-order complex polynomial $D(z)$ in (1), and let $\hat{D}(z) = z^n \bar{D}(\bar{z}^{-1})$, where 'bar' denotes the complex conjugate.

Lemma 1 below was proved in [8] for real polynomials. However it can be shown that the same conclusion also holds for complex polynomials. On the other hand Theorem 1 is an extension of [Theorem 1, 9] to complex polynomials

Lemma 1: $|D(z)| = |\hat{D}(z)|$ for $|z| = 1$.

Proof: Let

$$D(z) = d_n \sum_{i=1}^{n} (z - \alpha_i) \tag{11}$$

Then

$$\hat{D}(z) = \bar{d}_n \sum_{i=1}^{n} (1 - \bar{\alpha}_i z) \tag{12}$$

After some manipulations, we have

$$|z - \alpha_i|^2 - |1 - \bar{\alpha}_i z|^2 = (|z^2| - 1)(1 - |\alpha_i|^2)$$

and the result follows.

Theorem 1: For any complex number $\gamma (|\gamma| < 1)$, the following statements are equivalent.

(i) $D(z)$ is Schur stable.

(ii) $D(z) + \gamma \hat{D}(z)$ is Schur stable.

Proof: Consider the family of polynomials

$$D(\lambda, z) = D(z) + \lambda \gamma \hat{D}(z), \quad \lambda \in [0,1]$$

Clearly $D(0,z) = D(z)$ and $D(1,z) = D(z) + \gamma \hat{D}(z)$. Now assume that one of these two polynomials is Schur stable, whereas the other one is not. Then, by the boundary crossing theorem [10], there exists $\lambda \in [0,1]$ such that $D(\lambda, z)$ has a zero, say z_0, on the unit circle, i.e., $D(\lambda, z_0) = 0$ and $|z_0| = 1$. However this is impossible since $|D(z_0)| = |\hat{D}(z_0)|$ by Lemma 1 and $|\lambda\gamma| < 1$.

Now we are ready to state our main result which is an extension of the result in [7] to complex polynomials.

Theorem 2: For the complex polynomial $D(z)$ given in (1), form the table:

$$\begin{array}{ccccccc} d_n^{(n)} & d_{n-1}^{(n)} & d_{n-2}^{(n)} & \cdots & d_1^{(n)} & d_0^{(n)} & k_n \\ d_{n-1}^{(n-1)} & d_{n-2}^{(n-1)} & \cdots & d_1^{(n-1)} & d_0^{(n-1)} & & k_{n-1} \\ d_{n-2}^{(n-2)} & \cdots & d_1^{(n-2)} & d_0^{(n-2)} & & & k_{n-2} \\ \vdots & \vdots & \iddots & & & & \vdots \\ d_1^{(1)} & d_0^{(1)} & & & & & k_1 \\ d_0^{(0)} & & & & & & \end{array} \tag{13}$$

where $d_i^{(n)} = d_i$, $k_i = d_0^{(i)} / \bar{d}_i^{(i)}$ and

$$\begin{aligned} d_{n-i}^{(n-1)} &= d_{n-i+1}^{(n)} - k_n \bar{d}_{i-1}^{(n)}, \qquad i = 1, 2, \cdots, n \\ d_{n-i}^{(n-2)} &= d_{n-i+1}^{(n-1)} - k_{n-1} \bar{d}_{i-2}^{(n-1)}, \quad i = 2, 3, \cdots, n \\ &\vdots \\ d_0^{(0)} &= d_1^{(1)} - k_1 \bar{d}_0^{(1)} \end{aligned}$$

Assume $\mathrm{Re}\{d_n\} > 0$. Then $D(z)$ is Schur stable if and only if $\mathrm{Re}\{d_k^{(k)}\} > 0$ for $k = n-1, n-2, \cdots, 0$.

Proof: Consider a sequence of polynomials

$$D_k(z) = \tilde{d}_k^{(k)} z^k + \tilde{d}_{k-1}^{(k)} z^{k-1} + \cdots + \tilde{d}_1^{(k)} z + \tilde{d}_0^{(k)} \tag{14}$$

recursively obtained by

$$D_n(z) = D(z) \tag{15}$$

$$D_{k-1}(z) = \frac{1}{z}(D_k(z) + \gamma^{(k)} z^k \bar{D}_k(\bar{z}^{-1})) \tag{16}$$

where $\gamma^{(k)} = -\tilde{d}_0^{(k)} / \bar{\tilde{d}}_k^{(k)}$. Then, as in Section 2, we can show that the second row of (13) constitutes the coefficients of $D_{n-1}(z)$, and the third row of (2) corresponds to the coefficients of $D_{n-2}(z)$, etc. In other words, the sequence of polynomials in (14) recursively obtained from (15) and (16) can be written in terns of table (13) as

$$D_k(z) = d_k^{(k)} z^k + d_{k-1}^{(k)} z^{k-1} + \cdots + d_1^{(k)} z + d_0^{(k)} \tag{17}$$

for $k = n, n-1, \cdots, 1, 0$.

If $\mathrm{Re}\{d_k^{(k)}\} > 0$ and $D_k(z)$ is Schur stable, then $\left|\gamma^{(k)}\right| = \left|d_0^{(k)} / \bar{d}_k^{(k)}\right| < 1$ and $D_{k-1}(z)$ is Schur stable by Theorem 1. Furthermore

$$\begin{aligned} d_{k-1}^{(k-1)} &= d_k^{(k)} - \gamma^{(k)} \bar{d}_0^{(k)} \\ &= \frac{\left|d_k^{(k)}\right|^2 - \left|d_0^{(k)}\right|^2 d_k^{(k)}}{\left|d_k^{(k)}\right|^2} \end{aligned}$$

Hence we have $\mathrm{Re}\{d_{k-1}^{(k-1)}\} > 0$. Conversely if $\mathrm{Re}\{d_{k-1}^{(k-1)}\} > 0$ and $D_{k-1}(z)$ is Schur stable, then $\left|\gamma^{(k)}\right| < 1$ and $D_k(z)$ is Schur stable again by Theorem 1. Hence if $\mathrm{Re}\{d_k^{(k)}\} > 0$, the following two conditions are equivalent.

(i) $D_k(z)$ is Schur stable.

(ii) $\mathrm{Re}\{d_{k-1}^{(k-1)}\} > 0$ and $D_{k-1}(z)$ is Schur stable.

Now we are ready to complete the proof of Theorem 2 by showing that if $\mathrm{Re}\{d_n\} > 0$, (iii) and (iv) below are equivalent.

(iii) $D(z)$ is Schur stable.

(iv) $\mathrm{Re}\{d_k^{(k)}\} > 0$ for $k = n-1, n-2, \cdots, 0$.

Suppose $D_n(z) = D(z)$ is Schur stable. Then repeatedly applying the equivalence of (i) and (ii), we obtain (iv). Conversely assume (iv) holds. Since $\mathrm{Re}\{d_0^{(0)}\} > 0$ and $D_0(z) = d_0^{(0)}$ is Schur stable, $D_1(z)$ is Schur stable by the equivalence of (i) and (ii). Similarly $D_2(z)$ is Schur stable since $d_1^{(1)} > 0$ and $D_1(z)$ is Schur stable. Continuing the same process, we have that $D_n(z) = D(z)$ is Schur stable.

As noted in Section II, techniques for checking the Schur stability for complex polynomials are useful in determining whether a 2-D digital filter is stable or not [3,4]. For example consider a 2-D digital filter with a transfer function given as

$$T(z_1, z_2) = \frac{C(z_1, z_2)}{D(z_1, z_2)} \tag{18}$$

It is assumed that nonessential singularities of the second kind [11] does not exist. Then the digital filter given in (18) is stable if and only if the following two conditions are satisfied [12]:

(a) $D(z_1, 0) \neq 0$ for $|z_1| \leq 1$

(b) $D(z_1, z_2) \neq 0$ for $|z_1| = 1$, $|z_2| \leq 1$

The condition (a) can be easily checked using the Jury test [2] for the reciprocal of $D(z_1, z_2)$ with respect to z_1. The condition (b) can be checked using the techniques for Schur stability test for complex polynomials. To this end, express $D(z_1, z_2)$ as a function in z_2 and then form the table (13) for the reciprocal with respect to z_2. In this case the entries of (13) are rational functions of z_1 and it is not easy to handle. To ameliorate this problem, we can obtain a division-free table as in [3,4]. This can be accomplished by modifying the recursive algorithm (16) as

$$D_{k-1}(z) = \frac{\bar{\tilde{d}}_k^{(k)}}{z}(D_k(z) + \gamma^{(k)} z^k \bar{D}_k(\bar{z}^{-1})) \tag{19}$$

or

$$D_{k-1}(z) = \frac{1}{z}(\bar{\tilde{d}}_k^{(k)} D_k(z) - \tilde{d}_0^{(k)} z^k \bar{D}_k(\bar{z}^{-1})) \tag{20}$$

Even if (19) or (20) is used, the proof of Theorem 2 still holds with slight modification. Hence we have Theorem 3 below.

Theorem 3: For the complex polynomial $D(z)$ given in (1), form the table:

$$\begin{array}{cccccc} d_n^{(n)} & d_{n-1}^{(n)} & d_{n-2}^{(n)} & \cdots & d_1^{(n)} & d_0^{(n)} \\ d_{n-1}^{(n-1)} & d_{n-2}^{(n-1)} & \cdots & d_1^{(n-1)} & d_0^{(n-1)} & \\ d_{n-2}^{(n-2)} & \cdots & d_1^{(n-2)} & d_0^{(n-2)} & & \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & & & \\ d_1^{(1)} & d_0^{(1)} & & & & \\ d_0^{(0)} & & & & & \end{array} \tag{21}$$

where $d_i^{(n)} = d_i$ and

$$\begin{aligned} d_{n-i}^{(n-1)} &= d_{n-i+1}^{(n)} \bar{d}_n^{(n)} - d_0^{(n)} \bar{d}_{i-1}^{(n)}, \quad i = 1, 2, \cdots, n \\ d_{n-i}^{(n-2)} &= d_{n-i+1}^{(n-1)} \bar{d}_{n-1}^{(n-1)} - d_0^{(n-1)} \bar{d}_{i-2}^{(n-1)}, \quad i = 2, 3, \cdots, n \\ &\vdots \\ d_0^{(0)} &= \left|d_1^{(1)}\right|^2 - \left|d_0^{(1)}\right|^2 \end{aligned}$$

Then $D(z)$ is Schur stable if and only if $d_k^{(k)} > 0$ for $k = n-1, n-2, \cdots, 0$.

IV. EXAMPLES

Example 1: Consider a third order complex polynomial given by

$$D(z) = (1 + j)z^3 + (0.3 - j0.3)z^2 - (0.1 - j0.4)z + j0.4 \tag{22}$$

Then the table (13) is given by

$1 + j$	$0.3 - j0.3$	$-0.1 + j0.4$	$j0.4$	$k_3 = -0.2 + j0.2$
$0.92 + j0.92$	$0.2 - j0.36$	$0.02 + j0.4$		$k_2 = -0.21 + j0.23$
$0.83 + j0.83$	$0.32 - j0.33$			$k_1 = 0.39 - j0.005$
$0.70 + j0.70$				

Since $\mathrm{Re}\{d_k^{(k)}\} > 0$, for $k = 0,1,2,3$, we can conclude that $D(z)$ is Schur stable. On the other hand table (21) is computed by

$$\begin{array}{llll} 1+j & 0.3-j0.3 & -0.1+j0.4 & j0.4 \\ 1.84 & -0.16-j0.56 & 0.42+j0.38 & \\ 3.0648 & -0.0144-j1.2048 & & \\ 7.9412 & & & \end{array}$$

Since $d_k^{(k)} > 0$, for $k = 0,1,2$, we can also conclude that $D(z)$ is Schur stable. In fact $D(z)$ has zeros at $z_1 = 0.5 - j0.5$, $z_2 = j0.8$ and $z_3 = -0.5$.

Example 2: Consider a 2–D digital filter with the characteristic polynomial [13]

$$D(z_1,z_2) = (z_1^2 + 5z_1 + 6)z_2 + (2z_1^2 + 10z_1 + 12) \qquad (23)$$

It is clear that $D(z_1,0) \neq 0$ for $|z_1| \leq 1$. To check if $D(z_1,z_2) \neq 0$ for $|z_1| = 1$, $|z_2| \leq 1$, take the reciprocal of $D(z_1,z_2)$ with respect to z_2, say $D_2(z_1,z_2)$. Then

$$D_2(z_1,z_2) = (2z_1^2 + 10z_1 + 12)z_2 + (z_1^2 + 5z_1 + 6) \qquad (24)$$

and it is necessary to check if $D_2(z_1,z_2) \neq 0$ for $|z_1| = 1$, $|z_2| \geq 1$. Since $D_2(z_1,z_2)|_{|z_1|=1}$ is a function of z_2 with complex coefficients, Theorem 3 can be used. For $D_2(z_1,z_2)$ in (23), form the table

$$\begin{array}{ll} 2z_1^2 + 10z_1 + 12 & z_1^2 + 5z_1 + 6 \\ \bar{z}_1^2 + 5\bar{z}_1 + 6 & 2\bar{z}_1^2 + 10\bar{z}_1 + 12 \\ d_0^{(0)}(z_1) & \end{array}$$

where

$$d_0^{(0)}(z_1) = 24(z_1^2 + \bar{z}_1^2) + 140(z_1 + \bar{z}_1) + 248$$

Since $|z_1| = 1$, let $z_1 = e^{j\theta}$, $\theta \in [0, 2\pi]$. Then

$$d_0^{(0)}(\theta) = 48\cos 2\vartheta + 280\cos\theta + 248$$

or

$$d_0^{(0)}(x) = 96x^2 + 280x + 248$$

where $x = \cos\theta$. It is easily seen that $d_0^{(0)}(x) > 0$ for $|x| \leq 1$. Hence $D(z_1,z_2)$ is stable.

V. CONCLUSIONS

This paper concerned the Schur stability of complex polynomials. A simplified Jury's table for real polynomials [7] was extended to complex polynomials. A division-free stability table was also presented, which is appropriate for the stability test of 2-D digital filters. Contributions of this paper can be summarized as follows: (i) It is meaningful in academic point of view. (ii) The division-free table obtained in this paper takes a similar form to those of [3,4]. However a rigorous but simple proof is provided whereas it is not the case in [3,4].

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